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Existence of Solutions for Generalized Caputo Periodic and Non-Local Boundary Value Problems

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Dédicace

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ملخص

تستعمل المعادلات التفاضلية ذات مشتقات من الدرجة الكسرية المعممة لكابيتو في مجموعة متنوعة من مجالات التطبيقات البيولوجية والفيزيائية والهندسية، وقد حظيت هذه المعادلات باهتمام كبير في السنوات الأخيرة. تطرقنا في هذه المذكرة إلى وجود حلول عشوائية لمسائل القيمة الحدية الدورية وغير المحلية ذات مشتقة من الدرجة الكسرية المعممة لكابيتو. سيتم الحصول على نتائجنا عن طريق نظريات النقاط الثابتة وتقنية قياس المجموعات الغير متراصة

Résumé

Les équations différentielles fonctionnelles apparaissent dans divers domaines d'applications biologiques, physiques et techniques, et ces équations ont reçu beaucoup d'attention ces dernières années. Dans ce mémoire, nous avons étudie l'existence de solutions aléatoires pour des problèmes avec condition aux limites périodiques et non locales , avec dérivée fractionnaire de Caputo généralisée. Nos résultats seront base sur la théorie du points fixes et par la technique des mesures de non-compacité.

Abstract

Functional differential equations occur in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. In this memoir, we touched on the existence of solutions and random solutions for generalized Caputo periodic and non-local boundary value Problems, with generalized Caputo fractional derivative. Our results will be obtained by means of fixed points theorems and by the technique of measures of noncompactness.

Key words and phrases : Functional differential equations, generalized Caputo, existence, solutions, random solutions, periodic and non-local boundary value Problems, measure of non-compactness, fixed point.

AMS Subject Classification : 34A08, 34K32, 34K37, 34F05.

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Introduction

Fractional differential equations are rapidly expanded in present for applications in modeling and the physical explanation of natural phenomena. Indeed, it also has applications in biophysics, quantum mechanics, wave theory, polymers, and continuum mechanics. The noninteger derivatives of fractional order have been applied successfully to the generalization of fundamental laws of nature, especially in the transport phenomena. We refer the reader to the monographs [4, 5, 6, 22, 26, 32, 33, 35], and the references therein.

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Kuratowski [28], Darbo [17] and was developed by Bana's and Goebel [13] and many researchers in the literature. The applications of the measure of noncompactness (for the weak case, the measure of weak noncompactness developed by De Blasi [18]) can be seen in the wide range of applied mathematics : theory of differential equations (see [9, 10, 11, 14, 15] and references therein).

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random differential equations, used in many on cases, to describe phenomena in biology, physics, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. We refer the reader to the monographs [16, 29, 34], the papers [1, 2, 3, 7, 8] and references therein.

In the following we give an outline of our memoiry organization consisting of three chapters. The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this memoiry.

In Chapter 2, we establish the existence and uniqueness of random solutions for the following fractional boundary value problem :

$$ {}^{C}D^{\nu;\rho}_{0^+}\left(u(t,w) - g(t,u(t,w),w)\right) = f(t,u(t,w),{}^{C}D^{\nu;\rho}_{0^+}u(t,w),w),\ t \in J := [0,2\pi], \quad (1) $$

$$ u(0,w) = u(2\pi,w) \ \ and \ \ \sum_{k=1}^{m} a_k u(\tau_k,w) = d(w), \quad (2) $$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi$, $1 < \nu \leq 2$, ${}^{C}D^{\nu;\rho}_{0^+}$ is the generalized Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \times \Psi \to \mathbb{R}$ and $g : J \times \mathbb{R} \times \Psi \to \mathbb{R}$, are given functions with $g(0,u(0,w),w) = g(2\pi,u(2\pi,w),w) = 0$, $a_k \neq 0$ for all $k = 1,\ldots,m$, and Ψ is the sample space in a probability space and w is a random variable.

We present existence and uniqueness results for the problem (1)-(2) that are founded on the Banach contraction principle and Krasnoselskii fixed point theorem.

In Chapter 3,, we establish, the existence of solutions for generalized Caputo fractional differential equations in Banach space with periodic and non-local boundary value problems.

$$ {}^{C}D_{0^+}^{\nu;\rho}(x(t)-\psi_x(t,x(t)))=f\left(t,x(t),{}^{C}D_{0^+}^{\nu;\rho}x(t)\right), t\in J:=[0,2\pi] \quad (3) $$

$$ x(0)=x(2\pi) \text{ and } \sum_{k=1}^{m}\lambda_k x\left(\tau_k\right)=d, \quad (4) $$

where $0=\tau_0<\tau_1<\tau_2<\cdots<\tau_m<\tau_{m+1}=2\pi, 1<\nu\leq 2, {}^{C}D_{0^+}^{\nu,\rho}$ is the generalized Caputo fractional derivative, $f: J\times E\times E\rightarrow E$ and $\psi_x: J\times E\rightarrow E$, are given functions, λ_k are real constants such that $\sum_{k=1}^{m}\lambda_k\neq 0$. For the sake of simplicity, we assume that $\psi_x\left(\tau_k, x\left(\tau_k\right)\right)=0; k=0,1,\ldots,m+1$.

We present existence results for the problem (3)-(4), is based on the method associated with the technique of measures of non compactness and the fixed point theorems of Darbo and Mönch.

Chapitre 1

Preliminaries

1.1 Notations and Definitions

In this part, we present notations and definitions we will use throughout this work. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ equipped with the norm

$$\|u\|_{[0,2\pi]} = \sup\{|u(t)| : 0 \leq t \leq 2\pi\}.$$

Let $L^1(J)$, be the Banach space of measurable functions $u : J \to E$ which are Bouchner integrable, equipped with the norm

$$\|u\|_{L^1} = \int_a^T \|u(t)\|dt$$

Consider the space $X_b^p(0,2\pi)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions u on J for which $\|u\|_{X_b^p} < \infty$, where the norm is defined by :

$$\|u\|_{X_b^p} = \left(\int_0^{2\pi} |t^b u(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}},\ \ (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 1.1 *(Generalized Riemann-Liouville integral [25, 27]) Let $\upsilon \in \mathbb{R}, b \in \mathbb{R}$ and $u \in X_b^p(0,2\pi)$, the generalized RL fractional integral of order υ is defined by :*

$$(I_{0^+}^{\upsilon,\rho} u)(t) = \frac{\rho^{1-\upsilon}}{\Gamma(\upsilon)} \int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\upsilon-1} u(s)ds, \qquad t > a, \rho > 0 \tag{1.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(\upsilon) = \int_0^\infty s^{\upsilon-1}e^{-s}ds,\ \upsilon > 0.$$

Definition 1.2 *([24]) Let $0 \leq a < t$. The generalized fractional derivative, corresponding to* *the fractional integral (1.1), is defined by :*

$$D_{0^+}^{\upsilon,\rho}u(t) = \frac{\rho^{1-n+\upsilon}}{\Gamma(n-\upsilon)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\upsilon}}u(s)ds \tag{1.2}$$

$$= \delta_\rho^n (I_{0^+}^{n-\upsilon,\rho}u)(t),$$

where $\delta_\rho^n = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Definition 1.3 *([24, 30]) The Caputo-type generalized fractional derivative* ${}^C D_{0^+}^{\upsilon,\rho}$ *is defined by*

$$({}^C D_{0^+}^{\upsilon,\rho}u)(t) = \left(D_{0^+}^{\upsilon,\rho}\left[u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right]\right). \tag{1.3}$$

Lemma 1.4 *([24]) Let* $\upsilon, \rho \in \mathbb{R}^+$*, then*

$$(I_{0^+}^{\upsilon,\rho}{}^C D_{0^+}^{\upsilon,\rho}u)(t) = u(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \tag{1.4}$$

for some $c_k \in \mathbb{R}$*,* $n = [\upsilon] + 1$.

Lemma 1.5 *([36]) If* $x > n$*, then we have*

$$\left[I_{0^+}^{\upsilon,\rho}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right](x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\upsilon)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\upsilon+\alpha-1}. \tag{1.5}$$

1.2 Random operators

We denote the σ-algebra of Borel subsets of $\mathbb{R}$ by $B_\mathbb{R}$. A mapping $N : \Psi \to \mathbb{R}$ is said to be measurable if for any $D \in B_\mathbb{R}$, one has

$$N^{-1}(D) = \{w \in \Psi : N(w) \in D\} \subset \mathcal{A}.$$

Definition 1.6 *A mapping* $N : \Psi \times \mathbb{R} \to \mathbb{R}$ *is called jointly measurable if for any* $D \in B_\mathbb{R}$*, one has*

$$N^{-1}(D) = \{(w,t) \in \Psi \times \mathbb{R} : N(w,t) \in D\} \subset \mathcal{A} \times B_\mathbb{R},$$

where $\mathcal{A} \times B_\mathbb{R}$ *is the product of the* σ*-algebras* $\mathcal{A}$ *defined in* Ψ *and* $B_\mathbb{R}$.

