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The condition of non-commutativity of the squares of two vector fields in a nilpotent distribution

Présenté par :
Mlle. Mimouni Assia Elbatoule

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Devant le jury composé de :

Mme. GUERRARRA Siham	M.C.A	Université de Ghardaïa	Président
Mme.KHELLAF Yasmina	M.A.A	Université de Ghardaïa	Encadreuse
M.CHIKH SALAH Abdelouahab	M.C.B	Université de Ghardaïa	Examinateur 1
M.NAIMI Abdelouahab	M.A.A	Université de Ghardaïa	Examinateur 2

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DEDECATION

Praise be to God for His countless blessings . I have reached the end of my journey , where I prayed to God to grant me a moment of joy in my success , and God amazed me with His generosity .

I dedicate my graduation to :

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ملخص

في هذه المذكرة، نسعى إلى تحديد بعض الخصائص المتعلقة بالتوزيع النيلبوتيني $\mathcal{D} = span\{X,Y\}$ على المجال $\mathbb{R}^n$، هذه الخاصية تحدد الشرط الضروري لقوس لاي $[X^2,Y^2] \neq 0$، تقدم هذه النتيجة حسابات أقل ووصفًا ممتازًا لحساب $[X^2,Y^2]$

الكلمات المفتاحية

توزيع ، حقل متجهي ، قوس لاي ، النيلبوتيني ...

Abstract

In this Memory , we aim to define certain properties regarding the nilpotent distribution $\mathcal{D} = span\{X,Y\}$ in the space $\mathbb{R}^n$, This property determines the necessary condition for the Lie bracket $\left[X^2,Y^2\right] \neq 0$. This result offers less computation and excellent description for the calcul of $\left[X^2,Y^2\right]$.

Keywords:

Distribution , Vector fields , Lie bracket , nilpotent ...

Résumé

Dans ce mémoire , nous visons à définir certaines propriétés concernant la distribution nilpotente $\mathcal{D} = span\{X,Y\}$ dans un l'espace $\mathbb{R}^n$, Cette propriété détermine la condition nécessaire pour le crochet de Lie $\left[X^2,Y^2\right] \neq 0$. Ce résultat offre moins de calculs et une excellente description pour le calcul de $\left[X^2,Y^2\right]$.

Les mots clés:

Distribution , Champ de vecteur , Crochet de lie , nilpotent ...

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INTRODUCTION

The theory of subelliptic operators $L = X_1^2 + \cdots + X_1^m$ to $\mathbb{R}$, where $X_1^2 + \cdots + X_1^m$ are vector fields , plays a fundamental role in sub-Riemannian geometry . The main question addressed in this paper is the theory of finding heat kernels . The method for finding heat kernels depends on the commutativity of the squares of the vector fields $X_1^2 + \cdots + X_m^2$. If the operators commute , then the value of the heat kernel is the product of the heat kernels of the operators . Otherwise , the Toranter formula will be presented using the method of integration by parts...

The goal se in this memory is to present the result cited in [13] , [6] , defining a very important property in sub-Riemannian geometry . This property aims to verify the non-commutativity of the square of the vector fields X and Y in a nilpotent distribution $\mathcal{D} = span\{X,Y\}$ of class 2 and class 3 .

Our work revolves around three essential points

The first chapter : It is dedicated to recalling the essential basic concepts necessary for understanding differential geometry , It provides the mathematical framework , deliberately very detailed , in which this document is situated , we definde champ vecteur , distribution , sub-Riemannian structur , Lie bracket , ...

Second chapter : The kernel of this work , is dedicated to defining a very important notion regarding the nilpotent distribution $\mathcal{D} = span\{X,Y\}$ of class 2 and class 3 , which are necessary conditions for the non-commutativity of squar of the vector field X , Y .

Third chapter : We illustrate our work with two excelente examples in sub-Riemannian geometry those of the Heisenberg distribution and Martinet distribution .

this work shows that the condition of non-commutativity of the squares of two vector fields in a nilpotent distribution is a fundamental aspect of the study of sub-Riemannian geometry . This property highlights the uniqueness of nilpotent distributions and their impact on the geometric and topological structure of the studied spaces . A deep understanding of these conditions opens new avenues for research in this field , enhancing our knowledge of the interactions between geometry and analysis .

CHAPTER 1

PRELIMINARY

1.1 Vector fields in $\mathbb{R}^n$

We use any of the notation

$$\partial_j \quad , \quad \partial_{x_j} \quad , \quad \frac{\partial}{\partial_{x_j}} \quad , \quad \partial/\partial_{x_j} \quad ,$$

to indicate the partial derivative operator with respect to the j-th coordinate of $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be an open (and non-empty) set .

Definition 1.1 *[15] Given an N-tuple of a scalar function $a_1,...,a_N$*

$$a_j : \Omega \to \mathbb{R} \quad , \quad j \in \{1,...,N\} \quad .$$

the first order linear differential operator

$$X = \sum_{j=1}^{N} a_j \partial_j \quad . \tag{1.1}$$

will be called a vector field on Ω with component functions (or simply , components) $a_1,...,a_N$. If $f : \Omega \longrightarrow \mathbb{R}^m$ is a differentiable function , we denote Xf the function on Ω by

$$Xf(x) = \sum_{j=1}^{N} a_j(x)\partial_j f(x) \quad , \quad x \in \Omega \quad .$$

Occasionally , we shall also use the notation Xf when

$$f : \Omega \longrightarrow \mathbb{R}^m \quad .$$

is a vector-valued function , to mean the component-wise action of X . More precisely , we set

$$Xf(x) = \begin{pmatrix} Xf_1(x) \\ \vdots \\ Xf_m(x) \end{pmatrix} \quad for \quad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \quad .$$

Furthermore , given a differentiable function $f:\Omega\longrightarrow\mathbb{R}^m$ *, we shall denote by*

$$\mathfrak{J}_f(x) \quad , \quad x\in\Omega \quad .$$

the Jacobian matrix of f at x .

Let $C^\infty(\Omega,\mathbb{R})$ *for brevity ,* $C^\infty(\Omega)$ *be the set of smooth (i.e . infinitely-differentiable) real-valued functions . If the components* a_j *are smooth , we shall call X a smooth vector field and we shall often consider X as an operator acting on smooth functions ,*

$$X: C^\infty(\Omega)\to C^\infty(\Omega), f\mapsto Xf \quad .$$

We shall denote by $T(\mathbb{R}^n)$ *the set of all smooth vector fields in* $\mathbb{R}^n$ *. Equipped with the natural operations ,* $T(\mathbb{R}^n)$ *is a vector space over* $\mathbb{R}$ *.*

We adopt the following notation : I will denote the identity map on $\mathbb{R}^n$ *and , if X is the vector field in (1.1) , then*

$$XI := \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}. \tag{1.2}$$

will be the column vector of the components of X .

