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Mémoire

Some Qualitative Analyses of Neutral Functional Differential Equations with and without Delay

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Abstract

In this memory, we first transform our main problems into an equivalent fractional integral equation. After that, based on some fixed point theorems namely, Banach's fixed point theorem and Krasnoselskii's fixed point theorem and by applying Schauder's fixed point theorem in generalized Banach spaces, and Perov's fixed point theorem combined with the Bielecki norm, the existence and uniqueness of solutions to the proposed problems are discussed theoretically. Furthermore, examples are presented to illustrate our theoretical findings.
Key words and phrases : Fractional differential equation, Coupled fractional differential system, ψ-Caputo fractional derivative, initial value problem, fixed-point theorems, existence, uniqueness, Bielecki norm, Banach space, Generalized Banach space.
AMS Subject Classification : 26A33, 34A08, 34B15.

Résumé

Dans cette mémoire, nous transformons d'abord nos principaux problèmes en une équation intégrale fractionnaire équivalente. Après cela, sur la base de certains théorèmes de point fixe, à savoir le théorème de point fixe de Banach et le théorème de point fixe de Krasnoselskii et en appliquant le théorème de point fixe de Schauder dans les espaces de Banach généralisés, et Le théorème du point fixe de Perov combiné à la norme de Bielecki, l'existence et l 'unicité des solutions aux problèmes proposés sont discutés théoriquement. De plus, des exemples sont présentés pour illustrer nos résultats théoriques.

الملخص

في هذه المذكرة، نقوم أولاً بتحويل مشاكلنا الرئيسية إلى معادلة تكاملية كسرية مكافئة. بعد ذلك، تم الاعتماد على بعض نظريات النقطة الثابتة وهي نظرية النقطة الثابتة لباناخ ونظرية النقطة الثابتة لكراسنوسيلسكي، وبتطبيق نظرية النقطة الثابتة لشودر في فضاءات باناخ المعممة، ونقطة بيروف الثابتة تمت مناقشة النظرية جنبًا إلى جنب مع قاعدة بيليكي، ووجود وتفرد الحلول للمشكلات المقترحة من الناحية النظرية. وعلاوة على ذلك، يتم عرض أمثلة لتوضيح النتائج النظرية التي توصلنا إليها

List of symbols

We use the following notations throughout this thesis

Acronyms

- FC : Fractional calculus.
- FD : Fractional derivative.
- FDE : Fractional differential equation.
- FI : Fractional integral.
- IVP : Initial value problem.

Notation

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{R}^n$: Space of n-dimensional real vectors.
- $\in$: belongs to.
- sup : Supremum.
- max : Maximum.
- $n!$: Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.
- $\Gamma(\cdot)$: Gamma function.
- $\mathrm{B}(\cdot,\cdot)$: Beta function.
- $I_{a^+}^{\alpha,\psi}$: The fractional ψ–integral of order $\alpha > 0$.
- ${}^{H}D_{a^+}^{\alpha;\psi}$: The ψ-Caputo fractional derivative of order $\alpha > 0$.
- $C(J,\mathbb{E})$: Space of continuous functions on J.
- $AC(J,\mathbb{E})$: Space of absolutely continuous functions on J.
- $L^1(J,\mathbb{R})$: space of Lebesgue integrable functions on J.
- $L^\infty(J,\mathbb{E})$: space of functions u that are essentially bounded on J .

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Introduction

Fractional calculus (FC) is a branch of mathematics that deals with derivatives and integrals of non-integer order. It has applications in a wide range of fields, including physics, engineering, economics, and biology. The concept of fractional calculus is based on the idea of extending the notion of differentiation and integration beyond integer order to fractional and complex orders. In traditional calculus, differentiation is performed by applying the derivative operator to a function $f(x)$ to obtain its derivative $f'(x)$, which measures the rate at which $f(x)$ changes with respect to x. Fractional calculus has several important applications in science and engineering. For example, it has been used to model the behavior of viscoelastic materials, which exhibit both elastic and viscous properties. It has also been applied to the study of diffusion processes, where the rate of diffusion can be modeled using fractional derivatives [31, 1, 25]. Fractional calculus has also been used in the modeling of electrical circuits, where it is used to model the behavior of capacitors and inductors. In recent years, there has been a surge of interest in fractional calculus, and many new applications have been discovered. Research in this field is ongoing, and new techniques and methods are being developed to better understand and apply the principles of fractional calculus. Despite its complexity, fractional calculus has proven to be a powerful tool for modeling and understanding complex physical systems, and it is likely to remain an important area of research for years to come [13, 21, 22, 10].

Recently, there has been a trend of developing fractional order notions for the integrals and derivatives based on the singular and non singular kernels. In the past, the Riemann-Liuoville and Caputo's fractional differential operators were considered for a large number of physical problems to capture their scientific data. Both the theoretical and computational aspects were well studied for it. Now a days, the researchers in science and engineers have considered fractional operators with Mittag-Leffler functions for the modeling and simulations. Several analytical and semi analytical techniques were developed for the fractional operators with Mittag-Leffler kernel. Among these, ψ-Caputo fractional differential operator is one of the most popular fractional derivative.

In this thesis we are interesting by initial value problems (IVP for short) for fractional neutral functional differential equations with ψ-Caputo fractional derivative. Our results may be interpreted as extensions of previous results of C. Derbazi et al. [8], A. Boutiara et al. [5] obtained for fractional differential equations with ψ-Caputo derivative and those considered with fractional derivative and without delay. In fact, in the proof of our theorems we essentially use several fixed point techniques.

This thesis is arranged as follows :

Chapter 1, is reserved to expose some development and contributions to fractional calculus.

In **Chapter** 2, we give a technically precise overview of definitions, notations, lemmas and notions of fractional calculus, fixed point theorems that are used throughout this thesis.

Chapter 3, is reserved to expose some results of existence and uniqueness of solutions concerning a initial value problem for a fractional differential equation of the generalized Caputo type. The results are provided by the fixed point theorems of "Banach, Krasnoselskii and Schauder". We are interested in the existence and uniqueness of solutions for the following fractional initial value problem

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[z(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,(\tau))\right]=\mathbb{H}(\tau,z(\tau)),\tau\in \mathrm{J}:=[a,b],\\ z(a)=z_a, \end{cases} \tag{1}$$

where ${}^c\mathbb{D}_{a^+}^{\beta;\psi}$ is the ψ-Caputo FOD of order $\beta\in\{\nu,\varsigma\}\in(0,1]$, $\mathbb{I}_{a^+}^{\theta;\psi}$ is the ψ-RL fractional integral of order $\theta>0$, $\theta\in\{\sigma_1,\sigma_2,\dots,\sigma_m\}$, $\sigma_k>0, k=1,2,\dots,m$, $\mathbb{F}_k,\mathbb{H}:\mathrm{J}\times\mathscr{C}\longrightarrow\mathbb{R}$ $(k=1,2,\dots,m)$ are given continuous functions, where $\mathscr{C}=C([a,b],\mathbb{R})$.

Finally, an example is given at the end of this section to illustrate the theoretical results.

In **Chapter** 4,we deal with the existence and uniqueness results of solutions for to the following delayed coupled system of the fractional differential equations involving the ψ-Caputo derivative of the form :

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\nu;\psi}x)(\tau)=\mathbb{F}_1(\tau,x_\tau,y_\tau),\\ ({}^c\mathbb{D}_{a^+}^{\mu;\psi}y)(\tau)=\mathbb{F}_2(\tau,x_\tau,y_\tau), \end{cases}\qquad \tau\in\mathrm{J}, \tag{2}$$

along with the initial conditions

$$\begin{cases} x(\tau)=\alpha(\tau),\\ y(\tau)=\beta(\tau), \end{cases}\qquad \tau\in[a-\delta,a], \tag{3}$$

where $\delta>0$ is a constant delay and $\mathbb{F}_1,\mathbb{F}_2:\mathrm{J}\times C([-\delta,0],\mathbb{R}^n)\times C([-\delta,0],\mathbb{R}^n)\longrightarrow\mathbb{R}^n$, are given continuous functions and $\alpha,\beta:[a-\delta,a]\longrightarrow\mathbb{R}^n$ are two continuous functions. For any function z defined on $[a-\delta,a]$ and any $\tau\in\mathrm{J}$, we denote by z_τ the element of $C([-\delta,0],\mathbb{R}^n)$ defined by

$$z_\tau(\rho)=z(\tau+\rho),\quad \rho\in[-\delta,0].$$

Hence $z_\tau(\cdot)$ represents the history of the state from times $\tau-\delta$ up to the present time τ.

Finally, an example is also constructed to illustrate our results.

Chapitre 1

Development and contributions to fractional calculus

1.1 History of fractional calculus

Since the 60s of the last century Fractional Calculus has got a remarkable progress and now it is recognized to be an important domain for scientists especially mathematicians. Fractional Calculus (FC) started in 1695 with the ideas of Gottfried Leibniz generated from letter which was written by Antoine Marquez de L'Hopital asking him "what would happen if the order of the derivative was a real number instead of an integer ?". Leibniz responded : "It will lead to the paradox, from which beneficial consequences will one day be extracted". This exchange Between L'Hopital and Leibniz is generally considered the beginning of fractional calculus. However, the actual development of fractional calculus was in 1832. When Joseph Liouville introduced what is now called the Riemann-Liouville definition of the fractional derivative. Many other definitions of fractional integrals and derivatives are based on the Riemann-Liouville integral, other definitions extend the notion based on the differences of the kernel functions.

1.2 List of mathematician's contributions to fractional calculus :

In this section we address a list of mathematicians, who have provided important contributions to fractional calculus, for more detail see [30, 16].

P.S. Laplace (1812), proposed the idea of differentiation of non-integer order for the functions

Liouville (1835) derived the formula of the fractional integral and fractional derivative respectively of the form

$$D^{-\beta}g(x) = \frac{1}{(-1)^{\beta}\Gamma(\beta)}\int_0^{\infty} g(x+t)t^{\beta-1}, x \in \mathbb{R}, \quad \mathrm{Re}(\beta) > 0$$

and

$$D^{\beta}g(x) = \frac{1}{(-1)^{\beta}\Gamma(\beta)}\int_0^{\infty} \frac{d^n g(x+t)}{dx^n} t^{\beta-1} dt, x \in \mathbb{R}, \quad \mathrm{Re}(\beta) > 0$$

Riemann (1835) derived the formula of fractional integrals related with the Liouville fractional integral of the form

$$D^{-\beta}g(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} g(t)dt, \quad \mathrm{Re}(\beta) > 0$$

where $\mathrm{Re}(\beta) \in (n-1, n], n \in \mathbb{N}$ and $\mathrm{Re}(\cdot)$ denotes the real part of complex number.

Grünwald-Letnikov (1867-1868) introduce the operator called the Grünwald-Letnikov fractional operator of the form

$$D^{\beta}g(x)=\lim_{h\to 0}\frac{\Delta_h^{\beta}g(x)}{h^{\beta}},\quad \beta>0,h>0$$

where $\Delta_h^{\beta}g(x)$ is a difference of fractional order, given by

$$\Delta_h^{\beta}g(x)=\sum_{k=0}^{\infty}(-1)^k C_k^{\beta}g(x-kh)$$

Sonine (1872) introduced the Sonine fractional derivative, given as on the form

$$D^{\beta}g(x)=\frac{1}{\Gamma(\rho-\beta+1)}\int_a^x\frac{dg(t)}{dt}(x-t)^{\rho-\beta}dt$$

where $\mathrm{Re}(\rho)<\beta<\mathrm{Re}(\rho+1),\rho\in\mathbb{C}$.

Hadamard (1892) proposed the following fractional integral in the form

$$I^{\beta}g(x)=\frac{x^{\beta}}{\Gamma(\beta)}\int_0^1\frac{g(tx)}{(1-t)^{1-\beta}}dt,\quad \mathrm{Re}(\beta)>0$$

Weyl (1917) derived the left-sided and right-sided of the Weyl fractional integrals in the form

$$I_+^{\beta}g(x)=\tfrac{1}{\Gamma(\beta)}\int_{-\infty}^{x}\tfrac{g(t)}{(x-t)^{1-\beta}}dt,\quad 0<\beta<1$$
$$I_-^{\beta}g(x)=\tfrac{1}{\Gamma(\beta)}\int_{x}^{\infty}\tfrac{g(t)}{(x-t)^{1-\beta}}dt,\quad 0<\beta<1$$

Marchaud (1927) introduced the Marchaud fractional derivatives of the form

$$D^{\beta}g(x)=\frac{C}{\Gamma(\beta)}\int_x^{\infty}\frac{\Delta_t^l g(x)}{t^{1+\beta}}g(t)dt$$

where $\Delta_t^l g(x)$ is the finite difference of order l, for $l>\beta$ and $l\in N$. When $\Delta_t^l g(x)$, it is called the Weyl type finite difference for $l=1$ and $0<\beta<1$.

