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Predator-Prey System with the Beddington-DeAngelis Functional Response

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Dedication

I dedicate this modest work to
My dear parents for their sacrifices and who have never stopped to encourage me, that ALLAH keeps them to me.
My dear brothers and sisters.
All my family members, big and small, to my friends and to my colleagues in my promotion without exception.

ملخص :

تعتبر جملة المفترس-الفريسة مع إستجابة وظيفية من نوع بدنجتون-دي أنجيليس نموذجا ديناميكيا يصف واحدا من التفاعلات الحاصلة فيما بين المجموعات السكانية السكانية. الإستجابة الوظيفية لبدنجتون-دي أنجيليس مماثلة لإستجابة هولينج من النوع الثاني و لكنها يحتوي على طرف يصف التدخل المتبادل من قبل الحيوانات المفترسة.

يهدف هذا العمل إلى تقديم دراسة الحدود و الإستقرار الشامل عن السلوك النوعي لفئة من نماذج المفترس-الفريسة مع إستجابة وظيفية من نوع بدنجتون-دي أنجيليس، حيث تشتمل على نتائج حول حدود الحلول، وجود مجموعة جاذبة، و الإستقرار المحلي و الشامل للتوازن، و تم تعيين معايير الإستقرار من أنظمة هولينج من النوع الثاني.

الكلمات المفتاحية: نموذج المفترس-الفريسة، إستجابة بدنجتون-دي أنجيليس الوظيفية، إستقرار نقطة التوازن، الإستجابة الوظيفية لهولينج من النوع الثاني.

Résumé :

Système Prédateur-Proie avec réponse fonctionnelle de Beddington-DeAngelis est un modèle dynamique décrivant l'un de types d'interaction entre populations. La réponse fonctionnelle de Beddington-DeAngelis est similaire à la réponse fonctionnelle de Holling type-II, sauf qu'elle contient un terme décrivant les interférences mutuelles des prédateurs. Notre travail vise présenter une étude de la bornitude et la stabilité globale sur le comportement qualitatif d'une classe de modèles de prédateurs-proies de type Beddington-DeAngelis réponse fonctionnelle. Les résultats incluent sur la bornitude des solutions, l'existence d'un ensemble attractif et la stabilité locale et globale des points d'équilibres. Les critères de stabilité sont définis des systèmes de Holling-II.

Mots clés : modèle prédateur-proie, réponse fonctionnelle de type Beddington-DeAngelis, stabilité globale du point d'équilibre, réponse fonctionnelle de Holling de type-II.

Abstract :

Predator-Prey model with Beddington-DeAngelis-type functional response is considered, it is a dynamic model for the interaction of the population. The functional response of Beddington-DeAngelis is similar to the Holling type-II functional response, but it contains a term describing the mutual interference of predators. The work aims to present a study boundedness and global stability on the qualitative behavior of a class of predator-prey models with Beddington-DeAngelis type functional response. Including results the boundedness of solutions, existence of an attracting set and local and global stability of equilibria. Stability criteria are set from Holling-II systems.

Keywords predator-prey model, Beddington-DeAngelis type functional response, global stability of equilibria point, Holling type-II functional response.

Introduction

Mathematical modeling in biology and ecology involves using mathematics to describe or explain phenomenons and the dynamics of a population in the real world, it has having several applications in various fields.
The modeling process requires a good inderstanding of the factors governing the evolution of population over time such as available recources, populations in interaction, thretening dangers...

In this memory, we are interested in studing the work of W. Khellaf and N. Hamri [11] and contribute to explaining and analyzing this work.
Predator-Prey interaction is that relationship between organisms occupying the same environment, where the prey is a (principal) source of food for predators. Within animals (sharks and fish) or animals and plants (rabbits and plants) or viruses and cells (HIV virus attacks immune system cells).

The model that describes the dynamics populations of predator-prey over time is of the form

$$\begin{cases} \dfrac{dX}{d\tau} &= f(X)X - g(X,Y)Y, \\ \dfrac{dY}{d\tau} &= h(X,Y)Y, \end{cases} \tag{1}$$

in model (1) $X(\tau)$ and $Y(\tau)$ represent respectively the population densities (or biomasses) of the prey and predator at the τ moment. This model is based on the assumption thut functions: $g(X,Y)$ and $h(X,Y)$ satisfy $\dfrac{\partial g}{\partial Y} \geq 0, \dfrac{\partial h}{\partial X} \geq 0$. One well known Predator-Prey model is the Lotka-Volterra system ([12], [17])

$$\begin{cases} \dfrac{dx}{dt} &= ax - bxy, \\ \dfrac{dy}{dt} &= cxy - dy, \end{cases}$$

with $a, b, c, d > 0$, which imposes a per capita rate of predation depends on prey numbers only [18].