This notation is obviously consistent with our definition of the action of X on a vector-valued function . Thus , XI may also be regarded as a smooth map from $\mathbb{R}^n$ *to itself .*

Often , many authors identify X and XI . Instead , in order to avoid any confusion between a smooth vector field as a function belonging to $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ *and a smooth vector field as a differential operator from* $C^\infty(\mathbb{R}^n)$ *to itself , we prefer to use the different notation XI and X as described in (1.2) and (1.1) , respectively*

By consistency of notation , we may write

$$Xf = (\nabla f)\cdot XI \quad .$$

where $\nabla = (\partial_1,\cdots,\partial_N)$ *is the gradient operator in* $\mathbb{R}^n$ *, f is any real-valued smooth function on* $\mathbb{R}^n$ *and* $\cdot$ *denotes the row* $\times$ *column product .*

For example : *for the following two vector fields on* $\mathbb{R}^3$ *(whose points are denoted by* $x=(x_1,x_2,x_3)$*)*

$$X_1 = \partial_{x1} + 2x_2\partial_{x3} \quad , \qquad X_2 = \partial_{x2} - 2x_1\partial_{x3} \quad , \tag{1.3}$$

we have

$$X_1 I(x) = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix} \quad , \quad X_2 I(x) = \begin{pmatrix} 0 \\ 1 \\ -2x_1 \end{pmatrix} \quad , \tag{1.4}$$

1.2 Integral Curves

Definition 1.2 *[15]*

A path $\gamma : \mathfrak{D} \to \mathbb{R}^n$ being an interval of $\mathbb{R}$, will be said an integral curve of the smooth vector field X if

$$\dot{\gamma}(t) = XI(\gamma(t)) \qquad for \quad every \quad t \in \mathfrak{D} \quad .$$

If X is a smooth vector field , then , for every $x \in \mathbb{R}^n$, he Cauchy problem

$$\begin{cases} \dot{\gamma} = XI(\gamma) & . \\ \gamma(0) = x & . \end{cases} \tag{1.5}$$

has a unique solution

$$\gamma_X(.,x) : \mathfrak{D}(X,x) \to \mathbb{R}^n \quad .$$

Since X is smooth , $t \to \gamma_x(t,x)$ is a C^∞ function whose n-th Taylor expansion in a neighborhood of $t = 0$ is given by

$$\gamma_X(t,x) = x + tX^{(1)}I(x) + \frac{t^2}{2!}tX^{(2)}I(x) + \cdots + \frac{t^n}{n!}tX^{(n)}I(x) + \frac{1}{n!}\int_0^t (t-s)^n X^{(n+1)}I(\gamma_X(s,x))ds \tag{1.6}$$

Hereafter , for $k \in \mathbb{N}$, we denote by $X^{(k)}$ the vector field

$$X^{(k)} = \sum_{j=1}^{N} (X^{(k-1)}a_j)\partial_{xj} \quad .$$

being $X^0 = I$ (the identity map) and X^h , $h \geq 1$, the h-th order iterated of X , i.e .

$$X^h := \underbrace{X \circ \cdots \circ X}_{htimes} \quad .$$

Example 1.1 *For example , if X_1 is as in (1.3) , since*

$$X_1^{(1)}I = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix}, X_1^{(2)}I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = X_1^{(k)}I, \quad \forall k \geq 3 \quad .$$

we have

$$\gamma_{X1}(t,x) = x + tX_1^{(1)}I(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 + t \\ x_2 \\ x_3 + 2x_2 t \quad . \end{pmatrix}$$

Definition 1.3 *[15] Let X be a smooth vector field on $\mathbb{R}^n$. Following all the above notation , we set*

$$exp(tX)(x) := \gamma_X(t,x) \quad .$$

where $\gamma_X(\cdot,x)$ is the solution of (1.5) Then , being X smooth , for every $n \in \mathbb{N}$, we have the expansion

$$exp(tX)(x) = \sum_{k=0}^{n} \frac{t^k}{k!}X^k I(x) + \frac{1}{n!}\int_0^t (t-s)^n X^{n+1} I(exp(sX)(x))ds \quad .$$

In particular , for $n = 1$,

$$exp(tX)(x) = x + tX^1 I(x) + \int_0^t (t-s)X^2 I(exp(sX)(x))ds \quad .$$

Moreover , from the unique solvability of the Cauchy problem related to smooth vector fields we get: $t \in \mathfrak{D}(-X,x)$ iff $-t \in \mathfrak{D}(X,x)$ and

$$exp\,(tX)\,(x) := exp((t)X)(x) = exp(t(X))(x) \quad .$$
$$exp(tX)(exp(tX)(x) = x \quad .$$
$$exp((t+\tau)X)(x) = exp(tX)(exp(X)(x)) \quad .$$
$$exp((t\tau)X)(x) = exp(t(X))(x) \quad .$$

when all the terms are defined .

Remark 1.1 *[15] Let us consider a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ and the vector field in (1.1) . Then*

$$Xu(x) = \lim_{t\to 0} \frac{u(exp(tX)(x) - u(x)}{t} \forall x \in \mathbb{R}^n \quad . \tag{1.7}$$

Indeed , since $exp(tX)(x) = x + tXI(x) + \mathcal{O}(t^2)$, the limit on the right-hand side of (1.7) is equal to the following one :

$$\lim_{t\to 0} \frac{u(x+tXI(x)) - u(x)}{t} = \nabla u(x) \cdot XI(x) = Xu(x) \quad .$$

1.3 Lie Brackets of Vector Fields in $\mathbb{R}^n$

Definition 1.4 *[15]*

If $X = \sum_{j=1}^{N}, a_j \partial_j$ and $Y = \sum_{j=1}^{N}, b_j \partial_j$, a direct computation shows that the Lie bracket $[X,Y]$ is the vector field

$$[X,Y] = \sum_{j=1}^{N} (Xb_j - Ya_j)\partial_j \quad .$$

As a consequence , Given two smooth vector fields X and Y in $\mathbb{R}^n$, we define the Lie bracket [X,Y] as follows

$$[X,Y] := XY - YX \quad .$$

If $X = \sum_{j=1}^{N}, a_j \partial_j$ and $Y = \sum_{j=1}^{N}, b_j \partial_j$, a direct computation shows that the Lie bracket $[X,Y]$ is the vector field

$$[X,Y] = \sum_{j=1}^{N} (Xb_j - Ya_j)\partial_j \quad .$$

As a consequence ,

$$[X,Y]I = \begin{pmatrix} Xb_1 \\ \vdots \\ Xb_N \end{pmatrix} - \begin{pmatrix} Ya_1 \\ \vdots \\ Ya_N \end{pmatrix} = \mathfrak{I}_{YI} \cdot XI - \mathfrak{I}_{XI} \cdot YI \quad .$$

Example 1.2 *[15] If X_1, X_2 are as in 1.3 , we have*

$$[X_1, X_2] = (X_1(-2x_1) - X_2(2x_2))\partial_{x_3} = -4\partial_{x_3} \quad .$$

It is quite trivial to check that $(X,Y) \mapsto [X,Y]$ is a bilinear map on the vector space $T(\mathbb{R}^n)$ satisfying the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad .$$

for every $X,Y,Z \in T(\mathbb{R}^n)$

We now introduce some other notation on the algebras of vector fields . Given a set of vector fields $Z_1, \cdots, Z_m \in T(\mathbb{R}^n)$ *and a multi-index*

$$J = (j_1, \cdots, j_k) \in \{1, \cdots, m\}^k \quad .$$

we set

$$Z_J := [Z_{j_1}, \cdots [Z_{j_{k-1}}, Z_{j_k}] \cdots] \quad .$$

We say that Z_J *is a commutator of length (or height) k of* $Z_1, \cdots, Z_m$ *. If* $J = j_1$ *, we also say that* $Z_J := Z_{j1}$ *is a commutator of length 1 of* $Z_1, \cdots, Z_m$ *A commutator of the form* Z_J *will also be called nested , in order to emphasize its difference from , e.g . a commutator of the form We shall refer to* $T(\mathbb{R}^n)$ *(equipped with the above Lie bracket) as the Lie algebra of the vector fields on* $\mathbb{R}^n$ *. Any sub-algebra* $\mathfrak{g}$ *of* $T(\mathbb{R}^n)$ *will be called a Lie algebra of vector fields .*
More explicitly , $\mathfrak{g}$ *is a Lie algebra of vector fields if* $\mathfrak{g}$ *is a vector subspace of* $T(\mathbb{R}^n)$ *closed with respect to* $[,]$ *, i.e .* $[X,Y] \in \mathfrak{g}$ *for every* $X, Y \in \mathfrak{g}$ *.*

We now introduce some other notation on the algebras of vector fields . Given a set of vector fields $Z_1, \cdots, Z_m \in T(\mathbb{R}^n)$ *and a multi-index*

$$J = (j_1, \cdots, j_k) \in \{1, \cdots, m\}^k \quad .$$

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$$[[Z_1, Z_2], [Z_3, Z_4]] \quad .$$

Definition 1.5 (The Lie algebra generated by a set) *[15]*
If V is any subset of $T(\mathbb{R}^n)$ *, we denote by* $Lie\{V\}$ *the least sub-algebra of* $T(\mathbb{R}^N)$ *containing V , i.e .*

$$Lie\{V\} := \bigcap \mathfrak{h} \quad .$$

where $\mathfrak{h}$ *is a sub-algebra of* $T(\mathbb{R}^n)$ *with* $V \subseteq \mathfrak{h}$ *. We also define*

$$rank(Lie\{V\}(x)) := dim_{\mathbb{R}}\{ZI(x) | Z \in Lie\{V\} \quad .$$

Example 1.3 *Let* X_1 *and* X_2 *be as in (1.3) .*
Since $[X_1, X_2] = -4\partial_{x_3}$ *and since any commutator involving* X_1, X_2 *more than twice is identically zero then*

$$Lie\{X_1, X_2\} = span\{X_1, X_2, [X_1, X_2]\} \quad , \quad and \quad rank(Lie\{X_1, X_2\}(x)) = 3 \quad for \quad every \quad x \in \mathbb{R}^3 \quad .$$

The following result holds .