Hadamard (1927) introduced the Hadamard fractional integral, and the fractional derivative respectively was given by

$$I_a^{\beta}g(x)=\frac{1}{\Gamma(\beta)}\int_a^x\frac{g(t)}{\left(\ln\frac{x}{t}\right)^{1-\beta}}\frac{dt}{t},\quad 0<\beta<1$$

$$D_a^{\beta}g(x)=\frac{1}{\Gamma(1-\beta)}\int_0^x\frac{g(x)-g(t)}{\left(\ln\frac{x}{t}\right)^{\beta+1}}\frac{dt}{t},\quad 0<\beta<1$$

Hille and Tamarkin (1930) proposed the Abel type integral equation on the second kind

$$g(x)=h(x)-\frac{\xi}{\Gamma(\beta)}\int_0^x\frac{h(t)}{(x-t)^{1-\beta}}dt,\quad 0<\beta<1,\xi\in\mathbb{C}$$

with the solution (also called the Hille-Tamarkin fractional derivative,

$$h(x)=\frac{d}{dx}\int_0^x E_{\beta}\left[\xi(x-t)^{\beta}\right]g(t)dt,\quad 0<\beta<1$$

where $E_{\beta}\left[\xi(x-t)^{\beta}\right]$ is the Mittag-Leffler function, with one-parameter constant $\xi\in\mathbb{C}$ is defined as

$$E_\beta\left(\xi(x-t)^\beta\right)=\sum_{n=1}^{\infty}\frac{\xi(x-t)^{n\beta}}{\Gamma(n\beta+1)},\quad \beta>0$$

Love and young (1938) proposed the convergent fractional integral in the form

$$I_{+}^{\beta}g(x)=\frac{1}{\Gamma(\beta)}\lim_{m\to\infty}\int_0^m\frac{g(x-t)}{t^{1-\beta}}dt$$

In 1939, Hille introduced the Hille fractional differential operator in the form

$$R(\beta,\xi)g(x)=\frac{1}{\xi}\frac{d}{dx}\int_0^x E_\beta\left[\frac{(x-t)^\beta}{\xi}\right]f(t)dt,\quad \mathrm{Re}(\beta)>0,\xi\in\mathbb{R}^\star$$

where $R(\beta,\xi)$ is the resolvent of ${}^{RL}I^{\beta}$ and $E_\beta\left[\frac{(x-t)^\beta}{\xi}\right]$ is the one parameter Mittag-Leffler function, with $\beta>0$.

Erdélyi-Kober (1940) proposed the fractional integrals and derivatives in the form

$$I_{a+;\sigma,\eta}^{\beta}g(x)=\frac{\sigma t^{-\sigma(\beta+\eta)}}{\Gamma(\beta)}\int_a^x\frac{t^{\sigma(\eta+1)-1}g(s)}{(x^\sigma-s^\sigma)^{1-\beta}}ds$$

and the fractional derivatives as

$$D_{a+;\sigma,\eta}^{\beta}g(x)=t^{-\sigma\eta}\left(\frac{1}{\sigma t^{\sigma-1}}D\right)^k t^{\sigma(\beta+\eta)}\left(I_{a+;\sigma,\eta+\beta}^{\beta}g\right)(x)$$

where $\beta>0,\sigma>0,\eta\in\mathbb{R}$.

Cossar (1941) reported the Cossar fractional derivative in the form

$$D_{+}^{\beta}g(x)=-\frac{1}{\Gamma(1-\beta)}\lim_{m\to\infty}\frac{d}{dx}\int_0^m\frac{g(t)}{(t-x)^\beta}dt,\quad \beta>0$$

Riesz(1949) defined the fractional calculus based on the Fourier's work, which is called the Riesz fractional calculus in the form

$${}^{RZ}I_{R}^{\beta}g(x)=\frac{1}{2\Gamma(\beta)\cos(\pi\beta/2)}\int_{-\infty}^{\infty}\frac{g(s)}{|s-x|^{1-\beta}}ds,\quad \mathrm{Re}(\beta)>0$$

and

$${}^{RZ}D_{R}^{\beta}g(x)=\frac{1}{2\Gamma(n-\beta)\cos(\pi\beta/2)}\frac{d^n}{dx^n}\int_{-\infty}^{\infty}\frac{g(s)}{|s-x|^{\beta-n+1}}ds$$

Hille and Phillips (1957) introduced the integral in the form

$$I_{\xi}^{\beta}g(t)=\frac{\xi^{\beta+1}}{\Gamma(1+\beta)}\int_0^{\infty}t^\beta e^{\xi(x-t)}g(t)dt$$

Chen (1961) introduce the Chen fractional integrals and derivative respectively of the form

$$I^\beta g(x)=\frac{1}{\Gamma(\beta)}\int_a^x|x-t|^{\beta-1}g(t)dt,\quad x>a,\mathrm{Re}(\beta)>0$$

$$D^\beta g(x)=\frac{1}{\Gamma(1-\beta)}\int_a^x\frac{1}{|x-t|^\beta}\frac{dg(t)}{dt}dt,\quad x>a,\mathrm{Re}(\beta)>0$$

where $\text{Re}(\beta) \in (n-1,n], n \in \mathbb{N}$ and $\text{Re}(\cdot)$ denotes the real part of complex number. Srivastava (1964) proposed the fractional integral in the kernel of the confluent hypergeometric function was given as

$$I^{\beta} g(x) = \int_0^x \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} {}_1F_1(\alpha;\beta;x-t) g(t) dt, \quad \text{Re}(\beta) > 0$$

where ${}_1F_1(\alpha;\beta;x-t)$ is called the confluent hypergeometric function of the first kind, on the form

$${}_1F_1(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

is defined for $|z| < 1$ and α, β assumed arbitrarily real or complex values and $b \in \mathbb{Z}^+$.
Cooke (1965) proposed the Cooke fractional operator in the form

$$I_{a;\rho}^{\nu,\beta} g(x) = \begin{cases} \frac{2x^{-2(\nu+\beta)}}{\Gamma((\beta)} \int_a^b \frac{t^{2(\nu+1)-1}}{\left(x^2-t^2\right)^{1-\beta}} g(t) t^{2\nu-1} dt, & \beta > 0, \\ g(x), \beta = 0, \\ \frac{2x^{-2(\nu+\beta)-1}}{\Gamma(\beta+1)} \frac{d}{dx} \int_a^b \left(x^2-t^2\right)^{\beta} t^{2\nu+1} g(t) dt, & 0 < \beta < 1 \end{cases}$$

Saxena (1967) introduced the Saxena fractional integral within the kernel of the Gauss hypergeometric function, is defined by

$$I_a^{\beta} g(x) = \frac{x^{-\sigma-1}}{\Gamma(\beta)} \int_0^x {}_2F_1(1-\beta, \alpha+m; \alpha; t/x) g(t) t^{\sigma} dt, \quad \text{Re}(\beta) > 0$$

where ${}_2F_1(1-\beta, \alpha+m; \alpha; t/x)$ is the Gauss hypergeometric function is defined as

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is defined for $|z| < 1$ and α, β assume arbitrary real or complex values and $c \in \mathbb{Z}^+$. Kalisch (1967) proposed the left-sided and the right-sided of the Kalisch fractional derivative of purely imaginary order β, where $\beta = i\theta$, respectively in the form

$$I_{a^+}^{i\theta} g(x) = \frac{1}{\Gamma(1+i\theta)} \int_a^x (x-t)^{i\theta} g(t) dt$$

and

$$I_b^{i\theta} g(x) = \frac{1}{\Gamma(1+i\theta)} \int_x^b (t-x)^{i\theta} g(t) dt$$

Caputo (1967) introduced the Caputo fractional derivative in the form

$$D^{\beta} g(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{1}{(x-t)^{\beta}} g^{(n)}(t) dt, \quad x > 0, \text{Re}(\beta) > 0$$

Dzherbashyan (1967) proposed the Dzherbashyan fractional integral used the generalization of Hadamarod's idea and gave the fractional integral in the form

$$I^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^1 \frac{g(xz)}{(-\ln z)^{1-\beta}} dz, \quad \text{Re}(\beta) > 0$$

Srivastava (1968) proposed the Srivastava fractional operator which is related to the generalized Whittakar transform in the form

$$R^{-}_{\xi,\beta,n}g(x) = \frac{n}{\Gamma(\beta)}x^{\xi}\int_{x}^{\infty}(z^n - x^n)^{\beta-1}z^{-\xi-n\beta+n-1}f(z)dz, \quad x > 0$$

where $g \in L_p(0,\infty), \frac{1}{p} + \frac{1}{q} = 1, 0 < p < \infty, \beta > 0, \xi > \frac{1}{p}$.
Dzhrbashyan and Nersesyan (1968) proposed the Dzhrbashyan-Nersesyan fractional derivative in the form

$$D^{\beta}g(x) = \frac{1}{\Gamma(n-\beta)}\int_{0}^{\infty}\frac{1}{(x-t)^{n-\beta}}g^{n}(t)dt, \quad \mathrm{Re}(\beta)$$

Osler (1970) introduced Osler the fractional integral in the form

$$I^{\beta}_{a;k}g(x) = \frac{1}{\Gamma(\beta)}\int_{a}^{x}\frac{g(t)}{(k(x)-k(t))^{1-\beta}}k^{(1)}(t)dt, \quad 0 < \beta < 1$$

and the fractional derivative of the form

$$D^{\beta}_{a;k}g(x) = \frac{1}{\Gamma(1-\beta)}\frac{g(t)}{(k(x)-k(t))^{\beta}}\frac{\beta}{\Gamma(1-\beta)}\int_{a}^{x}\frac{g(x)-g(t)}{(k(x)-k(t))^{1+\beta}}k^{(1)}dt$$

where $x > 0, 0 < \alpha < 1, k \in C(I), k^{(1)}(t) \neq 0$.

Love (1971) considered the Love fractional integral and fractional derivative of purely imaginary order respectively as

$$D^{i\beta}_{a}g(x) = \frac{1}{\Gamma(1-i\beta)}\frac{d}{dx}\int_{a}^{x}\frac{g(t)}{(x-t)^{i\beta}}dt$$
$$I^{i\beta}_{a}g(x) = \frac{1}{\Gamma(i\beta)}\int_{0}^{\infty}\frac{g(t)}{(x-t)^{1-i\beta}}dt$$

Rafal'son (1971) introduced the Rafal'son type Bessel fractional integration and derivative respectively in the form

$$I^{\beta}_{-}g(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\infty}(x-t)^{\beta-1}e^{x-t}g(t)dt, \quad 0 < \beta < 1$$
$$D^{\beta}_{+}g(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\infty}(x-t)^{\beta-1}e^{x-t}g^{(\beta)}(t)dt, \quad 0 < \beta < 1.$$

Prabhakar (1972) introduced the Prabhakar type Humbert fractional integral in the form

$$I^{\beta}_{a}g(x) = \int_{0}^{x}\frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\Theta_1\left(\beta, b, c; 1-\frac{t}{x}, \xi(x-t)\right)g(t)dt$$

where $\Theta_1\left(\beta, b, c; 1-\frac{t}{x}, \xi(x-t)\right)$ is two variable hypergeometrique function or the
Humbert function is defined as

$$\Theta_1\left(\beta, b, c; 1-\frac{t}{x}, \xi(x-t)\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{n+m}(b)_m\left(1-\frac{t}{x}\right)^m(\xi(x-t))^n}{m!n!(c)_{m+n}}$$

where $(\beta)_n = \frac{\Gamma(\beta+n)}{\beta}, n = 0, 1, 2, \ldots$, and β, b, c are parameters which assume real or complex values.
Sneddon (1975) introduced the Sneddon fractional integral of the form

$$I^{\nu,\beta}_{a,\rho}g(x) = \frac{\rho x^{-\rho(\nu+\beta)}}{\Gamma(\beta)}\int_{a}^{x}\frac{t^{\rho(\nu+1)-1}}{(x^{\rho}-t^{\rho})^{1-\beta}}g(t)dt$$

where $\beta, \nu \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \rho > 0, t > 0$.

Saigo (1978) introduced the Saigo type Gauss hypergeometric fractional integral operator in the form

$$I_x^{\beta,\gamma,\nu} g(x) = \frac{x^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} {}_2F_1\left(\beta+\gamma, -\nu, \beta; 1-\frac{t}{x}\right) g(t)dt$$

where ${}_2F_1\left(\beta+\gamma,-\nu,\beta;1-\frac{t}{x}\right)$ is the Gauss hypergeometric function defined by

$${}_2F_1\left(\beta+\gamma,-\nu,\beta;1-\frac{t}{x}\right) = \sum_{n=0}^{\infty} \frac{(\beta+\gamma)_n(-\nu)_n}{(\beta)_n} \frac{\left(1-\frac{t}{x}\right)^n}{n!}$$

and $\beta,\gamma,\nu \in \mathbb{C}$.
Gearhart (1979) introduced the Rafal'son-Gearhart type Bessel fractional integral

$$I_-^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} (x-t)^{\beta-1} e^{\xi(x-t)} g(t)dt, \quad 0<\beta<1$$

Skornik (1980) reported the Skornik tempered fractional integral and derivative respectively of the form

$$I_a^{\beta} g(x) = e^{\frac{-x^2}{4}} \int_a^x \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} e^{\frac{t^2}{4}} g(t)dt, \quad 0<\beta<1$$

and

$$D_a^{\beta} g(x) = \frac{e^{\frac{-x^2}{4}}}{\Gamma(1-\beta)} \frac{d^n}{dx^n} \int_a^x \frac{e^{\frac{t^2}{4}}}{(x-t)^{-\beta}} g(t)dt, \quad 0<\beta<1$$

and

$$D_a^{\beta} I_a^{\beta} g(x) = g(x)$$

Peschanskii (1989) introduced the fractional integral operator involving the curvilinear convolution type in the form

$$I_x^{\beta,\gamma,\nu} g(x) = \frac{1}{2\pi i} \int_{\Gamma} {}_2F_1\left(1,1;1+\beta;\frac{t}{x}\right) g(t)\frac{dt}{t}, \quad \beta,\gamma,\nu \in \mathbb{C}$$

where ${}_2F_1\left(1,1;1+\beta;\frac{t}{x}\right)$ is the Gauss hypergeometric function.
Samko and Ross (1993) reported the variable-order fractional integral given by

$$I_a^{\beta(x)} g(x) = \frac{1}{\Gamma(\beta(x))} \int_a^x (x-t)^{-\beta(x)} g(t)dt, \quad \beta(x)>0$$

and the variable-order fractional derivative given as

$$D_a^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \frac{d}{dx} \int_a^x (x-t)^{-1-\beta(x)} g(t)dt, \quad 0<\beta(x)<1$$

Hilfer (2000) introduced the Hilfer fractional derivative by

$$D^{\beta,\gamma} g(x) = I_0^{\gamma(1-\beta)} \cdot D^1 \cdot I_0^{(1-\gamma)(1-\beta)} g(x)$$

where $0<\alpha<1, 0\le\beta\le1, g\in L^1\left(\mathbb{R}^+\right)$and $D^{(1)}g(x) = \frac{dg(x)}{dx}, I^{\gamma(1-\beta)}$ and $I^{(1-\gamma)(1-\beta)}$ are the Riemann-Liouville fractional integrals.