The Beddington-DeAngelis response can be generated by a number of natural mechanisms [2], [6], and because it admits rich but biologically reasonable dynamics [4], it is worthy for us to further study the Beddington- DeAngelis model.

more presesily, the following predator-prey model with the Beddington-DeAngelis is considered

$$\begin{aligned} \frac{dX}{d\tau} &= \left(a_1 - b_1 X - \frac{m_1 Y}{\alpha_1 X + \beta_1 Y + \gamma_1}\right) X, \\ \frac{dY}{d\tau} &= \left(a_2 - \frac{m_2 Y}{X + k_1}\right) Y, \end{aligned} \tag{2}$$

with the initial values $X(0) \geq 0$ and $Y(0) \geq 0$. The constants $a_1, a_2, b_1, m_1, m_2, \alpha_1, \beta_1, \gamma_1$, and k_1 are the parameters of model and are assumed to be positive.
These parameters are defined as follows: a_1 (resp., a_2) describes the growth rate of prey (resp., of predator), b_1 measures the strength of competition among individuals of prey's species, m_1 is the maximum value which per capita reduction rate of prey can attain, γ_1 (resp., k_1) measures the extent to which environment provides protection to prey (resp., to predator), and $m2$ has a similar meaning to $m1$. The functional response in (2) was introduced by Beddington [2] and DeAnglis et al. [7]. It is similar to the well-known Holling type-II functional response [2] but has an extra term $\beta_1 Y$ in the first right term equation modeling mutual interference among predators. Hence this kind of type functional response given in (2) is affected by both predator and prey.

A simpler Beddington-DeAngelis predator-prey model is obtained by change for parameters of system (2), $\tau = a_1 t, X(\tau) = x(t)/(b_1/a_1), Y(\tau) = y(t)/(m_2 b_1/a_1 a_2)$ hence for $\dfrac{dX}{d\tau}$,

$$\begin{aligned} \frac{dX}{d\tau} &= \left(a_1 - b_1 X - \frac{m_1 Y}{\alpha_1 X + \beta_1 Y + \gamma_1}\right) X, \\ \frac{dx(t)/(b_1/a_1)}{dt/a_1} &= \left(a_1 - b_1(x(t)/(b_1/a_1)) - \frac{m_1 y(t)/(m_2 b_1/a_1 a_2)}{\alpha_1 x(t)/(b_1/a_1) + \beta_1 y(t)/(m_2 b_1/a_1 a_2) + \gamma_1}\right) \frac{x(t)}{(b_1/a_1)}, \\ \frac{dx(t)}{dt} &= \left(a_1 - a_1 x(t) - \frac{((m_1 b_1)/a_1) y(t)}{\alpha_1 (m_2/a_2) x(t) + \beta_1 (a_2/m_2) y(t) + \gamma_1 (b_1/a_1)^2 (m_2/a_2)}\right) \frac{x(t)}{a_1}. \end{aligned}$$

and for $\dfrac{dY}{d\tau}$,

$$\begin{aligned}
\frac{dY}{d\tau} &= \left(a_2 - \frac{m_2 Y}{X+k_1}\right) Y, \\
\frac{dy}{dt}.\frac{a_1}{(m_2 b_1/a_1 a_2)} &= \left(a_2 - \frac{m_2\left(\dfrac{y(t)}{(m_2 b_1/a_1 a_2)}\right)}{\dfrac{x(t)}{(b_1/a_1)}+k_1}\right) \frac{y(t)}{(m_2 b_1/a_1 a_2)}, \\
\frac{dy}{dt} &= \left(a_2 - \frac{m_2\left(\dfrac{y(t)}{(m_2 b_1/a_1 a_2)}\right)}{\dfrac{x(t)}{(b_1/a_1)}+k_1}\right) \frac{y(t)}{(m_2 b_1/a_1 a_2)}.\frac{(m_2 b_1/a_1 a_2)}{a_1}, \\
\frac{dy}{dt} &= \frac{1}{a_1}\left(a_2 - \frac{y(t)/(b_1/a_1 a_2)}{(x+k_1(b_1/a_1))/(b_1/a_1)}\right) y(t), \\
\frac{dy}{dt} &= \frac{1}{a_1}\left(a_2 - a_2 \frac{y}{x+k_1(b_1/a_1)}\right)
\end{aligned}$$