1.4 Lie groups on $\mathbb{R}^n$

The Lie Algebra of a Lie Group on $\mathbb{R}^n$

We first recall a well-known definition.

Definition 1.6 *[15] Let $\circ$ be a given group law on $\mathbb{R}^n$ and suppose that the map*

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto y^{-1} \circ x \in \mathbb{R}^n \quad .$$

is smooth . Then $\mathbb{G} := (\mathbb{R}^n, \circ)$ is called a Lie group on $\mathbb{R}^n$.

Fixed $\alpha \in \mathbb{G}$, we denote by $\tau_\alpha(x) := \alpha \circ x$ the left-translation by α on $\mathbb{G}$. A (smooth) vector field X on $\mathbb{R}^n$ is called left-invariant on $\mathbb{G}$ if

$$X(\varphi \circ \tau_\alpha) = (X_\varphi) \circ \tau_\alpha \quad .$$

For every $\alpha \in \mathbb{G}$ and for every smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$. We denote by $\mathfrak{g}$ the set of the left-invariant vector fields on $\mathbb{G}$. It is quite obvious to recognize that

for every $X,Y \in \mathfrak{g}$ and for every $\lambda, \mu \in \mathbb{R}$ we have $\lambda X + \mu Y \in \mathfrak{g}$ and $[X,Y] \in \mathfrak{g}$.

Then , $\mathfrak{g}$ is a Lie algebra of vector fields , sub-algebra of $T(\mathbb{R}^n)$. It will be called the Lie algebra of $\mathbb{G}$.

Example 1.4 (First Heisenberg group $\mathbb{H}^1$) *The map*

$$(x_1,x_2,x_3) \circ (y_1,y_2,y_3) = (x_1+y_1, x_2+y_2, x_3+y_3+2(x_2y_1-x_1y_2)) \quad .$$

are left invariant w.r.t . $\circ$. Consequently, $X_1, X_2, [X_1,X_2] \in \mathfrak{h}^1$, say , the Lie algebra of $\mathbb{H}^1$.

Precisely ,

$$\mathfrak{h}^1 = span\{X_1, X_2, [X_1,X_2]\} = Lie\{X_1, X_2\}.$$

From the theorem of differentiation of composite functions , we easily get the following characterization of left-invariant vector fields on $\mathbb{G}$.

1.5 Distribution

Definition 1.7 *[14]*

A smooth distribution Δ of rank $m \leq n$ ($m \geq 1$) on M is a rank m subbundle of the tangent bundle TM , that is a smooth map that assigns to each point x of M a linear subspace $\Delta(x)$ of the tangent space T_xM f dimension m . In other terms , for every $x \in M$, here are an open neighborhood $\mathcal{V}_x$ of x in M and m smooth vector fields $X_x^1, \ldots, X_x^m$ linearly independent on $\mathcal{V}_x$ such that

$$\Delta(y) = Span\{X_x^1(y), \ldots, X_x^m\} \quad \forall y \in \mathcal{V}_x \quad .$$

*Such a family of smooth vector fields is called a **local frame** in $\mathcal{V}_x$ for the distribution $\Delta(x)$. All the distributions which will be considered later will be smooth with constant rank $m \in [1,n]$. Thus , from now on , "distribution" always means " smooth distri-bution with constant rank " . A co-rank k distribution on M is a distribution of rank $m = n-k$ and any smooth vector field X on M such that $X(x) \in \Delta(x)$ or any $x \in M$ is called a section of Δ .*

1.6 Sub-Riemannian Geometry

Definition 1.8 *[9]*

Let M be a manifold , together with a subbundle $\mathcal{H}$ of the tangent bundle . Then a ***sub-Riemannian manifold*** *is a triple (M,$\mathcal{H}$,h) where* ***h*** *is a fiber-metric on $\mathcal{H}$.*

In this setting , the subbundle $\mathcal{H}$ s called the ***horizontal distribution*** and ***h*** *is called the sub-Riemannian metric .*

We will use the letter d to refer to the rank of $\mathcal{H}$, and n to refer to the dimension of the manifold . The vector fields in $\Gamma(\mathcal{H})$ are called ***horizontal vector fields*** *. Given a sub-Riemannian metric* ***h*** *we can define the analogue of the flat and sharp operator as in Riemannian geometry . The definition of flat with respect to* ***h*** *then becomes*

$$\flat^h : \mathcal{H} \longrightarrow \mathcal{H}^* \quad , \quad \flat^h(v) = h(v,.) \quad .$$

for $v \in \mathcal{H}$, while the sharp operator with respect to ***h*** *is defined to be*

$$\sharp^h : T^*M \longrightarrow \mathcal{H}, \sharp^h(\omega) = (\flat^h)^{-1} \quad (\omega \mid \mathcal{H}) \quad .$$

for $\omega \in T^*M$.
Using the sharp operator we can define the ***cometric*** *with respect to* ***h*** *by*

$$h^* : T^*M \times T^*M \longrightarrow T^*M, < \alpha, \beta >_{h^*} = \alpha(\sharp^h(\beta)) \quad .$$

The cometric h^* is symmetric , since given any two elements $\alpha, \beta \in T^*M$ such that $\sharp^h(\alpha) = v$ and $\sharp^h(\beta) = w$, we have that

$$< v, w >_h = < v, . >_h (w) = \alpha \mid_{\mathcal{H}} (\sharp^h \beta) = < \alpha, \beta >_{h^*} \quad .$$

Note that $\sharp^h(\alpha) = h^*(\alpha, .)$. An additional property of the cometric is that h^* is zero on the annihilator of $\mathcal{H}$ in T^*M , i.e . the subbundle of T^*M given by

$$\{\alpha \in T^*M : \alpha(v) = 0 \quad \forall v \in \mathcal{H}\} \quad .$$

It is also possible to define the cometric by the following two properties : h^* is zero on the annihilator of H , and

$$< \sharp^h \alpha, \sharp^h \beta >_h = < \alpha, \beta >_{h^*} \quad .$$

for all $\alpha, \beta \in T^*M$ The main advantage of working with the cometric instead of the sub-Riemannian metric is that the cometric is defened on the entire T^*M instead of a subbundle . Later we will also encounter the cometric as the symbol of the sub-Laplacian , which gives us another reason for preferring the cometric rather than the sub-Riemannian metric .

Let $\gamma : [a,b] \longrightarrow M$ be a continuous curve such that $\dot{\gamma} \in \mathcal{H}$ almost everywhere . We say that γ is a ***horizontal absolutely continuous*** *curve if it satisfy*

$$\frac{d}{dx} \int_a^x \|\dot{\gamma}(t)\|_h dt = \|\dot{\gamma}(x)\|_h \quad .$$

almost everywhere.The ***length*** *of γ is then defined to be*

$$l(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_h dt \quad .$$

Denote by $C_{ac}^{\mathcal{H}}(A,B)$ the set of all horizontal absolutely continuous curves connecting the two points A and B on the manifold . We define the ***distance*** *between A and B , denoted d(A,B) to be*

$$d(A,B) = \inf_{C_{ac}^{\mathcal{H}}(A,B)} l(\gamma) \quad .$$

where we use the convention $inf\varnothing = \infty$. There are many cases in which the distance between two points may be infinite .