Coimbra (2003) introduced the variable-order fractional integral in the form

$$D^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \int_0^x \frac{1}{(x-t)^{\beta(x)}} \frac{dg(t)}{dt} dt + \frac{g\left(0^+ - g\left(0^-\right)\right.}{\Gamma(1-\beta(x))(x-t)^{\beta(x)}}$$

where $0 < \beta(x) < 1$.

Kilbas, Saigo and Saxena (2004)introduced the following general fractional derivative defined by

$$D_{+}^{\beta} g(x) = \frac{d^n}{dx^n} \int_a^x (s-t)^{\mu+n-\nu-1} E_{\beta,\mu+n-\nu}^{\varphi} \left(\varpi (x-t)^{\beta} \right) g(t) dt$$

where $\beta, \mu, \nu, \varpi \in \mathbb{C}, \mathrm{Re}(\beta) \in (n-1, n], n \in \mathbb{N}$ and $E_{\beta,\mu+n-\nu}^{\varphi} \left(\varpi (x-t)^{\beta} \right)$ is two parameter generalized Mittag-Leffler function with parameters $\alpha, \beta > 0$ defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n\alpha + \beta}$$

Agrawal (2007) developed the fractional derivative and integral operators in terms of the Riesz fractional derivative, in the form

$$\begin{aligned} I^{\beta} g(x) &= \tfrac{1}{\Gamma(n-\beta)} \tfrac{d^n}{dx^n} \int_a^b \tfrac{g(t)}{|x-t|^{n-\beta}} dt, \quad \mathrm{Re}(\beta) \in (n-1, n] \\ D^{\beta} g(x) &= \tfrac{1}{\Gamma(n-\beta)} \int_a^b \tfrac{1}{|x-t|^{\beta}} \tfrac{d^n g(t)}{dt^n} dt, \quad \mathrm{Re}(\beta) \in (n-1, n] \end{aligned}$$

and

$$I^{\beta} g(x) = \frac{1}{2\Gamma(\beta)} \int_a^b |x-t|^{\beta-1} g(t) dt, \quad 0 < \beta < 1$$

Gajda and Magdziarz (2010) introduced the fractional derivative for Fokker Planck equation (FFPE) in the form

$$D_{+}^{\beta} g(x) = \frac{d}{dx} \int_0^x M(x-t) g(t) dt, \quad \beta > 0$$

where the memory kernel $M(t)$ is defined via its Laplace transform denoted by $M(p)$, is

$$M(p) = \frac{1}{(p+\xi)^{\beta} + \xi^{\beta}}$$

Garra et al.(2014) introduced the fractional derivative of the form

$$D_{a^+}^{\beta,\mu,\mu,\omega} g(x) = \int_a^x (t-\mu)^{\mu-1} E_{\beta,\mu}^{\varphi} \left(\omega (x-t)^{\beta} \right) g^{(n)}(t) dt, \quad \mathrm{Re}(\mu) > 0$$

where $\beta, \mu, \varphi, \omega \in \mathbb{C}$, and $\mathrm{Re}(\beta) \in (n-1, n], n \in \mathbb{N}, \mathrm{Re}(\mu) > 0, g \in AC^n[0,b], 0 < t < b < \infty$ and $E_{\beta,\mu}^{\varphi} \left(\omega (x-t)^{\beta} \right)$ is two generalized Mittag-Leffler function.

Caputo and Fabrizio (2015) introduced the Caputo-Fabrizio derivative in the form

$$D_{+}^{\beta} g(x) = \frac{M(\beta)}{(1-\beta)} \int_a^x \exp\left(-\frac{\beta}{(1-\beta)} (x-t) \right) g^{(1)}(t) dt$$

where $M(\beta)$ is a normalization function such that $M(0) = M(1) = 1, 0 < \beta < 1, g \in H^1(a,b), b > a$, $a < x < b$.

Zayernouri, Ainsworth and Karniadakis (2015) proposed the fractional derivatives in the form

$$D_{+}^{\beta} g(x) = \frac{e^{\xi x}}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x (x-t)^{-\beta-1} e^{-\xi t} g(t) dt, \quad x > a,$$

and

$$D_{-}^{\beta}g(x)=\frac{e^{\xi x}}{\Gamma(1-\beta)}\frac{d}{dx}\int_{x}^{b}(x-t)^{-1-\beta}e^{-\xi t}g(t)dt,\quad x<a$$

where $0<\beta<1,\xi\geq 0$.

Yang, Srivastava and Machado (2015) proposed the fractional derivative with the exponential function by

$$D_{x}^{\beta}g(x)=\frac{(2-\beta)M(\beta)}{2(1-\beta)}\frac{d}{dx}\int_{a}^{x}\exp\left(-\frac{\beta}{(1-\beta)}(x-t)\right)g(t)dt$$

where $0<\beta<1, M(\beta)$ is a normalization function such that $M(0)=M(1)=1$, $g\in H^{1}(a,b), b>a$, $a<x<b$.

Sabzikar, Meerschaert and Chen (2015) introduced the fractional integrals and derivatives respectively of the form

$$I_{-}^{(\beta)}g(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\infty}(x-t)^{\beta-1}e^{-\xi(x-t)}g(t)dt$$
$$D_{+}^{(\beta)}g(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\infty}(x-t)^{\beta-1}e^{-\xi(x-t)}g^{(\beta)}(t)dt$$

where $0<\beta<1,\xi\geq 0$.

Atangana and Baleanu (2016) proposed the Atangana-Baleanu fractional derivative with the Mittag-Leffler function in the form

$$D_{x}^{\beta}g(x)=\frac{M(\beta)}{(1-\beta)}\int_{a}^{x}E_{\beta}\left(-\frac{\beta}{(1-\beta)}(x-t)^{\beta}\right)\frac{dg(t)}{dt}dt.$$

where $0<\beta<1, M(\beta)$ is a normalization function such that $M(0)=M(1)=1$, $g\in H^{1}(a,b), b>a$, $a<x<b$, and $E_{\beta}\left(-\frac{\beta}{(1-\beta)}(x-t)^{\beta}\right)$ is one generalized MittagLeffler function.

Yang (2016) proposed the Yang fractional derivatives and variable fractional order as

$$D_{x}^{\beta}g(x)=\frac{1+\beta^{2}}{\sqrt{\pi^{\beta}(1-\beta)}}\int_{a}^{x}\exp\left(-\frac{\beta}{(1-\beta)}(x-t)^{2\beta}\right)\frac{dg(t)}{dt}dt$$

where $\beta\geq 0, g\in AC[a,b]$ and $a<x<b$.

$$D_{x}^{\beta}g(x)=\frac{M(\beta(x))}{(1-\beta(x))}\int_{a}^{x}\exp\left(-(x-t)^{\beta(x)}\right)\frac{dg(t)}{dt}dt$$

and

$$D_{x}^{\beta(x)}g(x)=\frac{1}{\Gamma(1-\beta(x))}\int_{a}^{x}E_{\beta(x)}\left(-(x-t)^{\beta(x)}\right)\frac{dg(t)}{dt}dt$$

where $E_{\beta(x)}\left(-(x-t)^{\beta(x)}\right)$ is the Mittag-Leffler function with one-parameter variable $0<\beta(x)<1$.

Li and Deng (2016) proposed the Li-Deng fractional derivatives in the form

$$D_{+}^{\beta}g(x)=\frac{e^{-\xi x}}{\Gamma(n-\beta)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}\frac{g(t)}{(x-t)^{\beta+1}}e^{\xi t}dt \tag{69}$$

and

$$D_{-}^{\beta}g(x)=\frac{(-1)^{n}e^{-\xi x}}{\Gamma(n-\beta)}\frac{d^{n}}{dx^{n}}\int_{x}^{b}\frac{g(t)}{(x-t)^{\beta+1}}e^{\xi t}dt \tag{70}$$

where $\text{Re}(\beta) \in (n-1, n], n \in \mathbb{N}$ and for any $\xi \geq 0$.

Torres (2017) introduced the Torres fractional derivatives in the form

$$D_{+}^{\beta} g(x) = \xi^{\beta} g(x) + \frac{\beta}{\Gamma(1-\beta)} \int_{-\infty}^{x} \frac{g(x)-g(t)}{(x-t)^{\beta+1}} e^{-\xi(x-t)} dt \tag{71}$$

and

$$D_{-}^{\beta} g(x) = \xi^{\beta} g(x) + \frac{\beta}{\Gamma(1-\beta)} \int_{x}^{\infty} \frac{g(x)-g(t)}{(x-t)^{\beta+1}} e^{-\xi(x-t)} dt \tag{72}$$

where $0 < \beta < 1$ and for any $\xi \geq 0$.

Sun, Hao, Zhang and Baleanu (2017) proposed the fractional derivative in the form

$$D_{x}^{\beta} g(x) = \frac{\Gamma(1+\beta)}{(1-\beta)^{\frac{1}{\beta}}} \int_{a}^{x} \exp\left(-\frac{\beta}{1-\beta}(x-t)^{\beta}\right) \frac{dg(t)}{dt} dt, 0 < \beta < 1 \tag{73}$$

Yang and Machado (2017) introduced the Yang-Machado variable-order fractional derivative with the another function by

$$D_{a^{+}}^{\beta(x),\varphi} g(x) = \frac{1}{\Gamma(1-\beta(t))} \int_{a}^{x} \frac{g_{\varphi}^{(1)}(t)}{(\varphi(x)-\varphi(t))^{\beta(x)}} dt \tag{74}$$

where $0 < \beta(x) < 1$, and $g, \varphi \in c^{1}[a,b], \varphi'^{(1)} \neq 0$.

Dehghan, Abbaszadeh and Deng (2017) presented the fractional derivative in the form

$$D_{+}^{\beta} g(x) = \frac{1}{\Gamma(\beta-1)} \int_{0}^{x} (x-t)^{\beta+1} e^{-\xi(x-t)} \frac{d^{\beta} g(t)}{d|t|^{\beta}} dt \tag{75}$$

where $\frac{d^{\beta} g(t)}{d|tt^{\beta}}$ is the Riesz fractional derivative, with $1 < \beta \leq 2, \xi > 0$.

Yang, Machado and Baleanu (2017) proposed the fractional derivatives in the form

$$D_{+}^{\beta} g(x) = \int_{a}^{t} E_{\beta,\nu}^{\varphi,\phi}\left(-(t-s)^{\beta}\right) \frac{dg(s)}{ds} ds \tag{76}$$

where $E_{\beta,\nu}^{\varphi,\phi}(z) = \sum_{n=0}^{\infty} \frac{(\varphi)_n}{\Gamma(n\beta+\nu)} \frac{z^n}{n+1}$, with $\varphi, \beta, \varphi, \phi, \nu \in \mathbb{C}$, and $\text{Re}(\beta) > 0, \max(0, \text{Re}(\beta)) \in \mathbb{R}_0^{+}, n \in \mathbb{N}$.