We have by placing (*) $a = (m_1 b_1)/a_1, b = a_2/a_1, \alpha = \alpha_1(a_1 m_2)/a_2), \beta = \beta_1(a_1 a_2)/m_2, \gamma = \gamma_1((b_1/a_1)^2(m_2 a_1/a_2)$, and $k = k_1(b_1/a_1)$.

$$\begin{cases}
\dfrac{dx}{dt} = x(1-x) - \dfrac{axy}{\alpha x + \beta y + \gamma}, \\
\dfrac{dy}{dt} = b\left(1 - \dfrac{y}{x+k}\right) y,
\end{cases}$$

So, the study of our model prompted us to organize this work in five chapters. Chapter 1 is devoted to recalling some basic concepts on differential equations and inequalities. Such as semiflows and invariant sets, limit sets and comparison lemma. In addition, we have stated some elementary results for studying equilibrium points and their stability to determine their nature. We also present other necessary mathematical results at the end of the chapter.

In the second chapter, we talk about mathematical modeling we give some basic models.

In chapter 3, we show the boundedness of solutions and existence attracting set in the first quadrant. we prove positive invariance and ultimate boundedness of solutions.

Then in chapter 4, we determine trivial and interior equilibrias and we study their stability, recarding the interior equilibria we discuss global asymptotic stability as well finally in Chapter 5, We mention the definitions we use to get permanence, we study uniform permanence.

Remark 0.1. *In [11] we find:* $a = (a_2/a_1)(m_1/m_2), b = a_2/a_1, \alpha = \alpha_1, \beta = \beta_1(a_2/m_2), \gamma = \gamma_1(b_1/a_1)$, *and* $k = k_1(b_1/a_1)$. *which is different from the conclution we are lead to in (*).*

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Chapter 1

Preliminaries

In this chapter, we give some necessary mathematical definitions that are used in the studying Beddington-DeAngelis predator-prey model. Let us consider a dynamical system of the form

$$\begin{cases} \dot{x}(t) &= f(x(t)) \\ x(0) &= x_0 \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n, f : \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a locally Lipschitz in x and is piecewise continuous in t.

1.1 Generalities on differential equations and inequality

1.1.1 Existence and uniqueness

Theorem 1.1. *[1] If the system 1.1 satisfy*

$$\| f(t,x) - f(t,y) \| \leq L \| x - y \|$$

$\forall x, y \in B(x_0, r) = \{x \in \mathbb{R}^n | \|x - x_0\| \leq r\}, \forall t \in [t_0, t_1]$. *Then there exists some* $\delta > 0$ *such that the state equation*

$$\dot{x} = f(t,x)$$

with $x(t_0) = x_0$ *has a unique solution over* $[t_0, t_0 + \delta]$.

1.1.2 Semiflow and Invariant set

Definition 1.1 (Semiflow). *[8] Let* Ω *be a subset of a complete metric space* Υ, *and let* $\mathbb{R} = (-\infty, \infty)$ *and* $\mathbb{R}^+ = [0, +\infty[$. *A mapping* $\sigma : \Omega \times [0, +\infty[\longrightarrow \Omega$, *is said to be semiflow on* Ω, *if*

1. $\sigma(u, 0) = u$, *for all* $u \in \Omega$.

2. $\sigma(0, t)$, *is defined for all* $t \in \mathbb{R}$.

3. *The semi-groupe proprety holds, i.e.,* $\sigma_t(\sigma_s(u)) = \sigma_{t+s}(u)$, *for all* $u \in \Omega$ *and* $t, s \in \mathbb{R}^+$.

4. *The mapping* $\sigma : \Omega \times [0, +\infty[\longrightarrow \Omega$ *is continuous.*

Remark 1.1. *If the mapping* $u \longrightarrow \sigma(u, t)$ *is not linear, then the semiflow (nonlinear semigroup) is the form* $\sigma(u, t) = S(t)u$. *In the case* $S(t) : \Omega \longrightarrow \Omega$, *with a continuous inverse* $S(t)^{-1}$, *such that* $S(-t) = S(t)^{-1}$ *means that flow (nonlinear groupe).*