Definition 1.7 *A function* $N : \Psi \times \mathbb{R} \to \mathbb{R}$ *is called jointly measurable if* $N(\cdot, t)$ *is measurable for all* $t \in \mathbb{R}$ *and* $N(w, \cdot)$ *is continuous for all* $w \in \Psi$.

N is called a random operator if $N(w,t)$ is measurable in w for all $t \in \mathbb{R}$, and it expressed as $N(w)t = N(w,t)$. We also say in this situation that $N(w)$ is a random operator on $\mathbb{R}$. $N(w)$ is called continuous (resp. completely continuous, compact and totally bounded) if $N(w,t)$ is continuous (resp. completely continuous, compact and totally bounded) in t for all $w \in \Psi$. The details and the properties of completely continuous random operators in Banach spaces are available in Itoh [23].

Definition 1.8 *([19]) Let $\mathcal{D}(X)$ be the family of all nonempty subsets of X and F be a mapping from Ψ into $\mathcal{D}(X)$. A mapping $N : \{(w,x) : w \in \Psi,\ x \in F(w)\} \to X$ is called random operator with stochastic domain F, if F is measurable (i.e., for all closed $B \subset X$, $\{w \in \Psi, F(w) \cap B \neq \emptyset\}$ is measurable) and for all open $D \subset X$ and all $x \in X$, $\{w \in \Psi : x \in F(w), N(w,x) \in D\}$ is measurable. N will be called continuous if every $N(w)$ is continuous. For a random operator N, a mapping $x : \Psi \to X$ is called a random (stochastic) fixed point of N if for $P-$almost all $w \in \Psi$, $x(w) \in F(w)$ and $N(w)x(w) = x(w)$, and for all open $D \subset X$, $\{w \in \Psi : x(w) \in D\}$ is measurable.*

Definition 1.9 *A function $u : J \times \mathbb{R} \times \Psi \to \mathbb{R}$ is called random Carathéodory if the following conditions are met :*

(i) The map $(s,w) \to u(s,t,w)$ is jointly measurable for all $t \in \mathbb{R}$, and

(ii) The map $t \to u(s,t,w)$ is continuous for all $s \in J$ and $w \in \Psi$.

1.3 Measure of Noncompactness and Auxiliary Results

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness .

Definition 1.10 *([13]) Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty)$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}.$$

Properties 1.11 *The Kuratowski measure of noncompactness satisfies the following properties (for more details see [13]).*

$\mu(B) = 0 \Longleftrightarrow \overline{B}$ *is compact (B is relatively compact).*

$\mu(B) = \mu(\overline{B})$.

$A \subset B \Longrightarrow \mu(A) \leq \mu(B)$.

$\mu(A+B) \leq \mu(A) + \mu(B)$.

$\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.

$\mu(conv\boldsymbol{B}) = \mu(B)$.

Lemma 1.12 *([21]) Let $V \subset C(I, E)$ is a bounded and equicontinuous set, then*

(i) the function $s \longmapsto \mu(V(s))$ is continuous on J, and

$$\mu_c(V) = \max_{s \in J} \mu(V(s)),$$

(ii)

$$\mu\left(\int_a^T x(s)ds : x \in V\right) = \int_a^T \mu(V(s))ds,$$

where

$$V(s) = \{x(s) : x \in V\}, \ s \in J.$$

and μ_c is the Kuratowski measure of noncompactness defined on the bounded sets of $C(J)$.

1.4 Some Fixed Point Theorems

Theorem 1.13 *(Krasnoselskii ,[20]). Let a bounded , convex set M such that $M \neq \emptyset$ and a mapping $Pz = Bz + Az$ such that :*
(i) $Bx + Ay \in M$ for each $x, y \in M$,
(ii) A is continuous and compact,
(iii) B is a contraction.
Then P has a fixed point.

Theorem 1.14 *(Darbo , [17]). Let D be a bounded, closed and convex subset of Banach space X. If the operator $N : D \to D$ is a strict set contraction, i.e there is a constant $0 \leq \lambda < 1$ such that $\mu(N(S)) \leq \lambda\mu(S)$ for any set $S \subset D$ then N has a fixed point in D.*

Theorem 1.15 *(Mönch , [31]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Longrightarrow \mu(V) = 0,$$

holds for every subset V of D, then N has a fixed point.

Chapitre 2

Random Solutions for Generalized Caputo Periodic and Non-Local Boundary Value Problems

2.1 Introduction

In [37], Sh. A. Abd El-Salam studied the existence of at least one solution to the boundary value problem with non-local and periodic conditions given by :

$$\begin{cases} u''(t) = f(t, u(t,w), u'(t)), \text{ for } t \in (0, 2\pi), \\ u(0) = u(2\pi) \ \ and \ \ \sum_{k=1}^{m} a_k u(\tau_k) = u_0, \end{cases}$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi$.

In this chapter, we investigate the existence and uniqueness of random solutions for the following fractional boundary value problem :

$$^{C}D_{0^+}^{\nu;\rho}\left(u(t,w) - g(t, u(t,w), w)\right) = f(t, u(t,w), {}^{C}D_{0^+}^{\nu;\rho}u(t,w), w), \ t \in \Theta := [0, 2\pi], \quad (2.1)$$

$$u(0,w) = u(2\pi, w) \ \ and \ \sum_{k=1}^{m} a_k u(\tau_k, w) = d(w), \quad (2.2)$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi$, $1 < \nu \leq 2$, ${}^{C}D_{0^+}^{\nu;\rho}$ is the generalized Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \times \Psi \to \mathbb{R}$ and $g : J \times \mathbb{R} \times \Psi \to \mathbb{R}$, are given functions with $g(0, u(0,w), w) = g(2\pi, u(2\pi, w), w) = 0$, $a_k \neq 0$ for all $k = 1, \ldots, m$, and Ψ is the sample space in a probability space and w is a random variable.

2.2 Existence of Solutions

Lemma 2.1 *Let* $1 < \nu \leq 2$, $g : J \times \mathbb{R} \times \Psi \to \mathbb{R}$, *with* $\xi(\tau_k, u(\tau_k, w), w) = 0, \quad for \quad k = 0, \ldots, m+1$, $a_k \neq 0$ *for all* $k = 1, \ldots, m$, *and* $h, \xi : J \times \Psi \to \mathbb{R}$ *be measurable functions. Then the linear problem*

$$ {}^{C}D^{\nu,\rho}_{0^+}\left(u(t,w) - \xi(t,w)\right) = h(t,w), \text{ for a.e. } t \in J := [0, 2\pi],\ w \in \Psi, \tag{2.3}$$

$$ u(0,w) = u(2\pi, w) \ \ and \ \sum_{k=1}^{m} a_k u(\tau_k, w) = d(w), \tag{2.4}$$

has a unique random solution, which is given by

$$\begin{aligned} u(t,w) &= \xi(t,w) + \frac{d(w) - \sum_{k=1}^{m} a_k \xi(\tau_k, w)}{\sum_{k=1}^{m} a_k} \\ &+ \left[\frac{\sum_{k=1}^{m} a_k \tau_k^{\rho}}{\sum_{k=1}^{m} a_k} - t^{\rho}\right] \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds \\ &- \frac{1}{\Gamma(\nu)\sum_{k=1}^{m} a_k} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds \\ &+ \frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds. \end{aligned}$$

proof 1 *From Lemma 2.1, we have*

$$ u(t,w) - \xi(t,w) =^{\rho} I^{\nu}_{0^+} h(s,w) + c_0 + c_1 \left(\frac{t^{\rho}}{\rho}\right), \tag{2.5}$$

where c_1 *and* $c_2 \in \mathbb{R}$.
Then

$$c_0 = u(0,w) = u(2\pi, w) = c_0 + c_1 \frac{(2\pi)^{\rho}}{\rho} + \tfrac{1}{\Gamma(\nu)} \int_a^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds,$$

and

$$\begin{aligned} d(w) &= \sum_{k=1}^{m} a_k \xi(\tau_k, w) + \sum_{k=1}^{m} a_k u(\tau_k) \\ &= c_0 \sum_{k=1}^{m} a_k + c_1 \sum_{k=1}^{m} a_k \frac{\tau_k^{\rho}}{\rho} + \frac{1}{\Gamma(\nu)} \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds, \end{aligned}$$

Therefore

$$ c_1 = \frac{-\rho}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s,w)ds,$$

$$
\begin{aligned}
c_0 &= \frac{d(w)-\sum_{k=1}^m a_k\xi(\tau_k,w)}{\sum_{k=1}^m a_k}+\frac{\sum_{k=1}^m a_k\tau_k^\rho}{(2\pi)^\rho\Gamma(\nu)\sum_{k=1}^m a_k}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\\
&- \frac{1}{\Gamma(\nu)\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\int_0^{\tau_k}\left(\frac{\tau_k^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds.
\end{aligned}
$$

Substitute the value of c_0 and c_1 in (2.5), we get equation (2.5).

$$
\begin{aligned}
u(t,w) &= \xi(t,w)+\frac{d(w)-\sum_{k=1}^m a_k\xi(\tau_k,w)}{\sum_{k=1}^m a_k}\\
&+ \left[\frac{\sum_{k=1}^m a_k\tau_k^\rho}{\sum_{k=1}^m a_k}-t^\rho\right]\frac{1}{(2\pi)^\rho\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\\
&- \frac{1}{\Gamma(\nu)\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\int_0^{\tau_k}\left(\frac{\tau_k^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\\
&+ \frac{1}{\Gamma(\nu)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds.
\end{aligned}
$$

Hence, the proof is complete.