For instance , if $\mathcal{H}$ is an integrable distribution , i.e . $[X,Y] \in \Gamma(\mathcal{H})$ for all $X,Y \in \Gamma(\mathcal{H})$.

1.7 Sub-Riemannian structures

Let M be a smooth n-dimensional manifold.

Definition 1.9 *[12]*
A sub-Riemannian structure on M is a pair (D,g) where D is a distribution and g is a Riemannian metric on D .

A sub-Riemannian manifold (M,D,g) is a smooth manifold M equipped with a sub-Riemannian structure (D,g) .

Recall that a distribution D of rank $m\,(m \leq n)$ is a family of m-dimensional linear subspaces $D_q \subset T_qM$ depending smoothly on $q \in M$. A Riemannian metric on D is a smooth function $g : D \to \mathbb{R}$ which restrictions g_q to D_q are positive definite quadratic forms .

Let (M,D,g) be a sub-Riemannian manifold . A horizontal curve $\gamma : I \subset \mathbb{R} \to M$ is an absolutely continuous curve such that $\dot{\gamma}(t) \in D_{\gamma(t)}$ for almost every $t \in I$.

We define the length of a horizontal curve , as in Riemannian geometry , by :

$$length(\gamma) = \int_I \sqrt{g_{\gamma(t)}(\dot{\gamma}(t))dt} \quad .$$

Definition 1.10 *[12] The sub-Riemannian distance on (M, D, g) is defined by*

$$d(p,q) = inf\{length(\gamma) : \gamma \quad horizontal \quad curve \quad \gamma \quad joins \quad p \quad to \quad q \quad \} \quad .$$

We use the convention

$$inf\emptyset = +\infty \quad .$$

Thus,if p and q can not be joined by a horizontal curve ,

$$d(p,q) = +\infty \quad .$$

1.8 Brackets-Generating Distributions

In the following we shall define a very important type of horizontal distributions . Let T_pM be to tangent space of the manifold M at p . For each given point $p \in M$ We shall construct the following *sequence of ascendant linear subspaces of the space* T_pM :

$$\begin{array}{ll} \mathcal{D}_p^1 = & \mathcal{D}_p \\ \mathcal{D}_p^2 = & \mathcal{D}_p^1 + [\mathcal{D}_p, \mathcal{D}_p^1] \\ \mathcal{D}_p^3 = & \mathcal{D}_p^2 + [\mathcal{D}_p, \mathcal{D}_p^2] \\ \vdots & \\ \mathcal{D}_p^{n+1} = & \mathcal{D}_p^n + [\mathcal{D}_p, \mathcal{D}_p^n] \end{array}$$

Where $[\mathcal{D}_p, \mathcal{D}_p^n] = \{[X,Y]; X \in \mathcal{D}_p, Y \in \mathcal{D}_p^n\}$ *, such that + is* $\cup$ *.*

Definition 1.11 *[5]*
The distributions $\mathcal{D}$ *is said bracket generating at the point* $p \in M$ *if there is an integer* $r \geq 1$ *such that* $\mathcal{D}_p^r = T_pM$ *. The integer r is called the step of the sub-Riemannian manifold* $(M, \mathcal{D}, g)$ *at the point* p *.*

Remark 1.2 *[5]*

1. *The step is a property of the distribution* $\mathcal{D}$ *and does not depend on the sub-Riemannian metric* g *.*
2. *We have rank* $\mathcal{D}_p^r = dimM$
3. *There are distribution where the step is the same for all points . They are called constant-step distribution*

1.9 Nilpotent Distibutions

We Difine the iterated commutator sets

$$\begin{aligned} \mathcal{C}^1 &= \{[X;Y]\,; X,Y \in \Gamma(\mathcal{D})\} \\ \mathcal{C}^2 &= \{[[X;Y]\,,Z]\,; X,Y,Z \in \Gamma(\mathcal{D})\} \\ &= \{\left[\mathcal{C}^1, Z\right]; Z \in \Gamma(\mathcal{D})\} \\ &\vdots \\ \mathcal{C}^{n+1} &= \{[\mathcal{C}^n, Z]\,; Z \in \Gamma(\mathcal{D})\} \end{aligned}$$

$\mathcal{C}^n$ *is the set of vector fields obtained by n iterated lie brackets of horizontal vector fields*

Definition 1.12 *[5] the distibutions* $\mathcal{D}$ *is called nilpotent if there is an integer* $n \geq 1$ *such that* $\mathcal{C}^n = 0$ *: i.e , all the n iterated Lie brackets vanish . The smallest integer n with this property is called the nilpotnce classe of* $\mathcal{D}$

Example 1.5 *Let's consider the following vector fields in* $\mathbb{R}^3$

$$X = \frac{\partial}{\partial_x} + y\frac{\partial}{\partial_z} \quad , \quad Y = \frac{\partial}{\partial_y} \quad .$$

Step-by-Step Verification of Nilpotency :

1. *First-order Lie bracket :*
   Let's compute the Lie bracket between the vector fields :

$$[X,Y] = \left[\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_z}, \frac{\partial}{\partial_y}\right] \quad .$$

   Using the definition of the Lie bracket :

$$[X,Y] = X(Y) - Y(X) \quad .$$

   - $X(Y)$ *represents the action of* X *on* Y *:*

$$X\left(\frac{\partial}{\partial_y}\right) = 0 \quad .$$

   - $Y(X)$ *represents the action of* Y *on* X *:*

$$Y\left(\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_z}\right) = \frac{\partial}{\partial_z} \quad .$$

   Thus ,

$$[X,Y] = -\frac{\partial}{\partial_z} \quad .$$

2. *Higher-order Lie brackets :*
   Let's compute second-order Lie brackets, i.e . brackets of the form $[X,[X,Y]], [Y,[X,Y]], etc.$
   Since $[X,Y] = -\frac{\partial}{\partial_z}$ *is a constant vector field :*

$$[X,[X,Y]] = \left[\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_z}, -\frac{\partial}{\partial_z}\right] = 0 \quad .$$

$$[Y,[X,Y]] = \left[\frac{\partial}{\partial_y}, -\frac{\partial}{\partial_z}\right] = 0 \quad .$$

   Then , the nilpotent distribution of class 2 .

Proposition 1.1 *[5] There are brackets-generating distibutions that are not nilpotent*

Proof 1.9.1 *[5] We shall provide an example . Let* $\mathcal{D} = span\{X_1, X_2\}$ *,where*

$$X_1 = \partial_{x1} + e^{x2}\partial_t \quad , \quad X_2 = \partial_{x2} \quad .$$

are vector fields on $\mathbb{R}^3_{(x,t)}$.

Since $[X_1, X_2] = -e^{x2}\partial_t$*, it follows that* $X_1, X_2, [X_1, X_2]$ *are lineary independent vector fields at every point* $(x_1, x_2, t) \in \mathbb{R}^3$.

Hence the distibution $\mathcal{D}$ *is brackets generating with constant step 2 .*

On the other hand , the distibution $\mathcal{D}$ *is non nilpotent since the iterated Lie brackets never vanish because of the expontial factor* e^{x_2} .