Yang, Gao, Machado and Baleanu (2017) proposed

$$D_{x}^{\beta} g(x) = \frac{\beta M(\beta)}{(1-\beta)} \int_{a}^{x} \sin c\left(-\frac{\beta(x-t)}{1-\beta}\right) \frac{dg(t)}{dt} dt \tag{77}$$

where $0 < \beta < 1, M(\beta)$ is a normalization function such that $M(0) = M(1) = 1$, $g \in H^{1}(a,b), b > a$, $a < x < b$,

$$\sin c(x) = \frac{\sin(\pi x)}{\pi x}, x \in \mathbb{R} \tag{78}$$

Almeida (2017) based on the Liouville-Sonine-Caputo fractional derivative, Almeida defined the Liouville-Sonine-Caputo fractional derivative with respect to another function in the form

$$
\begin{aligned}
L_{SC}D^{\beta}_{a+,h}g(x) &= \left(I^{n-\beta;\varphi}_{a^+}\right) g^{(n)}(x) \\
&= I^{n-\beta;\varphi}_{a^+}\left(\frac{1}{\varphi'(x)\frac{d}{dx}}\right)^n g(x) \\
&= \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{\varphi^{(1)}(s)}{(\varphi(x)-\varphi(s))^{\beta-n+1}}\left(\frac{1}{\varphi^{(1)(s)}}\frac{d}{ds}\right)^n g^n(s)ds \qquad (79)
\end{aligned}
$$

where $g,\varphi \in C^n(I), \varphi'(x) \neq 0, \beta > 0, n = [\beta]+1$.
Sousa and de Oliveira (2018) introduced the Sousa-Oliveira fractional derivative by

$$
{}^{RL}D^{\beta(t)}_{a^+}g(x) = \frac{M(\beta(t))}{1-\beta(t)}\left(\frac{1}{\psi'(t)}\frac{d}{dx}\right)\int_a^x \psi'(t)\mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t)g(t)dt, \qquad (80)
$$

and

$$
{}^{C}D^{\beta(t)}_{a^+}g(x) = \frac{M(\beta(t))}{1-\beta(t)}\int_a^x \psi'(t)\mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t)g'(t)dt \qquad (81)
$$

where

$$
\mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t) = \mathbb{E}\left[\frac{-\beta(t)(\psi(x)-\psi(t))^{\gamma}}{1-\beta(t)}\right]
$$

and $0 < \beta(t) < 1, 0 < \gamma,\delta < 1, \mathbb{E}(\cdot)$ is a Mittag-Leffler function, which is considered uniformly convergent on the interval $[a,b] = I, M(\beta(t))$ is a normalization function such that $M(0) = M(1) = 1$, and $\psi(\cdot)$ is a positive function and increasing monotone, such that $\psi(t)' \neq 0$.

Chapitre 2

Preliminaries and Background Materials

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this thesis.

2.1 Some Notations and Definitions

2.2 Special Functions

Before introducing the basic facts on fractional operators, we recall two types of functions that are important in Fractional Calculus : the Gamma and Beta functions. Some properties of these functions are also recalled. More details about these functions can be found in [11, 23, 21].

2.2.1 Gamma function

The Euler Gamma function is an extension of the factorial function to real numbers and is considered the most important Eulerian function used in fractional calculus because it appears in almost every fractional integral and derivative definitions.

Definition 2.1 ([23, 11])**.** The Gamma function, or second order Euler integral, denoted $\Gamma(\cdot)$ is defined as :

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \mathrm{dt}, \quad \alpha > 0. \tag{2.1}$$

For positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ and thus can be seen as an extension of the factorial function to real values.

Properties 1. The basic properties of the Gamma function are :

1. The function $\Gamma(\alpha)$ is continuos for $\alpha > 0$.
2. The integral (2.1) is convergent for $\alpha > 0$ and divergent for $\alpha \leq 0$.
3. An important property of the gamma function $\Gamma(\alpha)$ is that it satisfies :
$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0.$$
4. The following relations are also valid :
$$\begin{aligned} \Gamma(n+1) &= n!, \quad \forall n \in \mathbb{N}, \\ \Gamma(1) &= 1, \\ \Gamma(0) &= +\infty. \end{aligned}$$

5. Taking account that the Γ function can be written as $\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$, it results that the Γ function can be defined also for negative values of α, in the interval $-1 < \alpha < 0$.
6. The following particular values for Γ function can be useful for calculation purposes :

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}, \\
\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi}, \\
\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}, \\
\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}.
\end{aligned}$$

2.2.2 Beta Function

Definition 2.2 ([23, 11])**.** The Beta function, or the first order Euler function, can be defined as

$$\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, \quad p,q > 0.$$

In the following we will enumerate the basic properties of the Beta function :

Properties 2.

1. The following formula which expresses the Beta function in terms of the Gamma function :

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q > 0.$$

2. For every $p > 0$ and $q > 0$, we have :

$$\mathrm{B}(p,q) = \mathrm{B}(q,p).$$

3. For every $p > 0$ and $q > 1$, the Beta function B satisfies the property :

$$\mathrm{B}(p,q) = \frac{q-1}{p+q-1}\mathrm{B}(p,q-1).$$

4. For any natural numbers m,n we obtain :

$$\mathrm{B}(m,n) = \frac{(n-1)!(m-1)!}{(n+m-1)!}.$$

2.2.3 Mittag-Leffler Function

Definition 2.3. [11] For $\alpha > 0$ and $z \in \mathbb{R}$, the one-parameter Mittag-Leffler function (MLF) is defined as follows :

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

Especially, when $\alpha = 1$ the one-parameter Mittag-Leffler function coincides with the exponential function, that is

$$\mathbb{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Moreover, the one-parameter MLF plays an important role in solving fractional ordinary differential equations (FODEs). Indeed, as $u(t)=u_a e^{-\lambda t}$ is the unique solution of the ordinary differential equation (ODE)

$$\begin{cases} u'(t)+\lambda u(t)=0, t>a, \\ u(a)=u_a, \end{cases}$$

so the MLF

$$u(t)=u_a\mathbb{E}_\alpha\left(-\lambda(\psi(t)-\psi(a))^\alpha\right), t>a, \alpha\in(0,1),$$

solves the homogeneous linear FODE with constant coefficients

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi}u(t)+\lambda u(t)=0, t>a, \\ u(a)=u_a, \end{cases}$$

where ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}$ represents the ψ-Caputo fractional derivative. The previous equation was studied by Almeida [4]. It has been used to model some population growth and the proof of its solution is obtained by using the standard technique of successive approximation.

Remark 2.4. It is important to note that several generalizations of the one-parameter MLF are available in the literature; such as a two-parameter Mittag-Leffler function, three-parameter Mittag-Leffler function, and the Mittag-Leffler function for matrix arguments. They have been used to solve some FODEs. But here we are not interested in them.

Lemma 2.5. *[11] Let $\alpha\in(0,1)$ and $z\in\mathbb{R}$. Among the numerous properties of the MLF, we mention that*

(1) The function $\mathbb{E}_\alpha$ is nonnegative,

(2) $\mathbb{E}_\alpha(0)=1$

(3) $\mathbb{E}_\alpha(\cdot)$ is an increasing function on $\mathbb{R}_+$.

2.3 Elements from Fractional Calculus Theory

In this part, we will review some fundamental properties of fractional calculus theory that will be useful in our work.

Denote by $C(\mathbb{J},\mathbb{R})$ and $AC(\mathbb{J},\mathbb{R})$ the space of continuous functions and the space of absolutely continuous functions from $\mathbb{J}$ into $\mathbb{R}$, respectively. Also $C^1(\mathbb{J},\mathbb{R})$ denotes the space of functions $u:\mathbb{J}\to\mathbb{R}$, whose first derivative is continuous on $\mathbb{J}$.

Assume that $C([a,b],\mathbb{R}^n)$ is used for description of the Banach space of each continuous function $\varpi:[a,b]\to\mathbb{R}^n$ with the norm $\|\varpi\|=\sup_{\zeta\in[a,b]}\|\varpi(\zeta)\|$, where $\varpi,\omega\in\mathbb{R}^n$ with $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)$, $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$. Also, $\varpi\le\omega$ means $\varpi_i\le\omega_i, i=1,\ldots,n$, and if $c\in\mathbb{R}$, then $\varpi\le c$ means $\varpi_i\le c, i=1,\ldots,n$. Set $\mathbb{R}^n_+=\left\{\varpi\in\mathbb{R}^n: \varpi_i\in\mathbb{R}_+, i=1,\ldots,n\right\}$. Moreover, we take

$$|\varpi|=(|\varpi_1|,|\varpi_2|,\ldots,|\varpi_n|),$$
$$\max(\varpi,\omega)=(\max(\varpi,\omega),\max(\bar{\varpi},\bar{\omega}),\ldots,\max(\varpi_n,\omega_n)),$$

Notice that, the space $AC(\mathbb{J},\mathbb{R})$ coincides with the space of primitives of Lebesgue summable functions. Indeed, we have the following equivalence.

$$u\in AC(\mathbb{J},\mathbb{R}) \quad \textit{if and only if } u'(t) \textit{ exists for a.e. } t\in\mathbb{J}$$
$$\textit{with } u'\in L^1(\mathbb{J},\mathbb{R}) \textit{ and } u(t)-u(a)=\int_a^t u'(s)\mathrm{d}s \textit{ for all } t\in\mathbb{J}.$$

From here onwards, we assume that $\psi \in C^1(\mathbb{J},\mathbb{R})$ *be an increasing and positive function such that* $\psi'(t) > 0$, *for all* $t \in \mathbb{J}$.

Now let us define some function spaces associated with ψ*-fractional operators. The spaces* $L^1_\psi(\mathbb{J},\mathbb{R})$ *and* $C^1_\psi(\mathbb{J},\mathbb{R})$ *of functions* u *on* $\mathbb{I}$ *are defined respectively by*

$$L^1_\psi(\mathbb{J},\mathbb{R}) = \left\{u : \mathbb{J} \to \mathbb{R} : u \text{ is measurable and } \int_a^b |u(t)|\psi'(t)\mathrm{d}t < \infty\right\},$$
$$C^1_\psi(\mathbb{J},\mathbb{R}) = \{u \in C(\mathbb{J},\mathbb{R}) : \mathscr{D}\psi u \in C(\mathbb{J},\mathbb{R})\},$$

where $\mathscr{D}_\psi u(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{dt}}\right)u(t)$. *The derivative* $\mathscr{D}_\psi u$ *of a function* u *has the following limit form :*

$$\mathscr{D}_\psi u(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{dt}}\right)u(t) = \lim_{h\to 0}\frac{u(t+h)-u(t)}{\psi(t+h)-\psi(t)}.$$

Furthermore, the space $AC_\psi(\mathbb{J},\mathbb{R})$ *consists of those and only those functions* u *whose derivative with respect to* ψ *exists for a.e.* $t \in \mathbb{J}$ *with* $\mathscr{D}_\psi u \in L^1_\psi(\mathbb{J},\mathbb{R})$ *and* $u(t) - u(a) = \int_a^t \mathscr{D}_\psi u(s)\psi'(s)\mathrm{d}s$ *for all* $t \in \mathbb{J}$. *Analogoustly, for* $\alpha \in (0,1)$ *the function space* $C^\alpha_\psi(\mathbb{J},\mathbb{R})$ *is defined by*

$$C^\alpha_\psi(\mathbb{J},\mathbb{R}) = \left\{u \in C(\mathbb{I},\mathbb{R}) : {}^c\mathbb{D}^{\alpha;\psi}_{a^+}u \in C(\mathbb{I},\mathbb{R})\right\}$$

where ${}^c\mathbb{D}^{\alpha;\psi}_{a^+}$ *denotes the* ψ*-Caputo type fractional derivative wich will be defined in the forthcoming definitions.*

Definition 2.6. [3, 13] Let $\alpha > 0$. The ψ-Riemann-Liouville fractional integral of order α of a function $u \in L^1_\psi(\mathbb{J},\mathbb{R})$ with respect to ψ is defined for a.e. t by

$$\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)\mathrm{d}s$$

Moreover, for $\alpha = 0$, we set $\mathbb{I}^{\alpha;\psi}_{a^+}u := u$.

Remark 2.7. Notice that, for suitably chosen ψ, we obtain some well-known definitions of fractional integrals, for example,

— The Riemann-Liouville integral [13] when $\psi(t) = t$
— the Hadamard integral [13] when $\psi(t) = \ln t$,
— the fractional integral with Sigmoid function [15] when $\psi(t) = \frac{1}{1+e^{-t}}$,
— the fractional integral with exponential memory [9] when $\psi(t) = e^{-\sigma t}$.

Lemma 2.8. *[3, 4] Let* $\alpha, \beta > 0$, *and* $u \in C(\mathbb{J},\mathbb{R})$. *Then for each* $t \in \mathbb{J}$ *we have*

(1) $\mathbb{I}^{\alpha;\psi}_{a^+}$ maps $C(\mathbb{I},\mathbb{R})$ *into* $C(\mathbb{J},\mathbb{R})$,

(2) $\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(a) = \lim_{t\to a^+}\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(t) = 0$,

(3) $\mathbb{I}^{\alpha;\psi}_{a^+}$ *is a linear bounded operator from* $C(\mathbb{J},\mathbb{R})$ *into* $C(\mathbb{J},\mathbb{R})$ *and*

$$\left\|\mathbb{I}^{\alpha;\psi}_{a^+}u\right\|_\infty \leq \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\|u\|_\infty$$

(4) $\mathbb{I}^{\alpha;\psi}_{a^+}\mathbb{I}^{\beta;\psi}_{a^+}u(t) = \mathbb{I}^{\beta;\psi}_{a^+}\mathbb{I}^{\alpha;\psi}_{a^+}u(t) = \mathbb{I}^{\alpha+\beta;\psi}_{a^+}u(t)$,

(5) $\mathbb{I}^{\alpha;\psi}_{a^+}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\psi(t)-\psi(a))^{\alpha+\beta-1}$,

(6) $\mathbb{I}^{\alpha;\psi}_{a^+}\left(\mathbb{E}_\alpha\left(\lambda(\psi(t)-\psi(a))^\alpha\right) = \frac{1}{\lambda}\left(\mathbb{E}_\alpha\left(\lambda(\psi(t)-\psi(a))^\alpha - 1\right), \quad \lambda > 0\right.\right.$.