Lemma 2.2 *Let $f : J\times \mathbb{R}\times\mathbb{R}\times\Psi \longrightarrow \mathbb{R}$ be a random Carathéodory function. A function $u(\cdot,w)\in C(J,\mathbb{R})$ is random solution of the non-local and periodic problem* (2.1)-(2.2) *if and only if u satisfies the integral equation*

$$
\begin{aligned}
u(t,w) &= g(t,u(t,w),w)+\frac{d(w)-\sum_{k=1}^m a_k g(\tau_k,u(\tau_k,w),w)}{\sum_{k=1}^m a_k}\\
&+ \left[\frac{\sum_{k=1}^m a_k\tau_k^\rho}{\sum_{k=1}^m a_k}-t^\rho\right]\frac{1}{(2\pi)^\rho\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\\
&- \frac{1}{\Gamma(\nu)\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\int_0^{\tau_k}\left(\frac{\tau_k^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\\
&+ \frac{1}{\Gamma(\nu)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds,
\end{aligned}
$$

where $h\in C(J,\mathbb{R})$ satisfies the functional equation

$$
h(t,w)=f(t,u(t,w),{}^C D_{0^+}^{\nu,\rho}u(t,w),w).
$$

The following hypotheses will be used in the sequel :

(H_1) The functions $f : J\times\mathbb{R}\times\mathbb{R}\times\Psi\longrightarrow\mathbb{R}$ and $g : J\times\mathbb{R}\times\Psi\longrightarrow\mathbb{R}$ are random Caratheodory.

(H_2) There exist measurable and essentially bounded functions $p, q, b : \Psi \to (0, \infty)$ such that

$$|f(t, u_1, v_1, w) - f(t, u_2, v_2, w)| \leq p(w)|u_1 - u_2| + q(w)|v_1 - v_2|,$$

and

$$|g(t, u_1, w) - g(t, u_2, w)| \leq b(w)|u_1 - u_2|,$$

for $t \in J$, $w \in \Psi$ and each $u_i, v_i \in \mathbb{R}$, $i = 1, 2$.

Now, we state and prove our existence result for problem (2.1)-(2.2) based on the Banach contraction principle [20].

Theorem 2.3 *Assume (H_1) and (H_2) hold. If*

$$2b(w)+\left(\frac{|\sum_{k=1}^{m} a_k \tau_k^{\rho}|}{|\sum_{k=1}^{m} a_k|} + 2(2\pi)^{\rho}\right) \frac{p(w)(2\pi)^{\rho(\nu-1)}}{(1-q(w))\rho^{\nu}\Gamma(\nu+1)} + \frac{p(w)\left|\sum_{k=1}^{m} a_k \left(\frac{\tau_k^{\rho}}{\rho}\right)^{\nu}\right|}{(1-q(w))\Gamma(\nu+1)\left|\sum_{k=1}^{m} a_k\right|} < 1, \tag{2.6}$$

then the problem (2.1)-(2.2) has a unique solution.

proof 2 *Let the operator $T : C(J, \mathbb{R}) \times \Psi \longmapsto C(J, \mathbb{R})$ defined by*

$$\begin{aligned}(Tu)(t, w) = &\frac{d(w) - \sum_{k=1}^{m} a_k g(\tau_k, u(\tau_k, w), w)}{\sum_{k=1}^{m} a_k} \\ &+ \left[\frac{\sum_{k=1}^{m} a_k \tau_k^{\rho}}{\sum_{k=1}^{m} a_k} - t^{\rho}\right] \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s, w) ds \\ &+ g(t, u(t, w), w) - \frac{1}{\Gamma(\nu)\sum_{k=1}^{m} a_k} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s, w) ds \\ &+ \frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1} h(s, w) ds,\end{aligned} \tag{2.7}$$

where $h \in C(J, \mathbb{R})$ such that

$$h(t, w) = f(t, u(t), h(t, w), w).$$

By Lemma 2.2 it is clear that the fixed points of T are random solutions (2.1)-(2.2) .

Let $u_1(\cdot,w)$ and $u_2(\cdot,w) \in \Psi$. Then for $t \in J$, we have

$$\begin{aligned}
&|(Tu_1)(t,w) - (Tu_2)(t,w)| \le |g(t,u_1(t,w),w) - g(t,u_2(t,w),w)| \\
&+ \frac{|\sum_{k=1}^m a_k| \, |g(\tau_k, u_1(\tau_k,w),w) - g(\tau_k, u_2(\tau_k,w),w)|}{|\sum_{k=1}^m a_k|} \\
&+ \left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + t^\rho \right) \frac{1}{(2\pi)^\rho \Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w) - h_{u_2}(s,w)| ds \\
&+ \frac{1}{\Gamma(\nu) \, |\sum_{k=1}^m a_k|} \left| \sum_{k=1}^m a_k \right| \int_0^{\tau_k} \left(\frac{\tau_k^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w) - h_{u_2}(s,w)| ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w) - h_{u_2}(s,w)| ds.
\end{aligned} \tag{2.8}$$

By (H_2), we have

$$\begin{aligned}
|h_{u_1}(t,w) - h_{u_1}(t,w)| &= |f(t,u_1(t,w),h_{u_1}(t,w),w) - f(t,u_2(t,w),h_{u_2}(t,w),w)| \\
&\le p(w)|u_1(t,w) - u_2(t,w)| + q(w)|h_{u_1}(t,w) - h_{u_2}(t,w)|.
\end{aligned}$$

Then

$$|h_{u_1}(t,w) - h_{u_1}(t,w)| \le \frac{p(w)}{1-q(w)} |u_1(t,w) - u_2(t,w)|.$$

Therefore, for each $t \in J$, we have

$$\begin{aligned}
|(Tu_1)(t,w) - (Tu_2)(t,w)| \le& \left[2b(w) + \left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho \right) \frac{p(w)(2\pi)^{\rho(\nu-1)}}{(1-q(w))\rho^\nu \Gamma(\nu+1)} \right. \\
&+ \left. \frac{p(w) \left| \sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho} \right)^\nu \right|}{(1-q(w))\Gamma(\nu+1) \, |\sum_{k=1}^m a_k|} \right] \|u_1(\cdot,w) - u_2(\cdot,w)\|_{[0,2\pi]}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|(Tu_1)(\cdot,w) - (Tu_2)(\cdot,w)\|_{[0,2\pi]} \\
\le& \left[b(w) + \left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho \right) \frac{p(w)(2\pi)^{\rho(\nu-1)}}{(1-q(w))\rho^\nu \Gamma(\nu+1)} \right. \\
+& \left. \frac{p(w) \left| \sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho} \right)^\nu \right|}{(1-q(w))\Gamma(\nu+1) \, |\sum_{k=1}^m a_k|} \right] \|u_1(\cdot,w) - u_2(\cdot,w)\|_{[0,2\pi]}.
\end{aligned}$$

Hence, by the Banach contraction principle, T has a unique fixed point which is a unique random solution of the problem (2.1)-(2.2).

Our second result is based on Krasnoselskii fixed point theorem [20].

Theorem 2.4 *Assume* (H_1) *and* (H_2) *hold. If*

$$G := b(w) + \left[\left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho \right) \frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu \Gamma(\nu+1)} + \frac{\left|\sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|} \right] \frac{p(w)}{1-q(w)} < 1. \tag{2.9}$$

Then problem (2.1)-(2.2) has at least one random solution defined on J.

proof 3 *Consider the set*

$$B_{\eta^*(w)} = \{y \in \Psi : ||y(\cdot, w)||_{[0,2\pi]} \leq \eta^*(w)\},$$

where

$$\eta^*(w) \geq \frac{g^*(w)}{1-\mathcal{M}} + \left[\left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho \right) \frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu \Gamma(\nu+1)} \right.$$
$$\left. + \frac{\left|\sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|} \right] \frac{f^*(w)}{(1-\mathcal{M})(1-q(w))},$$

and $f^*(w) = ess\sup_{t\in\Theta} |f(t,0,0,w)|$, $g^*(w) = ess\sup_{t\in\Theta} |g(t,0,w)|$,

$$\mathcal{M} = b(w) + \left[\left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho \right) \frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu \Gamma(\nu+1)} + \frac{\left|\sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|} \right] \frac{p(w)}{1-q(w)}.$$

We define the operators P *and* Q *on* $B_{\eta^*(w)}$ *by*

$$(Pu)(t,w) = \frac{d(w) - \sum_{k=1}^m a_k g(\tau_k, u(\tau_k, w), w)}{\sum_{k=1}^m a_k}$$
$$+ \left[\frac{\sum_{k=1}^m a_k \tau_k^\rho}{\sum_{k=1}^m a_k} - t^\rho \right] \frac{1}{(2\pi)^\rho \Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} h(s,w) ds$$
$$+ g(t, u(t,w), w) - \frac{1}{\Gamma(\nu) \sum_{k=1}^m a_k} \sum_{k=1}^m a_k \int_0^{\tau_k} \left(\frac{\tau_k^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} h(s,w) ds, \tag{2.10}$$

$$(Qu)(t,w) = \frac{1}{\Gamma(\nu)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} h(s,w) ds, \tag{2.11}$$

Then the fractional integral equation (2.7) *can be written as operator equation*

$$(Tu)(t,w) = (Pu)(t,w) + (Qu)(t,w), \quad u(\cdot, w) \in \Psi.$$

The proof will be given in several steps.