CHAPTER 2
THE NON-COMMUTATIVITY ANALYSIS OF THE SQUARES OF TWO VECTOR FIELDS IN A NILPOTENT DISTRIBUTION

2.1 Introduction

In this chapter , we define a necessary condition for $\left[X^2, Y^2\right] \neq 0$ *such that* $\mathcal{D} = span\{X, Y\}$ *is a nilpotent distribution in* $\mathbb{R}^m$ *.*

Definition 2.1 *the vector fields X and Y be satisfing condition* $\mathfrak{R}_k$ *at point p if* $\left[X^k, Y^k\right]_p \neq 0$ *.*

Example 2.1 *[6] The vector fields*

$$X = \partial_x \quad , \quad Y = \partial_y + x\partial_z \quad .$$

The vector fields satisfy the condition everywhere on $\mathfrak{R}_2$ *. This follows from the relations*

$$[X, Y] = \partial_z \quad .$$
$$X^2 = \partial_x^2 \quad , \quad Y^2 = \partial_y^2 + 2x\partial_y\partial_z + x^2\partial_z^2 \quad .$$
$$\left[X^2, Y^2\right] = 4\partial_x\partial_y\partial_z + 2\partial_z^2 + 4x\partial_x\partial_z^2 \quad .$$

2.2 Posed the problem

Let X and Y be two vector fields on $\mathbb{R}^m$ *. Consider the differential operator of order n obtained by iterating the same vector field k times* $X^k = X \dots X$ *. We start by observing that if the vector fields X and Y commute , then the operators* X^k *and* X^k *k also commute . This can be written as the following set relation* [4] *:*

$$\left\{p; [X, Y]_p = 0\right\} \subseteq \left\{p; \left[X^k, Y^k\right]_p = 0\right\} \Longleftrightarrow \left\{p; \left[X^k, Y^k\right]_p \neq 0\right\} \subset \left\{p; [X, Y]_p \neq 0\right\} \quad .$$

Proof 2.2.1 *[11] We begin with the case where* $k = 2$ *, Using* $XY = YX$ *,*

Multiling the left :

$$
\begin{aligned}
& X^2Y^2 = XXYY = XYXY = YXXY = YXYX \\
& \quad = Y^2X^2 \quad . \\
& \quad \vdots \\
For \quad n{=}k \qquad & X^kY^k = X\cdots XY\cdots Y = X\cdots YX\cdots Y \\
& \quad \vdots \\
& \quad = Y\cdots YX\cdots X \\
& \quad = Y^k\cdots X^k \quad .
\end{aligned}
$$

Example 2.2 *Consider the following vector fields :*

$$X = \partial_x \quad , \quad Y = y\partial_y \quad .$$

We have :

$$[X,Y] = [\partial_x, y\partial_y] = XY - YX = y\partial_x\partial_y - y\partial_x\partial_y = 0 \quad .$$

Then we calculate

$$X^2 = \partial_x^2 \quad , \quad Y^2 = y^2\partial_y^2 \quad .$$

we calculate the commutator

$$\left[X^2, Y^2\right] = \left[\partial_x^2, y^2\partial_y^2\right] = X^2Y^2 - Y^2X^2 = y^2\partial_x^2\partial_y^2 - y^2\partial_x^2\partial_y^2 = 0 \quad .$$

So

$$([X,Y] = 0) \Longrightarrow \left(\left[X^2, Y^2\right] = 0\right) \quad .$$

For the other side of the equivalence :

Example 2.3 *:*

Consider the following vector fields :

$$X = \partial_x \quad , \quad Y = \partial_y + x\partial_z \quad .$$

$$
\begin{aligned}
[X,Y] &= XY - YX = \partial_x(\partial_y + x\partial_z) - (\partial_y + x\partial_z)\partial_x \\
&= \partial_x\partial_y + \partial_x(x\partial_z) - (\partial_y\partial_x + x\partial_z\partial_x) \\
&= (\partial_y\partial_x + \partial_z + x\partial_x\partial_z) - (\partial_y\partial_x + x\partial_z\partial_x) \\
&= \partial_y\partial_x + \partial_z + x\partial_x\partial_z \quad - \quad \partial_y\partial_x \quad - \quad x\partial_z\partial_x \\
&= \partial_z + x\partial_x\partial_z - x\partial_z\partial_x \\
&= \partial_z \\
&\neq 0 \quad .
\end{aligned}
$$

$$X^2 = \partial_x^2 \quad , \quad Y^2 = \partial_y^2 + x\partial_y\partial_z + x\partial_z\partial_y + x^2\partial_z^2$$

$$\begin{aligned}
[X^2,Y^2] &= X^2Y^2 - Y^2X^2 \\
&= \partial_x^2\left(\partial_y^2 + x\partial_y\partial_z + x\partial_z\partial_y + x^2\partial_z^2\right) - \left(\partial_y^2 + x\partial_y\partial_z + x\partial_z\partial_y + x^2\partial_z^2\right)\partial_x^2 \\
&= \partial_y^2\partial_x^2 + \partial_y\partial_z\partial_x + \partial_y\partial_z\partial_x + \partial_z\partial_x\partial_y + \partial_x\partial_z\partial_y + 2\partial_z^2 + 2x\partial_x\partial_z^2 - \left(\partial_y^2\partial_x^2 + \partial_y\partial_z\partial_x + \partial_z\partial_y\partial_x + 2x\partial_z^2\partial_x\right) \\
&= 2\partial_z^2 + 2x\partial_x\partial_z^2 - x\partial_y\partial_z\partial_x - x\partial_z\partial_y\partial_x \\
&\neq 0 \quad .
\end{aligned}$$

So

$$\left([X^2,Y^2] \neq 0\right) \Longrightarrow ([X,Y] \neq 0) \quad .$$

Remark 2.1 *The opposite of the property is not always true .*

counterexample

The vector fields

$$X = \partial_x \quad , \quad Y = \left(1 + x^2\right)\partial_z \quad .$$

We have :

$$[X,Y] = XY - YX \quad .$$

we calculate XY :

$$XY = \frac{\partial}{\partial x}\left(\left(1+x^2\right)\frac{\partial}{\partial z}\right) \quad .$$

Since $\left(\frac{\partial}{\partial z}\right)$ is just a coefficient function , we can take the derivative of the coefficient $\left(1+x^2\right)$ with respect to x :

$$XY = \frac{\partial}{\partial x}\left(1+x^2\right)\cdot\frac{\partial}{\partial z} = 2x\frac{\partial}{\partial z} \quad .$$

we calculate YX :

$$YX = \left(1+x^2\right)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\right) \quad .$$

Since $\left(\dfrac{\partial}{\partial x}\right)$ is a constant vector field , the derivative of $\left(\dfrac{\partial}{\partial x}\right)$ with respect to (z) is zero :

$$YX = \left(1+x^2\right)\cdot 0 = 0 \quad .$$

we calculate $[X,Y]$:

$$[X,Y] = \left[\partial_x, \left(1+x^2\right)\partial_z\right] = XY - YX = 2x\frac{\partial}{\partial z} - 0 = 2x\frac{\partial}{\partial z} \quad .$$

We have the square of vector fields is :

$$X^2 = \partial_x^2 \quad , \quad Y^2 = (1 + 2x^2 + x^4)(\partial_z)^2 \quad .$$

We have :

$$[X^2,Y^2] = X^2Y^2 - Y^2X^2 \quad .$$

we calculate X^2Y^2 :

$$X^2Y^2 = \frac{\partial^2}{\partial x^2}\left((1+x^2)^2\frac{\partial^2}{\partial z^2}\right) \quad .$$

Applying $\left(\frac{\partial^2}{\partial x^2}\right)$ *to* $\left((1+x^2)^2\right)$ *:*

$$\frac{\partial^2}{\partial x^2}(1+x^2)^2=\frac{\partial}{\partial x}\left(4x(1+x^2)\right)=4(1+x^2)+8x^2=4+12x^2 \quad .$$

Thus:

$$X^2Y^2=(4+12x^2)\frac{\partial^2}{\partial z^2} \quad .$$

Next , we calculate Y^2X^2

$$Y^2X^2=(1+x^2)^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial x^2}\right) \quad .$$

Since $\left(\frac{\partial^2}{\partial z^2}\right)$ *does not act on* x *, we get :*

$$Y^2X^2=(1+x^2)^2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2} \quad .$$

Therefore:

$$\left[X^2,Y^2\right]=X^2Y^2-Y^2X^2=\left(4+12x^2\right)\frac{\partial^2}{\partial z^2}-\left(1+x^2\right)^2\frac{\partial^2}{\partial z^2} \quad .$$