Definition 2.9. [3] Let $0<\alpha<1$. The ψ-Riemann-Liouville fractional derivative of a function $u\in L^1_\psi(\mathbb{J},\mathbb{R})$ of order α is defined when $\mathbb{I}^{1-\alpha;\psi}_{a^+}u\in AC_\psi(\mathbb{J},\mathbb{R})$ by

$$\begin{aligned}{}^{\mathrm{R-L}}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)&=\mathscr{D}_\psi\mathbb{I}^{1-\alpha;\psi}_{a^+}u(t)\\&=\frac{1}{\Gamma(1-\alpha)}\mathscr{D}_\psi\int_a^t\frac{\psi'(s)u(s)}{(\psi(t)-\psi(s))^\alpha}\mathrm{d}s,\text{ a.e. }t\in\mathbb{J}.\end{aligned}$$

Lemma 2.10. *[3] Let $0<\alpha<1$ and $\beta>0$. If $\mathrm{f}(t)=(\psi(t)-\psi(a))^{\beta-1}$. Then we have*

$$^{\mathrm{R-L}}\mathbb{D}^{\alpha;\psi}_{a^+}\mathrm{f}(t)=\begin{cases}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1},\\0,\quad\beta=\alpha.\end{cases}$$

One can draw the following interesting observation.

Remark 2.11. The Riemann-Liouville fractional derivative of the constant function $f(t)=1$ (i.e., $\beta=1$) is not identically zero, since for $t>a$:

$$^{\mathrm{R-L}}\mathbb{D}^{\alpha;\psi}_{a^+}1=\frac{(\psi(t)-\psi(a))^{-\alpha}}{\Gamma(1-\alpha)}.$$

Definition 2.12. [3] Let $0<\alpha<1$. The ψ-Caputo type fractional derivative of a function $L^1_\psi(\mathbb{J},\mathbb{R})$ of order α is given by

$$^{c}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)={}^{\mathrm{R-L}}\mathbb{D}^{\alpha;\psi}_{a^+}(u(t)-u(a))$$

Provided that $\mathbb{I}^{1-\alpha;\psi}_{a^+}(u-u(a))\in AC_\psi(\mathbb{J},\mathbb{R})$. Moreover if $u\in AC\psi(\mathbb{J},\mathbb{R})$ the ψ-Caputo type fractional derivative of order α of u is determined as

$$^{c}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)=\mathbb{I}^{1-\alpha;\psi}_{a^+}\mathscr{D}_\psi u(t)$$

Clearly, if $u(a)=0$ then we have

$$^{c}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)={}^{\mathrm{R-L}}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)$$

Lemma 2.13. *[3] Let $\alpha>0$.*
If $u\in C(\mathrm{J},\mathbb{R})$, then

$$^{c}\mathbb{D}^{\alpha;\psi}_{a^+}\mathbb{I}^{\alpha;\psi}_{a^+}u(t)=u(t),\ t\in\mathrm{J},$$

and if $u\in C^{n-1}(\mathrm{J},\mathbb{R})$, then

$$\mathbb{I}^{\alpha;\psi}_{a^+}\,{}^{c}\mathbb{D}^{\alpha;\psi}_{a^+}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{[k]}_\psi(a)}{k!}[\psi(t)-\psi(a)]^k,\quad t\in\mathrm{J}.$$

For the definition of higher order derivatives and their properties we refer the reader to [3].

Remark 2.14. Several special cases can be considered by choosing the function ψ as in Remark 2.8.

Lemma 2.15. *[3] Let $0<\alpha<1$ and $\beta>0$. If $\mathrm{f}(t)=(\psi(t)-\psi(a))^{\beta-1}$. Then we have*

$$^{\mathrm{c}}\mathbb{D}^{\alpha;\psi}_{a^+}\mathrm{f}(t)=\begin{cases}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}\\0,\qquad\qquad\beta=1.\end{cases}$$

Lemma 2.16. *Let $\alpha, \lambda > 0$. Then for all $t \in [a,b]$ we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} \leq \frac{e^{\lambda(\psi(t)-\psi(a))}}{\lambda^\alpha}.$$

Proof. *In the light of the definition of the ψ-Riemann-Liouville fractional integral, we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} e^{\lambda(\psi(s)-\psi(a))} \mathrm{ds}$$

Using the change of variables $\mathrm{y} = \psi(t) - \psi(s)$ *we get*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} = \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)} \int_0^{\psi(t)-\psi(a)} \mathrm{y}^{\alpha-1} e^{-\lambda \mathrm{y}} \mathrm{dy}$$

Using now the change of variables $\mathrm{v} = \lambda \mathrm{y}$ *in the above equation we get*

$$\begin{aligned}\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} &= \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)\lambda^\alpha} \int_0^{\lambda(\psi(t)-\psi(a))} \mathrm{v}^{\alpha-1} e^{-\mathrm{v}} \mathrm{dv} \\ &\leq \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)\lambda^\alpha} \int_0^{\infty} \mathrm{v}^{\alpha-1} e^{-\mathrm{v}} \mathrm{dv} \\ &= \frac{e^{\lambda(\psi(t)-\psi(a))}}{\lambda^\alpha}.\end{aligned}$$

This completes the proof.

□

Remark 2.17 ([28, 26]). On $C(\mathbb{J}_a^b, \mathbb{R}^n)$ we describe a Bielecki type norm $\|\cdot\|_{\mathfrak{D}}$ as follows

$$\|\varpi\|_{\mathfrak{D}} := \sup_{\varsigma \in \mathbb{J}_a^b} \frac{\|\varpi(\varsigma)\|}{e^{\vartheta(\Theta(\varsigma)-\Theta(a))}}, \quad \vartheta > 0. \tag{2.2}$$

Therefore, we possess the below characteristics.

1. $\left(C(\mathbb{J}_a^b, \mathbb{R}^n), \|\cdot\|_{\mathfrak{D}}\right)$ is a Banach space.
2. On $C(\mathbb{J}_a^b, \mathbb{R}^n)$, the norms $\|\cdot\|_{\mathfrak{D}}$ and $\|\cdot\|_\infty$ are equivalent, where $\|\cdot\|_\infty$ represented the Chebyshev norm on $C(\mathbb{J}_a^b, \mathbb{R}^n)$, i.e;
$$\iota_1 \|\cdot\|_{\mathfrak{D}} \leq \|\cdot\|_\infty \leq \iota_2 \|\cdot\|_{\mathfrak{D}},$$
where
$$\iota_1 = 1, \quad \iota_2 = e^{\vartheta(\Theta(b)-\Theta(a))}.$$

Additional informations of Bielecki type norms can be found in [7, 28, 26].

In the following lemma, we prove an auxiliary result for a linear variant of the problem (2.3) which is the basic tool to convert the problem (2.3) into an equivalent fixed point problem.

Lemma 2.18. *[3, 4] Let $0 < \alpha < 1$ and $h \in C(\mathbb{J}, \mathbb{R})$. A function $u \in C_\psi^\alpha(\mathbb{J}, \mathbb{R})$ is a solution of the fractional initial value problem*

$$\begin{cases} c\mathbb{D}_{a^+}^{\alpha;\psi} u(t) = h(t), t \in \mathbf{J} := [a,b], \\ u(a) = \theta_a, \end{cases} \tag{2.3}$$

if and only if $u \in C(\mathbb{J}, \mathbb{R})$ satisfies the integral equation

$$u(t) = \theta_a + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} h(s) \mathrm{d}s. \tag{2.4}$$

Proof. Suppose that $u \in C_{\psi}^{\alpha}(\mathbb{J},\mathbb{R})$ and the IVP (2.3). Using the definition of ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}u$ we obtain

$$\begin{aligned} {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}u(t) &= {}^{\mathrm{R-L}}\mathbb{D}_{a^+}^{\alpha;\psi}(u(t)-u(a)) \\ &= \mathscr{D}_{\psi}\mathbb{I}_{a^+}^{1-\alpha;\psi}(u(t)-u(a)) = h(t) \end{aligned}$$

Since

$$\mathbb{I}_{a^+}^{1;\psi}\mathscr{D}_{\psi}v(t) = v(t)-v(a)$$

we get

$$\mathbb{I}_{a^+}^{1-\alpha;\psi}(u(t)-u(a)) = \mathbb{I}_{a^+}^{1;\psi}h(t)$$

Taking the ψ-Riemann-Liouville fractional integral of order α to both sides of the above equation and using Lemma 2.8, it yields

$$\mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{1-\alpha;\psi}(u(t)-u(a)) = \mathbb{I}_{a^+}^{1;\psi}(u(t)-u(a)) = \mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{1;\psi}h(t) = \mathbb{I}_{a^+}^{1;\psi}\mathbb{I}_{a^+}^{\alpha;\psi}h(t),$$

and by differentiation with respect to ψ we deduce that

$$u(t) = \theta_a + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.$$

Conversely, suppose that $u \in C(\mathbb{J},\mathbb{R})$ satisfies (2.4). Then $\mathbb{I}_{a^+}^{1-\alpha;\psi}(u-\theta_a) \in C_{\psi}^{1}(\mathbb{D},\mathbb{R})$. Moreover, from Eq. (2.4), we get $u(a) = \theta_a$ and

$$\mathbb{I}_{a^+}^{1-\alpha;\psi}(u(t)-u(a)) = \mathbb{I}_{a^+}^{1;\psi}h(t)$$

Finally, applying $\mathscr{D}_{\psi}$ to both sides of the above equation, we get ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi}u(t) = {}^{\mathrm{R-L}}\mathbb{D}_{a^+}^{\alpha;\psi}(u(t)-u(a)) = h(t)$. This shows that u is solution of the IVP (2.3), which completes the proof.

□

Lemma 2.19. *[28] Let $\lambda > 0$ be a constant and $\alpha > 0$. Consider the space of continuous functions $C(\mathbb{J},\mathbb{R})$ coupled with a suitable norm, either*

the well-known supremum (uniform) norm

$$\|u\|_{\infty} := \sup_{t\in\mathbb{J}}|u(t)|$$

or the ψ-fractional Bielecki norm $\|\cdot\|_{\lambda;\alpha}$ defined as below

$$\|u\|_{\lambda;\alpha} := \sup_{t\in\mathbb{J}}\frac{|u(t)|}{\mathbb{E}_{\alpha}(\lambda(\psi(t)-\psi(a))^{\alpha})}.$$

Consequently, we have the following proprieties

1. *$\|\cdot\|_{\lambda;\alpha}$ is a norm and is equivalent to $\|\cdot\|_{\infty}$;*
2. *$\left(C(\mathbb{J},\mathbb{R}),\|\cdot\|_{\lambda;\alpha}\right)$ is a Banach space.*

2.4 Topics of Functional Analysis

Let's start with the following definitions

Definition 2.20. Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space. The operator $\mathbb{T}:\mathbb{X}\to\mathbb{X}$ is said to satisfy the Lipschitz condition, if there exists a positive real constant $\mathbb{L}$ such that for all for all $x,y\in\mathbb{X}$

$$\|\mathbb{T}x-\mathbb{T}y\|_{\mathbb{X}}\leq\mathbb{L}\|x-y\|_{\mathbb{X}}$$

Moreover, if $\mathbb{L}\in[0,1)$, the operator $\mathbb{T}$ is called a contraction mapping on $\mathbb{X}$.

Definition 2.21. A subset $\mathbb{M}$ of $C(\mathbb{I},\mathbb{R})$ is uniformly bounded if there exists a constant $\mathrm{k}>0$ such that $|\mathrm{m}(t)|\leq\mathrm{k}$ for each $t\in\mathbb{J}$ and each $\mathrm{m}\in\mathbb{M}$.

Definition 2.22. A subset $\mathbb{M}$ of $C(\mathbb{J},\mathbb{R})$ is equicontinuous if for every $\varepsilon>0$ there exists some $\delta>0$ (which depends only on ε) such that for all $\mathrm{m}\in\mathbb{M}$ and all $t_1,t_2\in\mathbb{J}$ with $|t_1-t_2|<\delta$, we have $|m(t_1)-m(t_2)|<\varepsilon$ In the following, we state the Ascoli-Arzela theorem.

Theorem 2.23. *A subset $\mathbb{M}$ of $C(\mathbb{I},\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous.*

We conclude this section, with the following fixed point theorems that are very useful to obtain our main results.

Theorem 2.24 (Banach fixed point theorem)**.** *Let $\mathbb{X}$ be a Banach space. Then any contraction operator $\mathbb{T}:\mathbb{X}\longrightarrow\mathbb{X}$ has a unique fixed point.*

Theorem 2.25 (Schauder fixed point theorem)**.** *Let $\mathbb{X}$ be a Banach space, Ω be a nonempty bounded closed and convex subset of $\mathbb{X}$, and $\mathbb{T}:\Omega\longrightarrow\Omega$ be a continuous mapping such that $\mathbb{T}(\Omega)$ is a relatively compact subset of $\mathbb{X}$. Then $\mathbb{T}$ has at least one fixed point in Ω.*

Theorem 2.26 (Krasnoselskii fixed point theorem)**.** *Let Ω be a closed, convex, non-empty subset of a Banach space $\mathbb{X}$. Suppose that $\mathbb{A}$ and $\mathbb{B}$ maps Ω into $\mathbb{X}$, such that the following hypotheses are fulfilled :*

(i) $\mathbb{A}u+\mathbb{B}v\in\Omega$ *for all* $u,v\in\Omega$*;*

(ii) $\mathbb{A}$ *is compact and continuous;*

(iii) $\mathbb{B}$ *is a contraction mapping.*

Then the operator equation $\mathbb{A}z+\mathbb{B}z=z$ has at least one solution on Ω.