Step 1 : *We prove that* $Pu_1(\cdot,w) + Qu_2(\cdot,w) \in B_{\eta^*(w)}$ *for any* $u_1(\cdot,w), u_2(\cdot,w) \in B_{\eta^*(w)}$. *For*

$t \in J$, we have

$$
\begin{aligned}
&|(Pu_1)(t,w)| \\
&\leq \left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + t^\rho\right) \frac{1}{(2\pi)^\rho \Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w)| ds \\
&+ |g(t, u_1(t,w))| + \frac{1}{\Gamma(\nu) |\sum_{k=1}^m a_k|} \left|\sum_{k=1}^m a_k\right| \int_0^{\tau_k} \left(\frac{\tau_k^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w)| ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} |h_{u_1}(s,w)| ds.
\end{aligned}
$$

By (H_2) we have

$$
\begin{aligned}
|h_{u_1}(t,w)| &= |f(t, u_1(t,w), h_{u_1}(t,w), w) - f(t,0,0,w) + f(t,0,0,w)| \\
&\leq |f(t, u_1(t,w), h_{u_1}(t,w), w) - f(t,0,0,w)| + |f(t,0,0,w)| \\
&\leq p(w)|u_1(t,w)| + q(w)|h_{u_1}(t,w)| + f^*(w).
\end{aligned}
$$

Then, we get

$$
|h_{u_1}(t,w)| \leq \frac{p(w)\eta^*(w) + f^*(w)}{1 - q(w)}. \tag{2.12}
$$

And, we have for each $t \in J$,

$$
\begin{aligned}
|g(t, u_1(t,w), w)| &= |g(t, u_1(t,w), w) - g(t,0,w) + g(t,0,w)| \\
&\leq |g(t, u_1(t,w), w) - g(t,0,w)| + |g(t,0,w)| \\
&\leq b(w)|u_1(t,w)| + g^*(w).
\end{aligned}
$$

Then for each $t \in J$, we obtain

$$
|g(t, u_1(t,w), w)| \leq b(w)\eta^*(w) + g^*(w). \tag{2.13}
$$

Thus, by (2.12) and (2.13), we have

$$
\begin{aligned}
|(Pu_1)(t,w)| \leq\ & b(w)\eta^*(w) + g^*(w) + \left[\left(\frac{|\sum_{k=1}^m a_k \tau_k^\rho|}{|\sum_{k=1}^m a_k|} + (2\pi)^\rho\right) \frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu \Gamma(\nu+1)}\right. \\
+\ & \left.\frac{\left|\sum_{k=1}^m a_k \left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1) |\sum_{k=1}^m a_k|}\right] \frac{p(w)\eta^*(w) + f^*(w)}{1 - q(w)}.
\end{aligned}
$$

Thus

$$\|(Pu_1)(\cdot,w)\|_{[0,2\pi]} \leq b(w)\eta^*(w) + g^*(w) + \left[\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|} + (2\pi)^\rho\right)\frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu\Gamma(\nu+1)} + \frac{\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}. \tag{2.14}$$

Now, for operator Q*, we have for* $t \in J$

$$|(Qu_2)(t,w)| \leq \frac{1}{\Gamma(\nu)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}|h_{u_2}(s,w)|ds.$$

Therefore

$$|(Qu_2)(t,w)| \leq \left[\frac{(2\pi)^{\rho\nu}}{\rho^\nu\Gamma(\nu+1)}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}.$$

Thus

$$\|(Qu_2)(\cdot,w)\|_{[0,2\pi]} \leq \left[\frac{(2\pi)^{\rho\nu}}{\rho^\nu\Gamma(\nu+1)}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}. \tag{2.15}$$

Linking (2.14) *and* (2.15) *for every* $u_1(\cdot,w), u_2(\cdot,w) \in B_{\eta^*(w)}$ *we obtain*

$$\begin{aligned}
&\|(Pu_1)(\cdot,w) + (Qu_2)(\cdot,w)\|_{[0,2\pi]} \\
&\leq b(w)\eta^*(w) + g^*(w) + \left[\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu\Gamma(\nu+1)} \right.\\
&\left. + \frac{\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}.
\end{aligned}$$

Since

$$\eta^* \geq \frac{g^*(w) + \left[\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu\Gamma(\nu+1)} + \frac{\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\right]\frac{f^*(w)}{1-q(w)}}{1 - b(w) - \left[\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|} + 2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\nu-1)}}{\rho^\nu\Gamma(\nu+1)} + \frac{\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\right]\frac{p(w)}{1-q(w)}},$$

we have

$$\|(Pu_1)(\cdot,w) + (Qu_2)(\cdot,w)\|_{[0,2\pi]} \leq \eta^*(w).$$

which infers that $Pu_1(\cdot,w) + Qu_2(\cdot,w) \in B_{\eta^*(w)}$

Step 2 : *P is a contraction.*
Let $u_1(\cdot,w), u_2(\cdot,w)\in\Psi$. Then for $t\in J$, we have

$$
\begin{aligned}
&|(Pu_1)(t,w)-(Pu_2)(t,w)| \le |g(t,u_1(t,w),w)-g(t,u_2(t,w),w)|\\
&+\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|}+t^\rho\right)\frac{1}{(2\pi)^\rho\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}|h_{u_1}(s,w)-h_{u_2}(s,w)|ds\\
&+\frac{1}{\Gamma(\nu)\,|\sum_{k=1}^m a_k|}\left|\sum_{k=1}^m a_k\right|\int_0^{\tau_k}\left(\frac{\tau_k^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}|h_{u_1}(s,w)-h_{u_2}(s,w)|ds.
\end{aligned}
$$

Therefore, for each $t\in J$, we have

$$
\begin{aligned}
&\|(Pu_1)(\cdot,w)-(Pu_2)(\cdot,w)\|_{[0,2\pi]}\\
\le\ &\Bigg[b(w)+\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|}+(2\pi)^\rho\right)\frac{p(w)(2\pi)^{\rho(\nu-1)}}{(1-q(w))\rho^\nu\Gamma(\nu+1)}\\
+\ &\frac{p(w)\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{(1-q(w))\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\Bigg]\|u_1(\cdot,w)-u_2(\cdot,w)\|_{[0,2\pi]}.
\end{aligned}
$$

By (2.9), the operator P is a contraction.
Step 3 : *Q is compact and continuous.*
The continuity of Q follows from the continuity of f. Next we prove that Q is uniformly bounded on $B_{\eta^*}(w)$. Let any $u_2(\cdot,w)\in B_{\eta^*}(w)$. Then by (2.15) we have

$$
\|(Qu_2)(\cdot,w)\|_{[0,2\pi]}\le\left[\frac{(2\pi)^{\rho\nu}}{\rho^\nu\Gamma(\nu+1)}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}.
$$

This means that Q is uniformly bounded on $B_{\eta^*}(w)$. Next, we show that $QB_{\eta^*}(w)$ is equiconti-

nuous. Let any $u(\cdot,w)\in B_{\eta^*(w)}$ *and* $0<\tau_1<\tau_2\le 2\pi$*. Then*

$$
\begin{aligned}
&|(Qu)(\tau_2,w)-(Qu)(\tau_1,w)|\\
&\le \left|\frac{1}{\Gamma(\nu)}\int_0^{\tau_2}\left(\frac{\tau_2^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds-\frac{1}{\Gamma(\nu)}\int_0^{\tau_1}\left(\frac{\tau_1^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}h(s,w)ds\right|\\
&\le \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\frac{\tau_2^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}|h(s,w)|ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^{\tau_1}\left|\left(\frac{\tau_2^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}-\left(\frac{\tau_1^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}\right||h(s,w)|ds\\
&\le\left[\frac{(\tau_2^\rho-\tau_1^\rho)^\nu}{\rho^\nu\Gamma(\nu+1)}\right]\frac{p(w)\eta^*(w)+f^*(w)}{1-q(w)}\\
&+\frac{p(w)\eta^*(w)+f^*(w)}{\Gamma(\nu)(1-q(w))}\int_0^{\tau_1}\left|\left(\frac{\tau_2^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}-\left(\frac{\tau_1^\rho-s^\rho}{\rho}\right)^{\nu-1}s^{\rho-1}\right|ds
\end{aligned}
$$