Simplifying further:

$$\left[X^2,Y^2\right]=\left(4+12x^2-1-2x^2-x^4\right)\frac{\partial^2}{\partial z^2}=\left(3+10x^2-x^4\right)\frac{\partial^2}{\partial z^2} \quad .$$

Thus, the commutator $\left[X^2,Y^2\right]$ *is:*

$$\left[X^2,Y^2\right]=\left(3+10x^2-x^4\right)\frac{\partial^2}{\partial z^2} \quad .$$

At the point $(0,y,z)$ *,We have*

$$[X,Y]_p=0 \quad , \quad and \quad \left[X^2,Y^2\right]_p\neq 0 \quad .$$

- *In the following part , we will define the vector fields X and Y in a nilpotent distribution (whether it be of class 2 or class 3) in such a way that the opposite of this property holds true .*

2.3 The necessary condition For $\left[X^2,Y^2\right]\neq 0$

2.3.1 The case of nilpotent distribution of class 2

Theorem 2.3.1 *[13]*
Any distribution $\mathcal{D}=span\{X,Y\}$ *of nilpotency class equal to 2 is a* $\mathfrak{R}_2$*-distribution .*

Proof 2.3.1 *See [6]*

2.3.2 The case of nilpotent distribution of class 3

Theorem 2.3.2 *[13] Any distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class equal to 3 is a $\Re_2$-distribution , i.e , $\left[X^2,Y^2\right] \neq 0$.*

To prove this theorem , we use several lemmas , and we recall the following . The distribution $\mathcal{D} = span\{X,Y\}$ is nilpotent of class 3 meaning that

$$[X,Y] \neq 0 \quad . \tag{2.1}$$

$$[X\,[X,Y]] \neq 0 \quad or \quad [Y\,[X,Y]] \neq 0 \quad .$$

$$and \qquad [X\,[X\,[X,Y]]] = 0, \quad [Y\,[Y\,[X,Y]]] = 0 \quad . \tag{2.2}$$

$$[X\,[Y\,[X,Y]]] = 0, \quad [Y\,[X\,[X,Y]]] = 0 \quad . \tag{2.3}$$

Lemma 2.1 *[13] In a distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class 3 , we have*

$$\left[X^2,Y^2\right] = 0 \Longrightarrow (XY)^2 = (YX)^2 \quad .$$

Proof 2.3.2 *[13] By developing the first equation of (2.5)*

$$[X,[Y,[X,Y]]] = 0 \Longleftrightarrow X^2Y^2 - Y^2X^2 - 2\,(XY)^2 + 2\,(YX)^2 = 0$$
$$\Longrightarrow X^2Y^2 - Y^2X^2 = 2\left((XY)^2 - (YX)^2\right) \quad .$$

or

$$\left[X^2,Y^2\right] = 0 \quad .$$

then

$$(XY)^2 = (YX)^2 \quad .$$

Lemma 2.2 *[13] In a distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class 3 , we have*

$$\left[X^2,Y^2\right] = 0 \Longrightarrow XYX^2Y^2 = X^2Y^2XY \quad . \tag{2.4}$$

Proof 2.3.3 *[13] The expansion of the equations (2.3.1) gives*

$$X^3Y - 3X^2YX + 3XYX^2 - YX^3 = 0 \quad . \tag{2.5}$$

$$Y^3X - 3Y^2XY + 3YXY^2 - XY^3 = 0 \quad . \tag{2.6}$$

Multiplying the right-hand side , then the left-hand side of the relation (2.5) by Y^2 and therelation (2.6) by X^2 , we obtain

$$X^3Y^3 - 3X^2YXY^2 + 3XYX^2Y^2 - YX^3Y^2 = 0 \quad . \tag{2.7}$$

$$X^3Y^3 - 3Y^2XYX^2 + 3YXY^2X^2 - XY^3X^2 = 0 \quad . \tag{2.8}$$

$$X^3Y^3 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2 = 0 \quad . \tag{2.9}$$

$$X^3Y^3 - X^2Y^3X + 3X^2Y^2XY - 3X^2YXY^2 = 0 \quad . \tag{2.10}$$

By the subtractions of these equations , (2.7)-(2.8),(2.7)-(2.9)-(2.10)-(2.9) we have found , respectively ,

$$X^3Y^3 - Y^3X^3 = 3X^2YXY^2 - 3XYX^2Y^2 + YX^3Y^2 - 3Y^2XYX^2 + 3YXY^2X^2 - XY^3Y^2 \quad . \tag{2.11}$$

$$X^3Y^3 - Y^3X^3 = 3X^2YXY^2 - 3XYX^2Y^2 + YX^3Y^2 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2 \quad . \tag{2.12}$$

$$X^3Y^3 - Y^3X^3 = X^2Y^3X - 3X^2Y^2XY + 3X^2YXY^2 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2 \quad . \tag{2.13}$$

Subtracting the equations (2.11)-(2.12) gives

$$-3XYX^2Y^2 + YX^3Y^2 + Y^2X^3Y - 3Y^2X^2YX - XY^3X^2 + 3YXY^2X^2 - X^2Y^3X + 3X^2Y^2XY = 0 \quad .$$

In view of the fact that $X^2Y^2 = Y^2X^2$ *, the last equation becomes*

$$-XYX^2Y^2 + XYX^2Y^2 + X^2Y^2XY - X^2Y^2XY = 0 \quad .$$

then

$$X^2Y^2[X,Y] = [X,Y]X^2Y^2 \quad . \tag{2.14}$$

On the other hand , subtracting the equations (2.13)-(2.11) gives

$$-3XYX^2Y^2 + YX^3Y^2 - X^2Y^3X + 3X^2Y^2XY = 0 \quad .$$

then

$$-[X,Y]X^2Y^2 + X^2Y^2[X,Y] + 2\left(X^2Y^2XY - XYX^2Y^2\right) \quad .$$

Using the relation (2.3) , we obtain

$$XYX^2Y^2 = X^2Y^2XY \quad .$$

Lemma 2.3 *[13] In a distribution* $\mathcal{D} = span\{X,Y\}$ *of nilpotency class 3 , we have*

$$\left[X^2,Y^2\right] = 0 \Longrightarrow X^2Y^2 = 3(XY)^2 \quad . \tag{2.15}$$

Proof 2.3.4 *[13] Multiplying the equation (2.5) in the proof of Lemma (2.2) by Y on two sides , we obtain*

$$YX^3Y^2 - 3YX^2YXY + 3YXYX^2Y - Y^2X^3Y = 0 \quad . \tag{2.16}$$

Lemma(2.2) proves that

$$XYX^2Y^2 = X^2Y^2XY \quad .$$

and interchanging X and Y , we get

$$YXX^2Y^2 = X^2Y^2YX \quad .$$

then (2.16) becames

$$X^2Y^2[X,Y] - 3\left((XY)^3 - (YX)^3\right) = 0 \quad .$$

this implies that

$$\left(X^2Y^2 - 3(XY)^2\right)[X,Y] = 0 \quad .$$

but $[X,Y] \neq 0$ *, then*

$$X^2Y^2 = 3(XY)^2 \quad .$$

Proof 2.3.5 (Proof of theorem 2.3.1) *[13]*

We shall prove this theorem by contradiction , i.e , we assume that

$$\left[X^2, Y^2\right] = 0 \quad . \tag{2.17}$$

By developing $[X,Y]^3$ *and using Lemma(2.1) we get*

$$\begin{aligned}[X,Y]^3 &= (XY)^3 - (XY)^2(XY) - (XY)(YX)(XY) + (XY)(YX)^2 \\ &\quad - (YX)(XY)^2 + (YX)(XY)(YX) + (YX)^2(XY) - (YX)^3 \\ &= 3(XY)^3 - 3(YX)^3 - (XY)(YX)(XY) + (YX)(XY)(YX) \quad .\end{aligned} \tag{2.18}$$

Using Lemma(2.3) , we get

$$\begin{aligned}(XY)(YX)(XY) &= XY^2X^2Y \\ &= 3X(YX)^2Y \\ &= 3XYXYXY = 3(XY)^3. \\ (YX)(XY)(YX) &= YX^2Y^2Y \\ &= 3Y(XY)^2X \\ &= 3YXYXYX = 3(YX)^3.\end{aligned}$$

The equation (2.18) becomes

$$[X,Y]^3 = 3(XY)^3 - 3(YX)^3 - 3(XY)^3 + -3(YX)^3 = 0 \quad .$$

then $[X,Y] = 0$ *is a contradiction . It turns out that (2.17) cannot hold . It follows that the vector fields* X *and* Y *span a* $\mathfrak{R}_2$ *distribution .*

CHAPTER 3
APPLICATION TO THE SPECIFIC DISTRIBUTION

In this section , we introduce two examples of the sub-Riemannian geometry : the Heisenberg distribution and Martinet distribution , where we apply the all proprties that we saw in the previous chapters .