We close this section by introducing the following fixed-point theorems in generalized Banach spaces that will be employed in the sequel.

Definition 2.27 ([17])**.** Let $\varpi\neq\emptyset$, thus the map $d\colon E\times E\to\mathbb{R}^m$, is said a vector-valued metric on E if the below properties met

(i) $\forall\varpi,w\in E$, $\mathrm{d}(\varpi,w)\geq0$, and if $\mathrm{d}(\varpi,w)=0$, thus $\varpi=w$;

(ii) $\forall\varpi,w\in E$, $\mathrm{d}(\varpi,w)=\mathrm{d}(w,\varpi)$;

(iii) $\forall\varpi,w\in E$, $\mathrm{d}(\varpi,\varpi)\leq\mathrm{d}(\varpi,w)+\mathrm{d}(w,\varpi)$.

The pair (E,d) is said a generalized metric space (GMS) with

$$\mathrm{d}(\varpi,w):=\begin{pmatrix}\mathrm{d}_1(\varpi,w)\\ \mathrm{d}_2(\varpi,w)\\ \vdots\\ \mathrm{d}_k(\varpi,w)\end{pmatrix}.$$

Note that d is a GMS on ϖ iff d_j, $j=1,\ldots,k$, are metrics on E.

Definition 2.28 ([29]). A real square matrix $\mathbb{A}$ convergent to 0 $(\mathbb{A} \to^{CV} 0)$ iff its spectral radius $\rho(\mathbb{A}) < 1$.

Theorem 2.29 ([29]). *The below proposition are equivalent :*

(1) *As* $k \to \infty$, $\mathbb{A}^k \to^{CV} 0$;

(2) $\forall \lambda \in \mathbb{C}$, *we get* $|\lambda| < 1$, *such that* $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$ *s.t.* $\mathbb{I}$ *represents the unit matrix of* $\mathbb{A}_{n\times n}(\mathbb{R})$.

(3) *the matrix* $\mathbb{I} - \mathbb{A}$ *is nonsingular and* $(\mathbb{I} - \mathbb{A})^{-1} = \sum_{k=1}^{\infty} \mathbb{A}^k$.

(4) *the matrices* $\mathbb{I} - \mathbb{A}$ *and* $(\mathbb{I} - \mathbb{A})^{-1}$ *are non-singular and non-negative, respectively.*

Example 2.30 ([19]). The matrix $\mathbb{A} \in \mathbb{A}_{2\times 2}(\mathbb{R})$ given by

$$\mathbb{A} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

converges to 0 in this cases :

(1) $f = g = 0, e, h > 0$, and $\max\{e, h\} < 1$.

(2) $g = 0, e, h > 0, e + h < 1$, and $-1 < f < 0$.

(3) $e + f = g + h = 0, e > 1, g > 0$, and $|e - g| < 1$.

Definition 2.31 ([20, 24]). Let $(\mathbb{E}, \mathrm{d})$ be a GMS. If there is a matrix $\mathbb{A}$ converging to zero, then the mapping $\mathbb{Z} \colon \mathbb{E} \to \mathbb{E}$ is contractive, where

$$\forall \varpi, w \in \mathbb{E}, \;\; \mathrm{d}(\mathbb{Z}(\varpi), \mathbb{Z}(w)) \leq \mathbb{A}\mathrm{d}(\varpi, w).$$

We concludes this part by presenting the below FPTs that will be used as helping tools in the sequel.

Theorem 2.32 ([18, 20]). *Let the mapping* $\mathbb{Z} \colon \mathbb{E} \to \mathbb{E}$ *is a contractive with Lipschitz matrix* $\mathbb{A}$, *and* $(\mathbb{E}, \mathrm{d})$ *be a complete GMS. Then* $\mathbb{Z}$ *possess one and only one fixed point* ϖ_0 *and* $\forall \varpi \in \mathbb{E}$

$$\mathrm{d}(\mathbb{Z}^k(\varpi), \varpi_0) \leq \mathbb{A}^k (\mathbb{I} - \mathbb{A})^{-1} \mathrm{d}(\varpi, \mathbb{Z}(\varpi)) \quad \forall k \in \mathbb{N}.$$

Theorem 2.33 ([17]). *Let* Π *be a nonempty* $(\Pi \neq \emptyset)$ *convex closed subset of a GBS* ϖ. *Let* $\mathbb{V}$ *and* $\mathbb{U}$ *map* Π *into* ϖ *and that*

(i) $\forall \varpi, w \in \Pi, \;\; \mathbb{U}\varpi + \mathbb{V}w \in \Pi$;

(ii) $\mathbb{V}$ *is an* $\mathbb{A}$*-contraction mapping.*

(iii) $\mathbb{U}$ *is continuous and compact ;*

Then the operator $\mathbb{U}\varpi + \mathbb{V}\varpi = \varpi$ *possess at least one solution on* Π.

Chapitre 3

Existence results for ψ–Caputo fractional neutral functional integro-differential equations

3.1 Introduction

In this chapter, we are concerned with the existence and uniqueness of solutions for new type of nonlinear fractional differential equations via ψ- fractional derivative. Sufficient and necessary conditions will be presented for the existence and uniqueness of the solution of fractional initial value problem, we investigate the problem of existence and uniqueness for a initial value problem for fractional differential equations by applying some standard fixed point theorems. examples are given to illustrate our results. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.

In this chapter, we concentrate on the following initial value problem of nonlinear fractional differential equation :

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[z(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,(\tau))\right]=\mathbb{H}(\tau,z(\tau)),\tau\in \mathrm{J}:=[a,b],\\ z(a)=z_a, \end{cases} \tag{3.1}$$

where ${}^{c}\mathbb{D}_{a^+}^{\beta;\psi}$ is the ψ-Caputo FOD of order $\beta\in\{\nu,\varsigma\}\in(0,1]$, $\mathbb{I}_{a^+}^{\theta;\psi}$ is the ψ-RL fractional integral of order $\theta>0$, $\theta\in\{\sigma_1,\sigma_2,\dots,\sigma_m\}$, $\sigma_k>0, k=1,2,\dots,m$, $\mathbb{F}_k,\mathbb{H}:\mathrm{J}\times\mathscr{C}\longrightarrow\mathbb{R}$ $(k=1,2,\dots,m)$ are given continuous functions, where $\mathscr{C}=C([a,b],\mathbb{R})$.

3.2 Existence of solutions

First, we prove a preparatory lemma for boundary value problem of nonlinear fractional differential equations with ψ-Caputo derivative.

Let us define exactly what we mean by a solution of problem (3.1).

Definition 3.1. A function $z\in\mathscr{C}_b$ is said to be a solution of (3.1) if z fulfills the equation ${}^{c}\mathbb{D}_{a^+}^{\nu}\left[z(\tau)-\sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z(\tau))\right]=\mathbb{H}(\tau,z(\tau))$ on J, and the condition $z(a)=z_a$.

To demonstrate the existence of solutions to (3.1), we need the following Lemma.

Lemma 3.2. *Let* $0<\nu\leq 1$, $\alpha(a)=0$, *and* $g,h:\mathrm{J}\to\mathbb{R}$ *are continuous functions with* $h(a)=0$. *The linear problem*

$${}^{c}\mathbb{D}_{a^+}^{\nu;\psi}[z(\tau)-h(\tau)]=g(\tau),\quad \tau\in \mathrm{J}, z(a)=z_a,$$

has a unique solution $z(t)$ defined by :

$$z(t) = h(\tau) + \mathbb{I}_{a^+}^{\nu,\psi} g(\tau) + z_a, \tau \in \mathrm{J},$$

For the proof of Lemma 3.2, it is useful to refer to [13, 2].

3.2.1 First result [1]

In the following, we prove the existence and uniqueness results, for the initial value problem (3.1) by using a variety of fixed point theorems.

Now, we give the following hypotheses.

(G1) The functions $\mathbb{H}, \mathbb{F}_k : \mathrm{J} \times \mathscr{C} \to \mathbb{R}$ are continuous.

(G2) There exist positive functions μ, υ_k, $k = 1, 2, \cdots, m$, with bounds $\|\mu\|$ and $\|\upsilon_k\|$, $k = 1, 2, \cdots, m$, respectively such that

$$|\mathbb{H}(\tau, z) - \mathbb{H}(\tau, \bar{z})| \leq \mu(\tau)\|z - \bar{z}\|,$$

and

$$|\mathbb{F}_k(\tau, z) - \mathbb{F}_k(\tau, \bar{z})| \leq \upsilon_k(\tau)\|z - \bar{z}\|.$$

for $\tau \in \mathrm{J}$ and $z, \bar{z} \in \mathscr{C}$.

(G3) There exist two constants $\varphi, \phi_k \geq 0$, for $k = 1, 2, \cdots, m$ such that

$$|\mathbb{H}(\tau, z)| \leq \varphi\|z\|_{\mathscr{C}}, \quad |\mathbb{F}_k(\tau, z)| \leq \phi_k, \quad \forall(\tau, z) \in \mathrm{J} \times \mathscr{C}.$$

For the sake of convenience, we setting the following symbols.

$$\Lambda_1 := \frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu + 1)}\|\mu\|,$$

$$\Lambda_2 := \sum_{k=1}^{m} \frac{(\psi(b) - \psi(a))^{\sigma_k}}{\Gamma(\sigma_k + 1)}\|\upsilon_k\|,$$

$$\Delta := \Lambda_1 + \Lambda_2. \tag{3.2}$$

Uniqueness result via Banach FPT

Theorem 3.3. *Suppose that assumptions (G1)–(G2) holds. If*

$$\Delta < 1 \tag{3.3}$$

where Δ is given by (3.2), then there exists a unique solution for (3.1) on the interval $[a, b]$.

Proof.

Define the set

$$U := \left\{ z \in \mathscr{C} ; {}^c\mathbb{D}_{a^+}^{\nu} z \in \mathscr{C} \right\},$$

and the operator $\mathscr{K} : U \to U$

1. **A. Boutiara**, M.S. Abdo, M. Benbachir, Existence results for ψ–Caputo fractional neutral functional integro-differential equations with finite delay, *Turk J Math*, (2020)(44) :2380-2401.

$$\mathscr{K}(z)(t) := \mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau)) + \mathbb{F}(\tau, z(\tau)) + z_a, \tau \in \mathrm{J}, \tag{3.4}$$

Notice that $\mathscr{K}$ is well defined. Indeed, for $z \in U$, the map $\tau \mapsto \mathscr{K}(z)(\tau)$ is continuous, for all $\tau \in [a,b]$. Also, for all $\tau \in \mathrm{J}$, ${}^c\mathbb{D}_{a^+}^{a,\psi}\mathscr{K}[z(\tau) - \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau))] = \mathbb{H}(\tau, z_\tau)$ exists and is continuous too due to continuity of $\mathbb{H}$. Now we need to show that $\mathscr{K}$ is a contraction map. Let $z, \bar{z} \in U$ and $\tau \in [a,b]$ and use (G2), it follows that

$$\begin{aligned}
|\mathscr{K}(z)(\tau) - \mathscr{K}(\bar{z})(\tau)| &\leq \mathbb{I}_{a^+}^{\nu;\psi}|\mathbb{H}(\tau, z(\tau)) - \mathbb{H}(\tau, \bar{z}(\tau))| + \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}|\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, \bar{z}(\tau))| \\
&\leq \|\mu\|\|z(\tau) - \bar{z}(\tau)\|_{\mathscr{C}}\mathbb{I}_{a^+}^{\nu;\psi}(1)(\tau) + \sum_{k=1}^{m}\|\upsilon_k\|\,\|z(\tau) - \bar{z}(\tau)\|\mathbb{I}_{a^+}^{\sigma_k;\psi}(1)(\tau) \\
&\leq \left(\frac{(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu+1)}\|\mu\| + \sum_{k=1}^{m}\frac{(\psi(b) - \psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|\upsilon_k\|\right)\|z(\tau) - \bar{z}(\tau)\| \\
&\leq (\Lambda_1 + \Lambda_2)\|z - \bar{z}\|,
\end{aligned}$$

which implies

$$\|\mathscr{K}(z) - \mathscr{K}(\bar{z})\| \leq \Delta\|z - \bar{z}\|.$$

Since $\Delta < 1$, the operator $\mathscr{K}$ is a contraction. Hence the theorem of Banach fixed point shows that $\mathscr{K}$ admits a unique fixed point.

□

Existence result via Kransnoselskii's FPT

Here, we apply the fixed point theorem of Krasnoselskii [14] to obtain the existence result.