Note that

$$|(Qu)(\tau_2,w)-(Qu)(\tau_1,w)|\to 0\quad as\quad \tau_2\to\tau_1.$$

This shows that Q *is equicontinuous on* Θ*. Therefore* Q *is relatively compact on* $B_{\eta^*(w)}$*. By Arzela-Ascoli Theorem* Q *is compact on* $B_{\eta^*(w)}$*.*
As a consequence of Krasnoselskii fixed point theorem, we deduce that T *has at least a fixed point which is a solution the problem (2.1)-(2.2).*

2.3 Examples

Example 2.5 *We equip the space* $\mathbb{R}_-^*:=(-\infty,0)$ *with the usual* σ*-algebra consisting of Lebesgue measurable subsets of* $\mathbb{R}_-^*$*. Consider the boundary value problem of generalized Caputo fractional differential equation :*

$$
\begin{cases}
{}^CD_{0^+}^{\frac{3}{2},\rho}(u(t,w)-g(t,u(t,w),w))=\dfrac{\sin(t)(u+1)}{100(w^2+1)\left(\left|{}^CD_{0^+}^{\frac{3}{2},\rho}v)(t,w)\right|+1\right)}, & t\in[0,2\pi],\\
u(0,w)=u(2\pi,w),\ \sum_{i=1}^{2}\frac{i}{3}u(\frac{i\pi}{3})=d(w).
\end{cases}
\tag{2.16}
$$

Set

$$f(t,u(t,w),({}^CD_{0^+}^{\frac{3}{2},\rho}u)(t,w),w)=\frac{\sin(t)(u(t,w)+1)}{100(w^2+1)\left(\left|{}^CD_{0^+}^{\frac{3}{2},\rho}u)(t,w)\right|+1\right)},\ \ t\in[0,2\pi],u\in\mathbb{R},$$

and

$$g(t,u(t,w),w)=\frac{(\sin(t)^2-\frac{\sqrt{3}}{2}\sin(t))u(t,w)}{1000(w^2+1)},\ \ t\in[0,2\pi],u\in\mathbb{R},i=1,2,$$

and $g(2\pi,u(2\pi,w),w)=g(0,u(0,w),w)=g(\tau_i,u(\tau_i,w),w)=0$, $i=1,2$, $\nu=\frac{3}{2},\rho=\frac{1}{5},\tau_i=\frac{i\pi}{3}$.

For each $u,\bar{u},v,\bar{v}\in\mathbb{R}$ *and* $t\in[0,2\pi]$, *we have*

$$\begin{aligned}|f(t,u,v,w)-f(t,\bar{u},\bar{v},w)| &\leq \left|\frac{\sin(t)(u+1)}{100(w^2+1)\,(|v|+1)}\right.\\ &- \left.\frac{\sin(t)(\bar{u}+1)}{100(w^2+1)\,(|\bar{v}|+1)}\right|\\ &\leq \frac{\sin(t)}{100(w^2+1)}|u-\bar{u}|,\end{aligned}$$

$$|g(t,u,w)-g(t,\bar{u},w)| \leq \frac{2+\sqrt{3}}{2000(w^2+1)}|u-\bar{u}|.$$

Therefore, (H_2) *is verified with* $p(w)=\dfrac{1}{100(w^2+1)},\ b(w)=\dfrac{2+\sqrt{3}}{2000(w^2+1)},\ q(w)=0$

The condition

$$\begin{aligned}&b(w)+\left(\frac{|\sum_{k=1}^m a_k\tau_k^\rho|}{|\sum_{k=1}^m a_k|}+2(2\pi)^\rho\right)\frac{p(w)(2\pi)^{\rho(\nu-1)}}{(1-q(w))\rho^\nu\Gamma(\nu+1)}+\frac{p(w)\left|\sum_{k=1}^m a_k\left(\frac{\tau_k^\rho}{\rho}\right)^\nu\right|}{(1-q(w))\Gamma(\nu+1)\,|\sum_{k=1}^m a_k|}\\ =\ &\frac{2+\sqrt{3}}{2000(w^2+1)}+\left(\frac{\sum_{k=1}^2\frac{k}{3}(\frac{k\pi}{3})^{\frac{1}{5}}}{\sum_{k=1}^2\frac{k}{3}}+2(2\pi)^{\frac{1}{5}}\right)\frac{(2\pi)^{\frac{1}{5}(\frac{2}{2}-1)}}{100(w^2+1)(\frac{1}{5})^{\frac{3}{2}}\Gamma(\frac{3}{2}+1)}\\ +\ &\frac{\sum_{k=1}^2\frac{k}{3}\left(\frac{(\frac{k\pi}{3})^{\frac{1}{5}}}{\frac{1}{5}}\right)^{\frac{3}{2}}}{100(w^2+1)\Gamma(\frac{3}{2}+1)\sum_{k=1}^2\frac{k}{3}}\approx\frac{0.43478131}{w^2+1}<1,\end{aligned}$$

is satisfied with $\nu=\frac{3}{2}$. *Hence all conditions of Theorem 2.3 are satisfied, it follows that the problem (2.16) admit a unique random solution.*

Example 2.6 *Consider the following problem :*

$$\begin{cases}{}^C D_{0^+}^{\frac{4}{3},\rho}(u(t,w)-g(t,u(t,w),w))=f(t,u(t,w),({}^C D_{0^+}^{\frac{4}{3},\rho}u)(t,w),w),\ \ t\in[0,2\pi],\\ \\ u(0,w)=u(2\pi,w),\ \ \sum_{i=1}^2 2iu\left(\tau_i\right)=d(w),\end{cases}\tag{2.17}$$

where

$$f(t,u,\bar{u},w)=\frac{|u|+|\bar{u}|+3}{411e^{t}(1+|u|+|\bar{u}|)(|w|+2)},\ \ t\in\Theta,\ u,\bar{u}\in \mathbb{R},$$

and

$$g(t,u,w)=\frac{(\cos(t)^{3}-\frac{\cos(t)}{2})|u|}{300(|w|+2)},\ \ t\in[0,2\pi],u\in \mathbb{R},i=1,2,$$

and $g(2\pi,u(2\pi,w),w)=g(0,u(0,w),w)=g(\tau_i,u(\tau_i,w),w)=0,\ \ i=1,2,$ $\nu=\frac{4}{3}$, $\rho=1$, $\tau_1=\frac{\pi}{4}$ *and* $\tau_2=\frac{7\pi}{4}$.

All conditions of Theorem 2.4 are satisfied with

$$p(w)=q(w)=\frac{1}{411(|w|+2)},\ b(w)=\frac{1}{200(|w|+2)},$$

and

$$\begin{aligned}\mathcal{M}&=\frac{1}{200(|w|+2)}+\left[\left(\frac{\frac{\pi}{2}+7\pi}{6}+4\pi\right)\frac{(2\pi)^{\frac{1}{3}}}{\Gamma(\frac{7}{3})}+\frac{2(\frac{\pi}{4})^{\frac{4}{3}}+4(\frac{7\pi}{4})^{\frac{4}{3}}}{6\Gamma(\frac{7}{3})}\right]\frac{1}{411|w|+821}\\&\approx\frac{1}{200(|w|+2)}+\frac{31.1975967219243}{411|w|+821}\\&<1.\end{aligned}$$

Then, it follows that the problem (2.17) admit at least one random solution.

Chapitre 3

Existence of solutions for generalized Caputo fractional differential equations with periodic and non-local boundary value problems in Banach Spaces

3.1 Introduction

In this chapter , we establish, the existence of solutions for generalized Caputo fractional differential equations in Banach space with periodic and non-local boundary value problems.

$$ {}^{C}D_{0^+}^{\nu;\rho}(x(t)-\psi_x(t,x(t))) = f\left(t, x(t), {}^{C}D_{0^+}^{\nu;\rho}x(t)\right), t \in J := [0, 2\pi] \tag{3.1} $$

$$ x(0) = x(2\pi) \text{ and } \sum_{k=1}^{m} \lambda_k x(\tau_k) = d, \tag{3.2} $$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi, 1 < \nu \leq 2, {}^{C}D_{0^+}^{\nu,\rho}$ is the generalized Caputo fractional derivative, $f : J \times E \times E \to E$ and $\psi_x : J \times E \to E$, are given functions, λ_k are real constants such that $\sum_{k=1}^{m} \lambda_k \neq 0$. For the sake of simplicity, we assume that $\psi_x(\tau_k, x(\tau_k)) = 0; k = 0, 1, \ldots, m+1$.