3.1 The Distribution of the Heisenberg

we introduce the Heisenberg group $\mathbb{H}^1$, a non-commutative group with underlying manifold $\mathbb{R}^3$. For the sake of transparency considering the type of problems we are dealing with in this dissertation , we only look at the one-dimensional Heisenberg group , and the extensions to higher dimensions are easy generalizations .

We identify points in $\mathbb{R}^2$ with points in $\mathbb{C}$ through the following law :

$$\mathbb{R}^2 \ni (x,y) \leftrightarrow z = xi + y \in \mathbb{C} \quad .$$

Let $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. Then for all points $(z,t),(w,s) \in \mathbb{H}^1$, we define the group law by

$$(z,t)\cdot(w,s) = \left(z+w, t+s+\frac{1}{4}[z,w]\right) \quad .$$

3.1.1 The left-invariant vector fields

A vector field V on $\mathbb{H}^1$ is said to be left-invariant if

$$\mathsf{V}\mathsf{L}_{(w,s)} = \mathsf{L}_{(w,s)}\mathsf{V}$$

for all $(w,s) \in \mathbb{H}^1$ where $\mathsf{L}_{(w,s)}$ is the left translation by (w,s) defined by

$$(\mathsf{L}_{(w,s)}f)(z,t) = f((w,s)\cdot(z,t)), (z,t) \in \mathbb{H}^1$$

We now introduce a particular Lie algebra, namely the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$

Theorem 3.1.1 *Let* $\mathfrak{b}^1$ *be the set of all left-invariant vector fields on* $\mathbb{H}^1$*.Then* $\mathfrak{b}^1$ *is a Lie algebra in which the Lie bracket[.,.]is the commutator given by*

$$[X,Y] = XY - YX$$

for all $X,Y \in \mathfrak{b}^1$.

Proof 3.1.1 *Linearity is obvious.Let* $X,Y \in \mathfrak{b}^1$*,and we need to show firstly that* $[X,Y] \in \mathfrak{b}^1$.
We write

$$X = a_1\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y} + c_1\frac{\partial}{\partial t}$$

and

$$Y = a_2\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + c_2\frac{\partial}{\partial t}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ *are* C^∞ *functions on* $\mathbb{H}^1$*.Then one can easily check that*
$XY = a_1a_2\frac{\partial^2}{\partial x^2} + b_1b_2\frac{\partial^2}{\partial y^2} + c_1c_2\frac{\partial^2}{\partial t^2} + (a_1b_2 + a_2b_1)\frac{\partial^2}{\partial x\partial y} + (b_1c_2 + b_2c_1)\frac{\partial^2}{\partial y\partial t} + (a_1c_2 + a_2c_1)\frac{\partial^2}{\partial t\partial x} + V_1$
where V_1 *is a vector field on* $\mathbb{H}^1$ *By switching subscripts in the second-order terms in XY,we get*

$$[X,Y] = XY - YX = V_1 - V_2$$

where V_2 *is another vector field on* $\mathbb{H}^1$ *To see that [X,Y] is left-invariant, let* $(w,s) \in \mathbb{H}^1$*,and we use the left-invariance of X,Y to check that*

$$L_{(w,s)}XY = XL_{(w,s)}Y = XYL_{(w,s)}$$

and

$$L_{(w,s)}YX = YL_{(w,s)}X = YXL_{(w,s)}$$

Thus, we have

$$[X,Y]L_{(w,s)} = L_{(w,s)}[X,Y]$$

and therefore $[X,Y] \in \mathfrak{b}^1$*,as desired. Secondly, we prove Jacobi's identity.*

$$\begin{aligned}
[X,[Y,Z]] &= [X,YZ - ZY] = XYZ - XZY - YZX + ZYX, \\
[Y,[Z,X]] &= [Y,ZX - XZ] = YXZ - YZX - ZXY + XZY, \\
[Z,[X,Y]] &= [Z,XY - YX] = ZXY - ZYX - XYZ + YXZ,
\end{aligned}$$

Thus,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

and therefore $\mathfrak{b}^1$ is a Lie algebra.

Theorem 3.1.2 *X,Y,T are vector fields on $\mathbb{H}^1$ defined as follows ,*

$$X=\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial t}$$

$$Y=\frac{\partial}{\partial x}-\frac{1}{2}x\frac{\partial}{\partial t}$$

$$T=\frac{\partial}{\partial t}$$

Then X,Y,T form a basis for $\mathfrak{b}^1$

Proof 3.1.2 *Firstly, we check that $X,Y,T\in\mathfrak{b}^1$ i.e.,*

$$XL_{(w,s)}=L_{(w,s)}$$

for all $(w,s)\in\mathbb{H}^1$.To see this, we write $w=(u,v)$,$z=(x,y)$ Then

$$(L_{(w,s)}f)(z,t)=f((w,s)\cdot(z,t))=f(u+x,v+y,s+t+\frac{1}{2}(vx-uy))$$

where$(z,t)\in\mathfrak{b}^1$.To simplify notation, we denote

$$(...)=(u+x,v+y,s+t+\frac{1}{2}(vx-uy)$$

Then, we have

$$(XL_{(w,s)}f)(z,t)$$

$$\begin{aligned}&=((\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial t})L_{(w,s)}f)(z,t)\\&=\frac{\partial f}{\partial x}(...)+\frac{1}{2}v\frac{\partial f}{\partial t}(...)+\frac{1}{2}y\frac{\partial f}{\partial t}(...)\\&=\frac{\partial f}{\partial x}(...)+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}(...)\end{aligned}$$

On the other hand ,

$$\begin{aligned}&(XL_{(w,s)}f)(z,t)\\&=(Xf)(...)\\&=\frac{\partial f}{\partial x}(...)+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}(...)\end{aligned}$$

Thus ,

$$XL_{(w,s)}=L_{(w,s)}X$$

So we have proved that $X\in\mathfrak{b}^1$, and similar arguments show that Y,T are also elements of $\mathfrak{b}^1$ Moreover , we know that the Lie algebra $\mathfrak{b}^1$ s isomorphic to $T_{(0,0,0)}\mathbb{H}^1$, the tangent space of the

Heisenberg group at the origin, and a proof can be found in [44] . Since $T_{(0,0,0)}\mathbb{H}^1$ is a three dimensional vector space , it remains to show that X,Y,T are linearly independent.The see this, we consider , the equation

$$aX + bY + cT = 0$$

where a,b,c are real numbers. for all of $\mathbb{H}^1$, we must show that

$$(aX + bY + cT)f = 0 \Leftrightarrow a,b,c = 0$$

But this is clear if we pick

$$\begin{aligned} f(x,y,t) &= x \\ f(x,y,t) &= y \\ f(x,y,t) &= t \end{aligned}$$

Therefore,X,Y,T is a basis for $\mathfrak{h}^1$
Lastly, we explain the choice of vector fields X,Y,T as a basis for $\mathfrak{h}^1$