Theorem 3.4. *Assume that (G1)–(G3) hold. Then (3.1) has at least one solution on* J*, provided*

$$\frac{(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi < 1. \tag{3.5}$$

Proof. By the assumption (G3), we can fix

$$\rho \geq \frac{z_a + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\phi_k}{1 - \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi},$$

where $B_\rho = \{z \in \mathscr{C}_b : \|z\| \leq \rho\}$. Let us split the operator $\mathscr{K} : \mathscr{C} \to \mathscr{C}$ defined by Equation (3.4) as $\mathscr{K} = \mathscr{K}_1 + \mathscr{K}_2$, where $\mathscr{K}_1$ and $\mathscr{K}_2$ are given by

$$\mathscr{K}_1(z)(t) := \mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau)) + z_a, \tau \in \mathrm{J},$$

$$\mathscr{K}_2(z)(t) := \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau)), \tau \in \mathrm{J}.$$

The proof will be split into numerous steps :
Step 1 : $\mathscr{K}_1(z) + \mathscr{K}_2(z) \in B_\rho$.

Let $z,\bar{z}\in B_\rho$ and $\tau\in$ J. Then

$$\begin{aligned}
|\mathscr{K}_1(z)(\tau)+\mathscr{K}_2(\bar{z})(\tau)| &\leq \mathbb{I}_{a^+}^{\nu;\psi}\left|\mathbb{H}(\tau,z(\tau))\right|+\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k,\psi}\left|\mathbb{F}_k(\tau,z_\tau)\right|ds+|z_a|\\
&\leq \varphi\|z(\tau)\|_{\mathscr{C}}\mathbb{I}_{a^+}^{\nu,\psi}(1)(\tau)+\sum_{k=1}^{m}\phi_k\mathbb{I}_{a^+}^{\sigma_k,\psi}(1)(\tau)+|z_a|\\
&\leq \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi\|z\|+\sum_{k=1}^{m}\frac{(\psi(\tau)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\phi_k+|z_a|\\
&\leq \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi\rho+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\phi_k+|z_a|\\
&\leq \rho.
\end{aligned}$$

Hence

$$\|\mathscr{K}_1(z)+\mathscr{K}_2(\bar{z})\|\leq\rho,$$

which shows that $\mathscr{K}_1 z+\mathscr{K}_2\bar{z}\in B_\rho$.

Step 2 : $\mathscr{K}_1$ is a contration map on B_ρ.

Due to the contractility of $\mathscr{K}$ as in Theorem 3.3, then $\mathscr{K}_1$ is a contraction map too.

Step 3 : $\mathscr{K}_2$ is completely continuous on B_ρ.

From the continuity of $\mathbb{F}_k\left(\cdot,z_{(\cdot)}\right)$, it follows that $\mathscr{K}_2$ is continuous.

Since

$$\|\mathscr{K}_2 z\|_{\mathscr{C}}=\sup_{\tau\in \mathrm{J}}|\mathscr{K}_2 z(\tau)|\leq\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\phi_k:=L,\ z\in B_\rho.$$

we get $\|\mathscr{K}_2 z\|\leq L$ which emphasize that $\mathscr{K}_2$ uniformly bounded on B_ρ.

Finally, we prove the compactness of $\mathscr{K}_2$.

For $z\in B_\rho$ and $\tau\in$ J, we can estimate the operator derivative as follows :

$$\begin{aligned}
\left|(\mathscr{K}_2 z)_{\psi}^{(1)}(\tau)\right| \leq \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k-1,\psi}\left|\mathbb{F}_k(\tau,z(\tau))\right| &\leq \sum_{k=1}^{m}\phi_k\mathbb{I}_{a^+}^{\sigma_k-1,\psi}(1)(\tau)\\
&\leq \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k-1}}{\Gamma(\sigma_k)}\phi_k:=\ell,
\end{aligned}$$

where we used the fact

$$D_{\psi}^{k}\,\mathbb{I}_{a^+}^{\alpha,\psi}=\mathbb{I}_{a^+}^{\alpha-k,\psi},\quad \omega_{\psi}^{(k)}(\tau)=\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{k}\omega(\tau)\text{ for }k=0,1,...,n-1.$$

Hence, for each $\tau_1,\tau_2\in$ J with $a<\tau_1<\tau_2<b$ and for $z\in B_\rho$, we get

$$|(\mathscr{K}_2 z)(t_2)-(\mathscr{K}_2 z)(t_1)|=\int_{\tau_1}^{\tau_2}\left|(\mathscr{K}_2 z)'(s)\right|ds\leq\ell(\tau_2-\tau_1).$$

which as $(\tau_2-\tau_1)\to 0$ tends to zero independent of z. So, $\mathscr{K}_2$ is equicontinuous. The equicontinuity for the case $\tau_1,\tau_2\in$ J is obvious. In view of the foregoing arguments along with Arzela–Ascoli theorem, we infer that $\mathscr{K}_2$ is compact on B_ρ. Thus, the hypotheses of Krasnoselskii fixed point theorem [14] holds, so there exists at least one solution of (3.1) on J.

□

Chapitre 4

Coupled System of Fractional Differential Equations in Generalized Banach Spaces

4.1 Introduction

The main objective of this chaptert is to establish various existence and uniqueness results of solutions to a Coupled system of ψ-Caputo fractional differential equations with delay in generalized Banach spaces. Existence and uniqueness results are obtained by applying Schauder's fixed point theorem in generalized Banach spaces, and Perov's fixed point theorem combined with the Bielecki norm. Finally, Some examples are given to illustrate the obtained results.

This chapter deals with the existence of solutions on a bounded interval for the following Coupled system of ψ-Caputo fractional differential equations with delay involving the ψ-Caputo derivative of the form :

$$\begin{cases}({}^c\mathbb{D}^{\nu;\psi}_{a^+}x)(\tau) = \mathbb{F}_1(\tau, x_\tau, y_\tau), \\ ({}^c\mathbb{D}^{\mu;\psi}_{a^+}y)(\tau) = \mathbb{F}_2(\tau, x_\tau, y_\tau), \end{cases} \quad \tau \in \mathrm{J}, \tag{4.1}$$

along with the initial conditions

$$\begin{cases} x(\tau) = \alpha(\tau), \\ y(\tau) = \beta(\tau), \end{cases} \quad \tau \in [a-\delta, a], \tag{4.2}$$

where $\delta > 0$ is a constant delay and $\mathbb{F}_1, \mathbb{F}_2 : \mathrm{J} \times C([-\delta, 0], \mathbb{R}^n) \times C([-\delta, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$, are given continuous functions and $\alpha, \beta : [a-\delta, a] \longrightarrow \mathbb{R}^n$ are two continuous functions. For any function z defined on $[a-\delta, a]$ and any $\tau \in \mathrm{J}$, we denote by z_τ the element of $C([-\delta, 0], \mathbb{R}^n)$ defined by

$$z_\tau(\rho) = z(\tau + \rho), \quad \rho \in [-\delta, 0].$$

Hence $z_\tau(\cdot)$ represents the history of the state from times $\tau - \delta$ up to the present time τ.

4.1.1 Existence of solutions

Now, we focus on the existence and uniqueness of solutions for the given problem (4.1)–(4.2). Before proceeding to the main results, we start by the following definition.

Definition 4.1. By a solution of problem (4.1)–(4.2) we mean a coupled function $(x, y) \in C([a-\delta, b], \mathbb{R}^n) \times C([a-\delta, b], \mathbb{R}^n)$ that satisfies the system

$$\begin{cases}({}^c\mathbb{D}^{\nu;\psi}_{a^+}x)(\tau) = \mathbb{F}_1(\tau, x_\tau, y_\tau), \\ ({}^c\mathbb{D}^{\mu;\psi}_{a^+}y)(\tau) = \mathbb{F}_2(\tau, x_\tau, y_\tau), \end{cases} \quad \tau \in \mathrm{J},$$

and the initial conditions

$$\begin{cases} x(\tau)=\alpha(\tau), \\ y(\tau)=\beta(\tau), \end{cases} \qquad \tau\in[a-\delta,a].$$

To prove the existence of solutions to (4.1)–(4.2), we need the following lemma that was proven in the recent work of Almeida [6].

Lemma 4.2 ([6]). *Let $g:[a,b]\times C([-\delta,0],\mathbb{R}^n)\longrightarrow\mathbb{R}^n$ be a continuous function. Then $z\in C([a-\delta,b],\mathbb{R}^n)$ is the solution of*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}z(\tau)=g\big(\tau,z_\tau\big), & \tau\in[a,b], \\ z(\tau)=\alpha(\tau) & \tau\in[a-\delta,a], \end{cases}$$

if and only if it is the solution of the integral equation

$$z(\tau)=\begin{cases} \alpha(a)+\int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}g\big(s,z_s\big)\mathrm{ds} & , \quad \tau\in[a,b], \\ \alpha(\tau) & , \quad \tau\in[a-\delta,a]. \end{cases} \tag{4.3}$$

As a consequence of Lemma 4.2 we have the following result which will be used in the sequel in the proofs of the main results.

Lemma 4.3. *Let $\nu,\mu\in(0,1]$ be fixed and $\mathbb{F}_1,\mathbb{F}_2:\mathrm{J}\times C([-\delta,0],\mathbb{R}^n)\times C([-\delta,0],\mathbb{R}^n)\longrightarrow\mathbb{R}^n$ are a given continuous functions. Then the coupled systems (4.1)–(4.2) is equivalent to the following integral equations*

$$x(\tau)=\begin{cases} \alpha(a)+\int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{F}_1(s,x_s,y_s)\mathrm{ds} & , \quad \tau\in[a,b], \\ \alpha(\tau) & , \quad \tau\in[a-\delta,a], \end{cases} \tag{4.4}$$

and

$$y(\tau)=\begin{cases} \beta(a)+\int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\mathbb{F}_2(s,x_s,y_s)\mathrm{ds} & , \quad \tau\in[a,b], \\ \beta(\tau), & , \quad \tau\in[a-\delta,a]. \end{cases} \tag{4.5}$$

We assume that the conditions given below stands hold throughout the remainder of the paper :

(C1) $\mathbb{F}_1,\mathbb{F}_2:\mathrm{J}\times C([-\delta,0],\mathbb{R}^n)\times C([-\delta,0],\mathbb{R}^n)\longrightarrow\mathbb{R}^n$ are continuous functions.

(C2) There exist positive constants $\mathbb{L}_i,\mathbb{M}_i, i=1,2$, such that

$$\|\mathbb{F}_i(\tau,x,y)-\mathbb{F}_i(\tau,\bar{x},\bar{y})\|\leq\mathbb{L}_i\|x-\bar{x}\|_{\mathfrak{C}_\delta}+\mathbb{M}_i\|y-\bar{y}\|_{\mathfrak{C}_\delta}$$

for all $\tau\in\mathrm{J}$ and each $(x,y),(\bar{x},\bar{y})\in\mathfrak{C}_\delta\times\mathfrak{C}_\delta$.

For the sake of brevity, we set

$$\mathbb{F}_i^*:=\sup_{\tau\in\mathrm{J}}\|\mathbb{F}_i(\tau,0,0)\|.$$

Define a square matrix $\mathbb{A}_\psi$ as

$$\mathbb{A}_\psi=\begin{pmatrix} \ell_\psi^\nu\mathbb{L}_1 & \ell_\psi^\nu\mathbb{M}_1 \\ \ell_\psi^\mu\mathbb{L}_2 & \ell_\psi^\mu\mathbb{M}_2 \end{pmatrix}. \tag{4.6}$$

Firstly, we prove the uniqueness result by means of the Perov's fixed point theorem.