3.2 Existence of Solutions

Definition 3.1 *A solution of problem (3.1) and (3.2) is a function $x(t) \in C(J, \mathbb{R})$ which satisfies the Equation (3.1) and the conditions (3.2).*

Lemma 3.2 *Let $1 < \nu \leq 2$ and $\kappa_x, \xi : J \to E$ be measurable functions, such that $\xi(\tau_k) = 0$; $k = 0, 1, \ldots, m+1$. Then, the linear problem*

$$
\begin{gathered}
{}^{C}D_{0^+}^{\nu,\rho}(x(t) - \xi(t)) = \kappa_x(t), \text{ for a.e. } t \in J, \\
x(0) = x(2\pi) \text{ and } \sum_{k=1}^{m} \lambda_k x(\tau_k) = d,
\end{gathered}
$$

has a solution given by

$$
\begin{aligned}
x(t) = \xi(t) &+ \frac{d}{\sum_{k=1}^{m} \lambda_k} \\
&+ \left[\frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right] \frac{1}{(2\pi)^{\rho} \Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds \\
&- \frac{1}{\Gamma(\nu) \sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^t \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds.
\end{aligned}
$$

Lemma 3.3 *Let $f : J \times E \times E \longrightarrow E$ be a Carathéodory function. A function $x(t) \in C(J, E)$ is a solution of the non-local and periodic problems (3.1) and (3.2) if, and only if, x satisfies the integral equation*

$$
\begin{aligned}
x(t) = \psi_x(t) &+ \frac{d}{\sum_{k=1}^{m} \lambda_k} \\
&+ \left[\frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right] \frac{1}{(2\pi)^{\rho} \Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds \\
&- \frac{1}{\Gamma(\nu) \sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^t \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa_x(s) ds,
\end{aligned}
$$

where $\kappa_x, \psi_x \in C(J, E)$ satisfies the functional equation

$$
\kappa_x(t) = f(\tau, x(t), \kappa_x(t)) \text{ and } \psi_x(t) = \psi_x(t, x(t)).
$$

The following hypotheses will be used in the sequel :

(H_1) The functions $f : J \times E \times E \longrightarrow E$ and $\psi_x : J \times E \longrightarrow E$ are Carathéodory.

(H_2) There exist measurable and essentially bounded functions $p, q, b : J \to L^{\infty}(\mathbb{R}^+)$, such that

$$|f(t,x_1,x_1') - f(t,x_2,x_2')| \leq p(t)|x_1 - x_2| + q(t)|x_1' - x_2'|,$$

and

$$|\psi_x(t,x_1) - \psi_x(t,x_2)| \leq b(t)|x_1 - x_2|,$$

for $t \in J$ and each $x_i, x_i' \in E; i = 1, 2$, with

$$p = \text{ ess } \sup_{t \in J} |p(t)|, \quad q = \text{ ess } \sup_{t \in J} |q(t)| < 1,$$

and

$$b = \text{ ess } \sup_{t \in J} |b(t)|.$$

(H_3) For each bounded set D_R in $\mathcal{C}$, the set $\{t \longrightarrow \psi(t, x(t)) : x \in D_N\}$ is equicontinuous in $C(J, \mathbb{R})$.

(H_4) for each bounded set $B_i \subset \mathcal{C}, i = 1, 2, 3$, and for each $t \in J$, ther exist a constant

$$p, q, b \in \mathbb{R}$$

$$\mu(f(t, B_1, B_2)) \leq p\mu(B_1) + q\mu(B_2),$$

and

$$\mu(\psi_x(t, B_3)) \leq b\mu(B_3),$$

for any bounded sets $B_1, B_2, B_3 \subseteq E$ and for each $t \in J$.

We are now in a position to state and prove our existence result for the problem (3.1) and (3.2) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Theorem 3.4 *Assume (H_1),(H_2) hold. If*

$$\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b < 1, \tag{3.3}$$

then the Problem (3.1) and (3.2) has at least one solution on J.

proof 4 *Transform the problem (3.1) and (3.2) into a fixed point problem. Define the operator $N : C(J, E) \to C(J, E)$ by :*

$$
\begin{aligned}
(Nx)(t) = {} & \frac{d}{\sum_{k=1}^{m}\lambda_k} + \left[\frac{\sum_{k=1}^{m}\lambda_k\tau_k^{\rho}}{\sum_{k=1}^{m}\lambda_k} - t^{\rho}\right]\frac{1}{(2\pi)^{\rho}\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\kappa_x(s)ds \\
& + \psi_x(t) - \frac{1}{\Gamma(\nu)\sum_{k=1}^{m}\lambda_k}\sum_{k=1}^{m}\lambda_k\int_0^{\tau_k}\left(\frac{\tau_k^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\kappa_x(s)ds \\
& + \frac{1}{\Gamma(\nu)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\kappa_x(s)ds,
\end{aligned}
$$

where $\kappa_x, \psi_x \in C(J, E)$ satisfies the functional equation

$$\kappa_x(t) = f(\tau, x(t), \kappa_x(t)) \text{ and } \psi_x(t) = \psi_x(t, x(t)).$$

According to Lemma 3.3, the fixed points of N are solutions to problem (3.1) and (3.2). We shall show that N satisfies the assumption of Darbo's fixed point Theorem. The proof will be given in several claims.

Claim 1 : N **is continuous**.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J, E)$. Then for each $t \in J$,

$$
\begin{aligned}
& \|(Nx_n)(t) - (Nx)(t)\| \leq \|\psi_{x_n}(\tau) - \psi_x(t)\| \\
& + \left(\frac{|\sum_{k=1}^{m}\lambda_k\tau_k^{\rho}|}{|\sum_{k=1}^{m}\lambda_k|} + t^{\rho}\right)\frac{1}{(2\pi)^{\rho}\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\|\kappa_{x_n}(s)-\kappa_x(s)\|\,ds \\
& + \frac{1}{\Gamma(\nu)|\sum_{k=1}^{m}\lambda_k|}\sum_{k=1}^{m}|\lambda_k|\int_0^{\tau_k}\left(\frac{\tau_k^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\|\kappa_{x_n}(s)-\kappa_x(s)\|\,ds \\
& + \frac{1}{\Gamma(\nu)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\|\kappa_{x_n}(s)-\kappa_x(s)\|\,ds.
\end{aligned}
$$

Where ,

$$\kappa_{x_n}(s) = f(s, x_n(s), \kappa_{x_n}(s)) \quad , \quad \psi_{x_n}(t) = \psi_{x_n}(t, x_n(t)),$$

and,

$$\kappa_x(s) = f(s, x(s), \kappa_x(s)) \;\; , \;\; \psi_x(t) = \psi_x(t, x(t)).$$

By (H_2) we have,

$$
\begin{aligned}
\|\kappa_{x_n}(s) - \kappa_x(s)\| &= \|f(s, x_n(s), \kappa_{x_n}(s)) - f(s, x(s), \kappa_x(s))\| \\
&\leq p(s)\|x_n(s) - x(s)\| + q(s)\|\kappa_{x_n}(s) - \kappa_x(s)\| \\
&\leq p\|x_n(s) - x(s)\| + q\|\kappa_{x_n}(s) - \kappa_x(s)\|.
\end{aligned}
$$

Then,

$$\|\kappa_{x_n}(s)-\kappa_x(s)\| \leq \frac{p}{1-q}\,\|x_n(s)-x(s)\|\,.$$

And,

$$\begin{aligned}\|\psi_{x_n}(t)-\psi_x(t)\| &= \|\psi_{x_n}(t,x_n(t))-\psi_x(t,x(t))\| \\ &\leq b(t)\,\|x_n(t)-x(t)\| \\ &\leq b\,\|x_n(t)-x(t)\|\,.\end{aligned}$$

Since $x_n \to x$, then we get $\kappa_{x_n}(s) \to \kappa_x(s)$ and $\psi_{x_n}(t) \to \psi_x(t)$ as $n \to \infty$, for each $s,t \in J$. Let $\eta > 0$ be such that, for each $s \in J$, we have $\|\kappa_{x_n}(s)\| \leq \eta$ and $\|\kappa_x(s)\| \leq \eta$.

Then we have,

$$\begin{aligned}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}\,\|\kappa_{x_n}(s)-\kappa_x(s)\| &\leq \left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}\left[\|\kappa_{x_n}(s)\|+\|\kappa_x(s)\|\right] \\ &\leq 2\eta\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}.\end{aligned}$$

For each $\nu \in J$, the function $s \mapsto 2\eta\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}$ is integrable on $[0,t]$, then by means of the Lebesgue Dominated Convergence Theorem has that,

$$\|N(x_n)(t)-N(x)(t)\| \to 0 \text{ as } n \to \infty.$$

Then,

$$\|N(x_n)-N(x)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let the constant R such that,

$$\frac{\left|\frac{d}{\sum_{k=1}^m \lambda_k}\right| + \frac{4M(2\pi)^{\nu\rho}}{\rho^\nu(1-q)\Gamma(\nu+1)} + M'}{1-\left[\frac{4p(2\pi)^{\nu\rho}}{\rho^\nu(1-q)\Gamma(\nu+1)}+b\right]} \leq R.$$

And,

$$\frac{4p(2\pi)^{\nu\rho}}{\rho^\nu(1-q)\Gamma(\nu+1)} < 1-b.$$

Define,

$$D_R = \{x \in C(J,E) : \|x\|_\infty \leq R\}\,.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J,E)$.