Theorem 3.1.3 *[8] Let e_1,e_2,e_3 be the coordinate axes and write them in their parameterized form*

$$\begin{aligned} e_1(s) &= (s,0,0), \quad s \in \mathbb{R} \\ e_2(s) &= (0,s,0), \quad s \in \mathbb{R} \\ e_3(s) &= (0,0,s), \quad s \in \mathbb{R} \end{aligned}$$

Then for all C^∞ functions f on $\mathbb{H}^1$, we have

$$\begin{aligned} (Xf)(z,t) &= \frac{d}{ds}|_{s=0} f((z,t)\cdot e_1(s)), \\ (Yf)(z,t) &= \frac{d}{ds}|_{s=0} f((z,t)\cdot e_2(s)), \\ (Tf)(z,t) &= \frac{d}{ds}|_{s=0} f((z,t)\cdot e_3(s)), \end{aligned}$$

for all $(z,t) \in \mathbb{H}^1$

Proof 3.1.3 *[8] Since*

$$\begin{aligned} &\frac{d}{ds}|_{s=0} f((z,t)\cdot e_1(s)) \\ &= \frac{d}{ds}|_{s=0} f(x+s,y,t+s+\frac{1}{2}sy) \\ &= \frac{\partial f}{\partial x}(x,y,t) + \frac{1}{2}y\frac{\partial}{\partial t}(x,y,t) \end{aligned}$$

We get

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$

as asserted.
Lastly , an observation can be made through the theorem below .

Theorem 3.1.4 *[8]* $[X,Y] = -T$, *and all other commutators among X,Y,T vanish .*

The vector fields X ,Y ,and their first-order commutator span the Lie algebra $\mathfrak{h}^1$ on the Heisenberg group. In fact, they are the so-called horizontal vector fields on $\mathbb{H}^1$,and T is known as the missing direction.
Now, we develop the sub-Laplacian on $\mathbb{H}^1$,which will later give rise to a family of linear operators known as the twisted Laplacians on $\mathbb{R}^3$.The sub-Laplacian $\mathfrak{L}$ on $\mathbb{H}^1$ is defined by

$$\mathfrak{L} = -(X^2 + Y^2)$$

More explicitly ,

$$\begin{aligned} X^2 &= (\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}) \\ &= \frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial t} + \frac{1}{4}y^2\frac{\partial^2}{\partial t^2} \end{aligned}$$

and

$$\begin{aligned} Y^2 &= (\frac{\partial}{\partial xy} - \frac{1}{2}y\frac{\partial}{\partial t})(\frac{\partial}{\partial y} - \frac{1}{2}y\frac{\partial}{\partial t}) \\ &= \frac{\partial^2}{\partial y^2} - x\frac{\partial^2}{\partial y \partial t} + \frac{1}{4}x^2\frac{\partial^2}{\partial t^2} \end{aligned}$$

Thus,

$$\mathfrak{L} = -\Delta - \frac{1}{4}\left(x^2 + y^2\right)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

3.1.2 The right-invariant vector fields

[3] Where as the right-invariant vector fields write :

$$\begin{aligned} \hat{X} &= \partial_x + y\partial_z \quad . \\ \hat{Y} &= \partial_y + x\partial_z \quad . \\ \hat{Z} &= \partial_z \quad . \end{aligned}$$

3.1.3 The degree of bracket generating distribution

The distribution $\mathcal{D} = span\{X,Y\}$ such that

$$X = \partial_x - y\partial_z \qquad , \qquad Y = \partial_y - x\partial_z \quad .$$

We have ,

$$[X,Y] = [\partial_x - y\partial_z, \partial_y - x\partial_z] = XY - YX = \partial_z \quad , \quad [[X,Y],Y] = 0 \quad , \quad [[X,Y],X] = 0 \quad .$$

According of chapter 1 , we have :

- $D^0 = 0$.
- $D^1 = D = span\{X_2, X_2\}$.
- $D^2 = D^1 + [D, D^1] = T_q\mathbb{R}^3$.

Then , the degree of bracket generating distribution of step 2 .

3.1.4 The class of nilpotence

We have:

$$\mathcal{C}^1 = \{[X,Y] = [\partial_x - y\partial_z, \partial_y - x\partial_z] = \partial_z; \quad X, Y \in \Gamma(\mathcal{D})\} \neq 0 \quad .$$

$$\mathcal{C}^2 = \{[[X;Y],Y] = 0, \quad [[X;Y],X] = 0; \quad X, Y \in \Gamma(\mathcal{D})\} = 0 \quad .$$

Then the class of nilpotent distibution is step 2 , according to the sited by chapter 2 , we have

$$[X^2, Y^2] \neq 0 \quad .$$

3.2 The Martinet distribution

We define the distribution $\mathcal{D} = span\{X, Y\}$ such that , X , Y two vector fields on $\mathbb{H}^1$ defined

3.2.1 The left invariant vector fields

$$X = \partial_x \quad , \quad Y = \partial_y + \frac{x^2}{2}\partial z \quad .$$

3.2.2 The right-invariant vector fields

Where as the right-invariant vector fields write :

$$X = \partial_x \quad , \quad Y = \partial_y - \frac{x^2}{2}\partial z \quad .$$

3.2.3 The degree of bracket generating

We have:

$$[X,Y] = x\partial_z \quad , \quad [X,[X,Y]] = 0 \quad , \quad [Y,[X,Y]] = 0 \quad .$$

The case of** $x = 0$ **:

The distribution $\mathcal{D} = span\{X, Y\}$ such that

$$X = \partial_x \quad , \quad Y = \partial_y + \frac{x^2}{2}\partial z \quad .$$

is bracket generating of step 3 .

The case of** $x \neq 0$ **:

Then the bracket generating of step 2 , we have

- $D^0 = 0$.
- $D^1 = D = span\{X,Y\}$.
- $D^2 = D^1 + \left[D, D^1\right] = \{X, Y, [X;Y]\}$.
- $D^3 = D^2 + \left[D, D^2\right] = \{X, Y, [X;Y], [[X;Y], X]\}$.

Then,the distribution $\mathcal{D} = span\{X,Y\}$ is bracket generating of step 3 .

3.2.4 The class of nilpotence

The distribution $\mathcal{D} = span\{X,Y\}$ is nilpotente of class 3 , We have
For** $x \neq 0$ **:

$$\mathcal{C}^1 = \{[X,Y] = x\partial_z; \quad X,Y \in \Gamma(\mathcal{D})\} \quad .$$

$$\mathcal{C}^2 = \{[[X,Y],X] = \partial_z, [[X,Y],Y] = 0; \quad X,Y \in \Gamma(\mathcal{D})\} \quad .$$

The distrbution is bracket generating of degree 2 , then the step of sub-elliptic operator is 3 .
For** $x = 0$ **:
Since the distrbution is bracket generating of degree 2 , then the step of sub-elliptic operator is 3

$$\mathcal{C}^3 = \{[[[X,Y],X],X] = 0, [[[X,Y],X],Y] = 0, [[[X,Y],Y],X] = 0, [[[X,Y],Y],Y] = 0; \quad X,Y \in \Gamma(\mathcal{D})\} \quad .$$

Then,the distribution $\mathcal{D} = span\{X,Y\}$ is bracket generating of step 3 .
According to the result of chapter 2 , we have directly

$$\left[X^2, Y^2\right] \neq 0 \quad .$$

CONCLUSION

In conclusion , the condition of non-commutativity of the squares of two vector fields in a nilpotent distribution constitutes a fundamental aspect in the study of sub-Riemannian geometries and nilpotent distributions . This non-commutativity , which manifests through the fact that the Lie brackets of the squares of the vector fields are not necessarily zero , reveals deep and complex structures within the geometric spaces studied .

We have presented two properties of a distribution in space $\mathbb{R}^n$ *, which give us a good description of the state of* $\left[X^2, Y^2\right]$ *. These properties facilitate or guide the future calculation of the heat kernel of the operator* $\mathfrak{L} = \left(X^2, Y^2\right)$.

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بصفتي مشرف والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـ

Quelques propriétés sur la distribution nilpotente de classe 2 et de classe 3

من انجاز الطالبة ميموني اسيا البتول

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