Theorem 4.4. *If the assumptions* (C1) *and* (C2) *are true along with the matrix $\mathbb{A}_\psi$ defined in* (4.6) *converges to zero. Then the coupled system* (4.1)–(4.2) *possesses a unique solution in the space* $C([a-\delta,b],\mathbb{R}^n)\times C([a-\delta,b],\mathbb{R}^n)$

Proof. Consider the Banach space $\mathfrak{C}=C([a-\delta,b],\mathbb{R}^n)$ equipped with the norm

$$\|z\|_{\mathfrak{C}}:=\sup_{\tau\in[a-\delta,b]}\|z(\tau)\|.$$

Consequently, the product space $\mathscr{X}:=\mathfrak{C}\times\mathfrak{C}$ is a generalized Banach space, endowed with the vector-valued norm

$$\|(x,y)\|_{\mathscr{X}}=\begin{pmatrix}\|x\|_{\mathfrak{C}}\\ \|y\|_{\mathfrak{C}}\end{pmatrix}.$$

We define an operator $\mathbb{K}=\big(\mathbb{K}_1,\mathbb{K}_2\big):\mathscr{X}\to\mathscr{X}$ by :

$$\mathbb{K}(x,y)=\big(\mathbb{K}_1(x,y),\mathbb{K}_2(x,y)\big). \tag{4.7}$$

where

$$\mathbb{K}_1(x,y)(\tau)=\begin{cases}\alpha(a)+\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{F}_1(s,x_s,y_s)\mathrm{ds} & ,\quad \tau\in[a,b],\\ \alpha(\tau) & ,\quad \tau\in[a-\delta,a].\end{cases} \tag{4.8}$$

and

$$\mathbb{K}_2(x,y)(\tau)=\begin{cases}\beta(a)+\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\mathbb{F}_2(s,x_s,y_s)\mathrm{ds} & ,\quad \tau\in[a,b],\\ \beta(\tau), & ,\quad \tau\in[a-\delta,a].\end{cases} \tag{4.9}$$

Now, we apply Perov's fixed point theorem to prove that $\mathbb{K}$ has a unique fixed point. To do this, it enough to show that $\mathbb{K}$ is $\mathbb{A}_\psi$-contraction mapping on $\mathscr{X}$. In fact, for all $\tau\in[a-\delta,b],(x,y),(\bar{x},\bar{y})\in\mathscr{X}$. When $\tau\in[a-\delta,a]$, we have

$$\|\mathbb{K}_1(x,y)(\tau)-\mathbb{K}_1(\bar{x},\bar{y})(\tau)\|=0.$$

On the other hand, keeping in mind the definition of the operator $\mathbb{K}_1$ on $[a,b]$ together with assumption (C2), we can get

$$\begin{aligned}&\big\|\mathbb{K}_1(x,y)(\tau)-\mathbb{K}_1(\bar{x},\bar{y})(\tau)\big\|\\ \leq&\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\|\mathbb{F}_1(s,x_s,y_s)-\mathbb{F}_1(s,\bar{x}_s,\bar{y}_s)\|\mathrm{ds}\\ \leq&\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\big(\mathbb{L}_1\|x_s-\bar{x}_s\|_{\mathfrak{C}_\delta}+\mathbb{M}_1\|y_s-\bar{y}_s\|_{\mathfrak{C}_\delta}\big)\mathrm{ds}\\ \leq&\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}}+\mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big)\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathrm{ds}\\ \leq&\frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}}+\mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big)\\ =&\ell_\psi^\nu\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}}+\mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big).\end{aligned}$$

Hence

$$\big\|\mathbb{K}_1(x,y)-\mathbb{K}_1(\bar{x},\bar{y})\big\|_{[a,b]}\leq\ell_\psi^\nu\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}}+\mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big).$$

By similar procedure, we get

$$\begin{cases}\|\mathbb{K}_2(x,y)-\mathbb{K}_2(\bar{x},\bar{y})\|_{[a-\delta,a]}=0\\ \big\|\mathbb{K}_2(x,y)-\mathbb{K}_2(\bar{x},\bar{y})\big\|_{[a,b]}\leq\ell_\psi^\mu\big(\mathbb{L}_2\|x-\bar{x}\|_{\mathfrak{C}}+\mathbb{M}_2\|y-\bar{y}\|_{\mathfrak{C}}\big).\end{cases}$$

Consequently,

$$\left\|\mathbb{K}(x,y)-\mathbb{K}(\bar{x},\bar{y})\right\|_{\mathscr{X}} := \left(\begin{array}{c} \|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} \\ \|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \end{array}\right) \leq \mathbb{A}_\psi \|(x,y)-(\bar{x},\bar{y})\|_{\mathscr{X}},$$

where $\mathbb{A}_\psi$ is the matrix given by (4.6). Since the matrix $\mathbb{A}_\psi$ converges to zero, then Theorem 2.32 implies that coupled system (4.1)–(4.2) has a unique solution in $\mathscr{X}$.

□

Next, the following result is based on Schauder's type fixed point theorem in generalized Banach spaces.

Theorem 4.5. *Let the assumptions* (C1) *and* (C2) *are satisfied. Then the coupled system* (4.1)–(4.2) *has at least one solution, provided that the spectral radius of the matrix $\mathbb{A}_\psi$ is less than one.*

Proof. In order to apply Schauder's fixed point theorem type in a generalized Banach space, we need to construct a nonempty closed bounded convex set $\mathbb{B}_r \subset \mathscr{X}$ such that

$$\mathbb{K}(\mathbb{B}_r) \subseteq \mathbb{B}_r, \tag{4.10}$$

where the operator $\mathbb{K} : \mathscr{X} \to \mathscr{X}$ defined in (4.7). Let us consider the set

$$\mathbb{B}_r = \left\{(x,y) \in \mathscr{X} : \|(x,y)\|_{\mathscr{X}} \leq r\right\},$$

where $r := (r_1, r_2) \in \mathbb{R}^2_+$ will be specified later. Now we try to find $r_1, r_2 \geq 0$ such that (4.10) holds. Indeed, for all $\tau \in [a-\delta, b], (x,y), \in \mathscr{X}$. When $\tau \in [a-\delta, a]$, we have

$$\|\mathbb{K}_1(x,y)(\tau)\| \leq \|\alpha\|_{[a-\delta,a]},$$

which yields

$$\|\mathbb{K}_1(x,y)\|_{[a-\delta,a]} \leq \|\alpha\|_{[a-\delta,a]}, \tag{4.11}$$

and if $\tau \in [a,b]$, we have

$$\begin{aligned} \|\mathbb{K}_1(x,y)(\tau)\| \leq & \|\alpha(a)\| + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \big(\|\mathbb{F}_1(s,x_s,y_s)-\mathbb{F}_1(s,0,0)\| \\ & + \|\mathbb{F}_1(s,0,0)\|\big)\mathrm{d}s \\ \leq & \|\alpha(a)\| + \ell_\psi^\nu \big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big). \end{aligned}$$

Hence, we get

$$\|\mathbb{K}_1(x,y)\|_{[a,b]} \leq \|\alpha(a)\| + \ell_\psi^\nu \big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big). \tag{4.12}$$

So from (4.11) and (4.12), we get

$$\begin{aligned} \|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} &\leq \|\mathbb{K}_1(x,y)\|_{[a-\delta,a]} + \|\mathbb{K}_1(x,y)\|_{[a,b]} \\ &\leq \|\alpha\|_{[a-\delta,a]} + \|\alpha(a)\| + \ell_\psi^\nu \big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big). \end{aligned}$$

In a similar way, we get

$$\|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \leq \|\beta\|_{[a-\delta,a]} + \|\beta(a)\| + \ell_\psi^\nu \big(\mathbb{L}_2\|x\|_{\mathfrak{C}} + \mathbb{M}_2\|y\|_{\mathfrak{C}} + \mathbb{F}_2^*\big).$$

Thus the above inequalities can be written in the vectorial form as follows

$$\|\mathbb{K}(x,y)\|_{\mathscr{X}} := \left(\begin{array}{c} \|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} \\ \|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \end{array}\right) \leq \mathbb{A}_\psi \left(\begin{array}{c} \|x\|_{\mathfrak{C}} \\ \|y\|_{\mathfrak{C}} \end{array}\right) + \left(\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \end{array}\right), \tag{4.13}$$

where $\mathbb{A}_\psi$ is the matrix given by (4.6), and

$$\begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} = \begin{pmatrix} \ell_\psi^\nu \mathbb{F}_1^* + \|\alpha(a)\| + \|\alpha\|_{[a-\delta,a]} \\ \ell_\psi^\mu \mathbb{F}_2^* + \|\beta(a)\| + \|\beta\|_{[a-\delta,a]} \end{pmatrix}.$$

Now we look for $r = (r_1, r_2) \in \mathbb{R}_+^2$ such that $\|\mathbb{K}(x,y)\|_{\mathfrak{C}} \le r$, for any $(x,y) \in \mathbb{B}_r$. To this end, according to (4.13), it is sufficient to show

$$\mathbb{A}_\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} \le \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Equivalently

$$\begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} \le (\mathbb{I} - \mathbb{A}_\psi) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{4.14}$$

Since the matrix $\mathbb{A}_\psi$ is convergent to zero. It yields, from Theorem 2.27 that the matrix $(\mathbb{I} - \mathbb{A}_\psi)$ is nonsingular and $(\mathbb{I} - \mathbb{A}_\psi)^{-1}$ has nonnegative elements. Therefore, (4.14) is equivalent to

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \ge (\mathbb{I} - \mathbb{A}_\psi)^{-1} \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix}. \tag{4.15}$$

Furthermore, if we denote

$$(\mathbb{I} - \mathbb{A}_\psi)^{-1} = \begin{pmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{pmatrix}.$$

Then (4.15) becomes

$$\begin{cases} r_1 \ge \kappa_1 \mathbb{P}_1 + \kappa_2 \mathbb{P}_2, \\ r_2 \ge \kappa_3 \mathbb{P}_1 + \kappa_4 \mathbb{P}_2. \end{cases}$$

Which means that $\mathbb{K}(\mathbb{B}_r) \subseteq \mathbb{B}_r$. Moreover, by a similar process used in [6], it is easy to show that the operator $\mathbb{K}$ is continuous and, $\mathbb{K}(\mathbb{B}_r)$ is relatively compact. Combining this facts, with Arzelà–Ascoli's theorem, we conclude that $\mathbb{K}$ is a compact operator. Invoking Schauder's fixed point theorem we get a fixed point of $\mathbb{K}$ in $\mathbb{B}_r$, which is just a solution of system (4.1)–(4.2). This completes the proof of the Theorem 4.5.

□

4.2 Applications

In this section, we provide some examples to illustrate our results constructed in the previous two sections

Example 4.6. Consider the following fractional delayed coupled system of the form :

$$\begin{cases} ({}^{CH}\mathbb{D}_{1^+}^{0.5} x)(\tau) = \mathbb{F}_1(\tau, x_\tau, y_\tau), \\ ({}^{CH}\mathbb{D}_{1^+}^{0.5} y)(\tau) = \mathbb{F}_2(\tau, x_\tau, y_\tau), \end{cases} \quad \tau \in \mathrm{J} := [1, e], \tag{4.16}$$

with initial conditions

$$\begin{cases} x(\tau) = \alpha(\tau) = (\alpha_1(\tau), \alpha_2(\tau)), \\ y(\tau) = \beta(\tau) = (\beta_1(\tau), \beta_2(\tau)), \end{cases} \quad \tau \in [1-\delta, 1], \tag{4.17}$$

where

$$\nu = \mu = 0.5, \psi(\tau) = \ln \tau, a = 1, b = e, \ell_\psi^\nu = \ell_\psi^\mu = \frac{2}{\sqrt{\pi}}.$$

and

$$\mathbb{F}_1(\tau, x_\tau, y_\tau) = \begin{pmatrix} \frac{|x_{1,\tau}|+|x_{2,\tau}|}{e^{\tau+1}} \\ \frac{\sin(|y_{1,\tau}|+|y_{2,\tau}|)}{\tau+9} \end{pmatrix}.$$

$$\mathbb{F}_2(\tau, x_\tau, y_\tau) = \frac{1}{(\tau+1)^2} \begin{pmatrix} \ln(1+|y_{1,\tau}|+|y_{2,\tau}|) \\ |x_{1,\tau}|+|x_{2,\tau}| \end{pmatrix}.$$

Clearly, the functions $\mathbb{F}_1, \mathbb{F}_2$ are continuous. Moreover, for any $x, y, \bar{x}, \bar{y} \in \mathfrak{C}_\delta$ and $\tau \in \mathrm{J}$ we have

$$\|\mathbb{F}_1(\tau, x, y) - \mathbb{F}_1(\tau, \bar{x}, \bar{y})\|_1 \leq \mathbb{L}_1 \|x - \bar{x}\|_{\mathfrak{C}_\delta} + \mathbb{M}_1 \|y - \bar{y}\|_{\mathfrak{C}_\delta}$$
$$\|\mathbb{F}_2(\tau, x, y) - \mathbb{F}_2(\tau, \bar{x}, \bar{y})\|_1 \leq \mathbb{L}_2 \|x - \bar{x}\|_{\mathfrak{C}_\delta} + \mathbb{M}_2 \|y - \bar{y}\|_{\mathfrak{C}_\delta},$$

Hence the hypothesis (C2) holds with

$$\mathbb{L}_1 = e^{-2}, \quad \mathbb{M}_1 = 0.1, \quad \mathbb{L}_2 = \mathbb{M}_2 = 0.25.$$

Furthermore, the matrix $\mathbb{A}_\psi$ given by (4.6) has the following form

$$\mathbb{A}_\psi = \frac{2}{\sqrt{\pi}} \begin{pmatrix} e^{-2} & 0.1 \\ 0.25 & 0.25 \end{pmatrix}.$$

Using the Matlab program we can get the eigenvalues of $\mathbb{A}_\psi$ as follows $\sigma_1 = 0.0276, \sigma_2 = 0.4072$, which show that $\mathbb{A}_\psi$ is converging to zero. Therefore, by Theorem 4.5 the coupled system (4.16)–(4.17) has a unique solution.

Conclusion and Perspective

This study discussed the existence and uniqueness of solutions for two kinds of nonlinear problems containing the ψ-Caputo fractional derivatives of order $0 < \alpha < 1$ subjected to initial conditions. Our technique was based on reducing the problems under consideration to a fractional integral equation and using some well-known fixed point theorems. Some suitable examples are also introduced to support the theoretical results. Moreover, according to the Remark 2.7, our proposed problems cover many of the corresponding problems in the literature, which are considered special cases.

At the same time, we know that there are still more works to do in the future. For example, how to apply other fixed point theorems to study the qualitative theories (existence, uniqueness, stability, attractivity, positivity, controllability, oscillation) of finite or infinite systems of FDEs involving other generalized fractional operators with singular/nonsingular kernels under some weak conditions ? Can we add some impulsive or delay effect to the IVP (2.3) ?

It would also be interesting to study the existence and uniqueness of extremal solutions for our current problems under the well-known monotone iterative technique together with the method of upper and lower solutions.

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قسم الرياضيات والإعلام الآلي

جامعـــة غرداية

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بصفتي مشرف والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـــــ

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