Claim $2 : N(D_R) \subset D_R$.

Let $x \in D_R$ we show that $Nx \in D_R$. We have, for each $t \in J$,

$$\begin{aligned}\|(Nx)(t)\| \leq & \left|\frac{d}{\sum_{k=1}^m \lambda_k}\right| + \left|\left[\frac{\sum_{k=1}^m \lambda_k \tau_k^\rho}{\sum_{k=1}^m \lambda_k} - t^\rho\right] \frac{1}{(2\pi)^\rho \Gamma(\nu)}\right| \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} \|\kappa_x(s)\| ds \\ & + \|\psi_x(t)\| + \left|\frac{1}{\Gamma(\nu)\sum_{k=1}^m \lambda_k}\right| \sum_{k=1}^m |\lambda_k| \int_0^{\tau_k} \left(\frac{\tau_k^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} \|\kappa_x(s)\| ds \\ & + \left|\frac{1}{\Gamma(\nu)}\right| \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1} \|\kappa_x(s)\| ds,\end{aligned}$$

where $\kappa_x, \psi_x \in C(J,E)$ such that ,

$$\kappa_x(s) = f(s, x(s), \kappa_x(s)), \quad \psi_x(t) = \psi_x(t, x(t)).$$

By (H_2) we have ,

$$\begin{aligned}\|\kappa_x(s)\| &= \|f(s, x(s), \kappa_x(s)) - f(s, 0, \kappa_0(s)) + f(s, 0, \kappa_0(s))\| \\ &\leq \|f(s, x(s), \kappa_x(s)) - f(s, 0, \kappa_0(s))\| + \|f(s, 0, \kappa_0(s))\| \\ &\leq p(s)\|x(s) - 0\| + q(s)\|\kappa_x(s) - \kappa_0(s)\| + \|f(s, 0, \kappa_0(s))\| \\ &\leq p\|x(s)\| + q\|\kappa_x(s)\| + \|f(s, 0, 0)\|.\end{aligned}$$

Where ,

$$x(s) = 0 \ , \ \kappa_0(s) = 0.$$

Then ,

$$\|\kappa_x(s)\| \leq \frac{1}{1-q}(p\|x(s)\| + \|f(s,0,0)\|) \leq \frac{1}{1-q}(p\|x\|_\infty + M) \leq \frac{pR+M}{1-q}.$$

Where,

$$M = \sup_{s \in J} \|f(s, 0, 0)\|.$$

And,

$$\begin{aligned}\|\psi_x(t)\| &= \|\psi_x(t,x(t))\| \\ &= \|\psi_x(t,x(t))-\psi_x(t,0)+\psi_x(t,0)\| \\ &\le b(t)\|x(t)-0\|+\|\psi_x(t,0)\| \\ &\le b\|x(t)\|+M' \le b\|x\|_\infty+M' \le bR+M'.\end{aligned}$$

Where ,

$$M' = \sup_{t\in J}\|\psi_x(t,0)\|.$$

Then ,

$$\begin{aligned}\|(Nx)(t)\| \le &\left|\frac{d}{\sum_{k=1}^m \lambda_k}\right| + \frac{pR+M}{1-q}\left|\left[\frac{\sum_{k=1}^m \lambda_k \tau_k^\rho}{\sum_{k=1}^m \lambda_k} - t^\rho\right]\frac{1}{(2\pi)^\rho\Gamma(\nu)}\right|\int_0^{2\pi}\left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}ds \\ &+ bRM' + \frac{pR+M}{1-q}\left|\frac{1}{\Gamma(\nu)\sum_{k=1}^m \lambda_k}\right|\sum_{k=1}^m |\lambda_k|\int_0^{\tau_k}\left(\frac{\tau_k^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}ds \\ &+ \frac{pR+M}{1-q}\left|\frac{1}{\Gamma(\nu)}\right|\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}ds,\end{aligned}$$

Since ,

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu-1} s^{\rho-1}ds &= \frac{1}{\nu\Gamma(\nu)}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu}\Bigg|_{s=0}^{s=t} \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{t^\rho}{\rho}\right)^\nu.\end{aligned}$$

Then we obtain ,

$$\begin{aligned}\|(Nx)(t)\| \le &\left|\frac{d}{\sum_{k=1}^m \lambda_k}\right| + \frac{pR+M}{1-q}\left[\left|\frac{\sum_{k=1}^m \lambda_k \tau_k^\rho}{\sum_{k=1}^m \lambda_k}\right| + |t^\rho|\right]\frac{1}{(2\pi)^\rho}\frac{1}{\Gamma(\nu+1)}\left(\frac{(2\pi)^\rho}{\rho}\right)^\nu \\ &+ bR + M' + \frac{pR+M}{1-q}\left|\frac{1}{\sum_{k=1}^m \lambda_k}\right|\sum_{k=1}^m |\lambda_k|\frac{1}{\Gamma(\nu+1)}\left(\frac{\tau_k^\rho}{\rho}\right)^\nu \\ &+ \frac{pR+M}{1-q}\frac{1}{\Gamma(\nu+1)}\left(\frac{t^\rho}{\rho}\right)^\nu.\end{aligned}$$

Since $t \leq 2\pi, \quad \tau_j \leq 2\pi$ *then we obtain,*

$$\begin{aligned}\|(Nx)(t)\| \leq& \left|\frac{d}{\sum_{k=1}^{m}\lambda_k}\right| + \frac{pR+M}{1-q}\left[\left|\frac{\sum_{k=1}^{m}\lambda_k(2\pi)^{\rho}}{\sum_{k=1}^{m}\lambda_k}\right| + (2\pi)^{\rho}\right]\frac{1}{(2\pi)^{\rho}}\frac{1}{\Gamma(\nu+1)}\left(\frac{(2\pi)^{\rho}}{\rho}\right)^{\nu} \\ &+ bR + M' + \frac{pR+M}{1-q}\left|\frac{1}{\sum_{k=1}^{m}\lambda_k}\right|\sum_{k=1}^{m}|\lambda_k|\frac{1}{\Gamma(\nu+1)}\left(\frac{(2\pi)^{\rho}}{\rho}\right)^{\nu} \\ &+ \frac{pR+M}{1-q}\frac{1}{\Gamma(\nu+1)}\left(\frac{(2\pi)^{\rho}}{\rho}\right)^{\nu} \\ \leq& \left|\frac{d}{\sum_{k=1}^{m}\lambda_k}\right| + \frac{4(pR+M)(2\pi)^{\nu\rho}}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + bR + M' \leq R,\end{aligned}$$

Then ,

$$N(D_R) \subset D_R.$$

Claim 3 *:* $N(D_R)$ *is bounded and equicontinuous.*

By Claim 2 we have $N(D_R) = \{N(x) : x \in D_R\} \subset D_R$. *Thus, for each* $x \in D_R$ *we have* $\|N(x)\|_{\infty} \leq R$ *which means that* $N(D_R)$ *is bounded. Let* $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$. *Assume that* H_3 *hold*

and let $x \in D_R$. *Then ,*

$$\begin{aligned}\|(Nx)(\tau_2) - (Nx)(\tau_1)\| \leq& \left\|\left[\tau_2^{\rho} - \tau_1^{\rho}\right]\frac{1}{(2\pi)^{\rho}\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa_x(s)ds\right\| \\ &+ \|\psi_x(\tau_2) - \psi_x(\tau_1)\| \\ &+ \left\|\frac{1}{\Gamma(\nu)}\int_0^{\tau_2}\left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1}s^{\rho-1}\kappa_x(s)ds\right. \\ &\left.- \frac{1}{\Gamma(\nu)}\int_0^{\tau_1}\left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa_x(s)ds\right\| \\ \leq& \left[\frac{1}{(2\pi)^{\rho}\Gamma(\nu)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\|\kappa_x(s)\|ds\right]|\tau_2^{\rho} - \tau_1^{\rho}| \\ &+ \|\psi_x(\tau_2) - \psi_x(\tau_1)\| \\ &+ \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\|\kappa_x(s)\|ds \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\tau_1}\left|\left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1} - \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\nu-1}\right|s^{\rho-1}\|\kappa_x(s)\|ds.\end{aligned}$$

With proposal change integral limit and continuity of the previous formulas, as $\tau_1 \to \tau_2$,*the right-hand side of the above inequality tends to zero.*

Claim 4 *: The operator* $N : D_R \to D_R$ *is a strict set contraction.*

Let $V \subset D_R$ and $t \in J$, then we have,

$$
\begin{aligned}
\mu(N(V)(t)) &= \mu((Nx)(t), x \in V) \\
&= \mu\Bigg(\frac{d}{\sum_{k=1}^{m} \lambda_k} + \left[\frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right] \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds \\
&+ \psi_x(t) - \frac{1}{\Gamma(\nu) \sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds, x \in V \Bigg) \\
&\leq \mu\Bigg(\frac{d}{\sum_{k=1}^{m} \lambda_k}, x \in V \Bigg) \\
&+ \mu\Bigg(\left[\frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right] \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds, x \in V \Bigg) \\
&+ \mu\Bigg(\psi_x(t), x \in V \Bigg) \\
&+ \mu\Bigg(- \frac{1}{\Gamma(\nu) \sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds, x \in V \Bigg) \\
&+ \mu\Bigg(\frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \kappa(s) ds, x \in V \Bigg) \\
&\leq \left| \frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right| \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \mu\Bigg(\kappa(s), x \in V \Bigg) ds \\
&+ \mu\Bigg(\psi_x(t), x \in V \Bigg) \\
&+ \frac{1}{\Gamma(\nu) \sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \mu\Bigg(\kappa(s), x \in V \Bigg) ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \mu\Bigg(\kappa(s), x \in V \Bigg) ds.
\end{aligned}
$$

Then,

$$\mu((Nx)(t), x \in V) \leq \left| \frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} - t^{\rho} \right| \frac{1}{(2\pi)^{\rho}\Gamma(\nu)} \int_0^{2\pi} \left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg)$$

$$+ b\mu\Bigg(x(t), x \in V \Bigg)$$

$$+ \frac{1}{\Gamma(\nu)\sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \int_0^{\tau_k} \left(\frac{\tau_k^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg) ds$$

$$+ \frac{1}{\Gamma(\nu)} \int_0^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\nu-1} s^{\rho-1} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg) ds$$

$$\leq \left[\left| \frac{\sum_{k=1}^{m} \lambda_k \tau_k^{\rho}}{\sum_{k=1}^{m} \lambda_k} \right| + |t^{\rho}| \right] \frac{1}{(2\pi)^{\rho}} \frac{1}{\Gamma(\nu+1)} \left(\frac{(2\pi)^{\rho}}{\rho} \right)^{\nu} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg)$$

$$+ b\mu\Bigg(x(s), x \in V \Bigg)$$

$$+ \frac{1}{\sum_{k=1}^{m} \lambda_k} \sum_{k=1}^{m} \lambda_k \frac{1}{\Gamma(\nu+1)} \left(\frac{\tau_k^{\rho}}{\rho} \right)^{\nu} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg)$$

$$+ \frac{1}{\Gamma(\nu+1)} \left(\frac{t^{\rho}}{\rho} \right)^{\nu} \frac{p}{1-q} \mu\Bigg(x(s), x \in V \Bigg).$$

Since, $t \leq 2\pi$, $\tau_k \leq 2\pi$ *then we obtain ,*

$$\mu((Nx)(t), x \in V) \leq \left(\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b \right) \mu\Bigg(x(s), x \in V \Bigg).$$

Then ,

$$\mu_c\left(N(V)\right) \leq \left[\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b \right] \mu_c(V).$$

So, by (3.3), the operator N is a set contraction. As a consequence of Darbo's Fixed Point Theorem, we deduce that N has a fixed point which is solution to the problem (3.1) and (3.2). This completes the proof.

Our next existence result for the problem (3.1) and (3.2) is based on concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 3.5 *Assume (H_1)-(H_2) and (3.3) hold. Then the problem (3.1) and (3.2) has at least one solution.*

proof 5 *Consider the operator N defined previous. According to Theorem 3.3, the operator N is bounded into itself, and equicontinuous.Now, we shall show that N satisfies the assumption of Mönch's fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that the implication :*

$$V = \overline{\text{conv}} N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every subset V *of* D_R*. Now let* V *be a subset of* D_R *such that* $V \subset \overline{conv}(N(V) \cup \{0\})$*.*$V$ *is bounded and equicontinuous and therefore the function* $t \to v(t) = \mu(V(t))$ *is continuous on* J*. By* (H_4) *and the properties of the measure* μ *we have for each* $t \in J$*,*

$$\begin{aligned} v(t) \leq \mu(N(V)(t) \cup \{0\}) &\leq \mu(N(V)(t)) \\ &\leq \mu\{(Nx)(t), x \in V\} \\ &\leq \left[\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b\right] \mu\{x(t), x \in V\} \\ &\leq \left[\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b\right] \mu\{V(t)\} \\ &\leq \left[\frac{p(2\pi)^{\nu\rho}((2\pi)^{\rho} + 2)}{\rho^{\nu}(1-q)\Gamma(\nu+1)} + b\right] v(t) \leq Lv(t). \end{aligned}$$

Because $L \leq 1$ *that's implies* $v(t) = 0$ *for each* $t \in J$*, and then* $V(t)$ *is relatively compact in* E*. In view of the Ascoli-Arzelà theorem,* V *is relatively compact in* D_R*. Applying now Mönch Theorem we conclude that* N *has a fixed point* $x \in D_R$*. Hence* N *has a fixed point which is solution to the problem (3.1) and (3.2). This completes the proof.*

3.3 Examples

Example 3.6 *Let*

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{k=1}^{\infty} |u_n| < \infty \right\},$$

be the Banach space with the norm

$$\|u\|_E = \sum_{k=1}^{\infty} |u_n|.$$

Consider the boundary value problem of generalized Caputo fractional differential equation :

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2},\rho}(u_n(t)) = \dfrac{(u_n(t) + 1)}{100\left(\left\|({}^C D_{0^+}^{\frac{3}{2},\rho} u_n)(t)\right\| + 1\right)}, \quad t \in [0, 2\pi], u_n(t) \in E, \\ \\ u_n(0) = u_n(2\pi), \quad \sum_{i=1}^{2} \frac{i}{3} u_n(\frac{i\pi}{3}) = d. \end{cases} \tag{3.4}$$

Set

$$f(t, u_n(t), ({}^C D_{0^+}^{\frac{3}{2},\rho} u_n)(t)) = \frac{(u_n(t)+1)}{100\left(\left\|({}^C D_{0^+}^{\frac{3}{2},\rho} u_n)(t)\right\|+1\right)}, \ \ t \in [0, 2\pi], u_n(t) \in E,$$

and

$$g(t, u_n(t)) = 0, \ \ t \in [0, 2\pi],$$

and $g(2\pi, u_n(2\pi)) = g(0, u_n(0)) = g(\tau_i, u_n(\tau_i)) = 0$, $i = 1, 2$, $\nu = \frac{3}{2}, \rho = \frac{1}{2}, \tau_i = \frac{i\pi}{3}$.
For each $u_n, \bar{u_n}, v_n, \bar{v_n} \in E$ *and* $t \in [0, 2\pi]$, *we have min*$\{\|v_n\|\} = 0$, *min*$\{\|\bar{v_n}\|\} = 0$ *then,*

$$\begin{aligned}\|f(t, u_n, v_n) - f(t, \bar{u_n}, \bar{v_n})\| &\leq \Bigg\| \frac{(u_n+1)}{100\left(\|v_n\|+1\right)} \\ &- \frac{(\bar{u_n}+1)}{100\left(\|\bar{v_n}\|+1\right)} \Bigg\| \\ &\leq \frac{1}{100}\|u_n - \bar{u_n}\|,\end{aligned}$$

$$\|g(t, u_n) - g(t, \bar{u_n})\| = 0.$$

Therefore, (H_2) *is verified with* $p = \dfrac{1}{100}, \ b = 0, \ q = 0,$

The condition

$$\frac{p(2\pi)^{\alpha\rho}((2\pi)^{\rho}+2)}{\rho^{\alpha}(1-q)\Gamma(\alpha+1)} + b = \frac{4 \times 2^{\frac{1}{4}}\pi^{\frac{1}{4}}(\sqrt{2}\sqrt{\pi}+2)}{75} \approx 0.3805357225 < 1$$

is satisfied with $\nu = \frac{3}{2}$. *Hence all conditions of Theorem 3.4 are satisfied, it follows that the problem (3.4) admit a unique solution.*

Conclusion and Perspective

In this memoiry, we have presented some results to the theory of the existence of solutions and uniqueness of fractional implicit differential equations with the derivatives of generalized-Caputo, and mention all the derivatives. The problem studied are with periodic and non-local boundary value Problems. The results obtained are based on some fixed point theorems and the measure of non-compactness. In future research, we plan to study some fractional differential and inclusions with impulses (instantaneous and not instantaneous) in fréchet spaces.

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Appendix

الجمهورية الجزائرية الديمقراطية الشعبية
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Département Des Mathématiques & de l'Informatique

جامعـــة غردايـــة

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أنا الأستاذ:

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بصفتي -مشرفوالمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـــــــ

Existence of Solutions for Generalized Caputo Periodic and Non-Local Boundary Value problems.

من انجاز الطالب(ة):

Benabderrahmane Hadj Slimane

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