Let σ *be a semiflow on* $\Omega \subset \Upsilon$. *For any* $u \in \Omega$ *the (positive) trajectory through* u *is defined as the set*

$$\gamma^+(u) = S(t)u : t \geq 0$$

If σ *is a flow on* Ω, *then the trajectory through* u *is the set*

$$\gamma(u) = \{S(t)u : t \in \mathbb{R}\}.$$

Definition 1.2 (Invariant set). *[8] A set* $A \subset \Omega$ *is said to be positively invariant if* $S(t)A \subset A$, *for all* $t \geq 0$, *and* A *is said to be an invariant set if* $S(t)A = A$, *for all* $t \geq 0$, *and the trajectories through* A *are given by*

$$\gamma^+(A) = \{S(t)u : u \in A, t \geq 0\}$$

Definition 1.3 (Forward invariant). *A set is called forward invariant if it is invariant for all* $t \in \mathbb{R}^+$.

1.1.3 Limit sets

Definition 1.4. *[8] The limit sets of a semiflow namely the omega limit set. The* ω*-limit set of a point* u *is defined as*

$$\omega(u) = \cap_{T\geq 0}\overline{\bigcup_{t\geq T} S(t)u}.$$

We define the ω*-limit set of* $\partial\Upsilon_0$ *(the boundary of* Υ_0*) as as follows:*

$$\omega(\partial\Upsilon_0) = \cap_{T\geq 0}\overline{\bigcup_{t\geq T} S(t)\partial\Upsilon_0},$$

where

$$S(t)\partial\Upsilon_0 = \bigcup_{u\in\partial\Upsilon_0} S(t)u.$$

Definition 1.5. *The semigroup* $S(t)$ *is said to be point dissipative in* Υ *if there is a bounded nonempty set* B *in* Υ *such that, for any* $u \in \Upsilon$ *there is a* $t_0 = t_0(\Upsilon, B)$ *such that* $S(t)u \in B$ *for* $t \geq t_0$.

1.1.4 Comparison lemma

Definition 1.6. *(Absolute Continuity)[19] A function $f : [a, b] \longrightarrow \mathbb{R}$, where $[a, b]$ is a finite closed intervale, is said absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of pairwise disjoint intervals $\{[a_k, b_k] : k = 1, ..., n\}$ conteined in $[a, b]$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$.*

Lemma 1. *[?] Let ϕ be an absolutely-continuous function satisfying the differential inequality*

$$\frac{d\phi(t)}{dt} + \alpha_1\phi(t) \leqslant \alpha_2, \qquad t \geq 0 \tag{1.2}$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2; \alpha_1 \neq 0$, then

$$\forall t \geq \bar{T} : \phi(t) \leqslant \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - \phi\left(\bar{T}\right)\right) e^{-\alpha_1(t-\bar{T})}. \tag{1.3}$$

Proof: Multiply both sides of (1.2) by $e^{\alpha_1 t}$ to get

$$\left(\frac{d\phi(t)}{dt} + \alpha_1\phi(t)\right) e^{\alpha_1 t} \leqslant \alpha_2 e^{\alpha_1 t}.$$

Then

$$\left(\frac{d\phi(t)}{dt} + \alpha_1\phi(t) - \alpha_2\right) e^{\alpha_1 t} \leqslant 0.$$

which is equivalent to

$$\frac{d}{dt}\left(\left(\phi(t) - \frac{\alpha_2}{\alpha_1}\right) e^{\alpha_1 t}\right) \leqslant 0.$$

Thus the function

$$\left(\phi(t) - \frac{\alpha_2}{\alpha_1}\right) e^{\alpha_1 t},$$

has a non-positive derivative and so is non-increasing for $t \geq 0$. Therefore, for all $t \geq \bar{T} \geq 0$,

$$\left(\phi(t) - \frac{\alpha_2}{\alpha_1}\right) e^{\alpha_1 t} \leqslant \left(\phi(\bar{T}) - \frac{\alpha_2}{\alpha_1}\right) e^{\alpha_1 \bar{T}},$$

and hence,

$$\phi(t) \leqslant \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - \phi(\bar{T})\right) e^{-\alpha_1(t-\bar{T})}.$$

Lemma 2. *Let $x(t)$ be a continuous function*

$$\begin{cases} \dfrac{dx}{dt} &= f(t,x), \\ x(t) &= x_0, \end{cases}$$

where $f(t,x)$ is continuous in t and locally Lipschitz in x, for all $t \in [t_0, T)$ and $x \in J \in \mathbb{R}$. Let $[t_0, T)$ (T can be ∞) be the maximal interval of existence of the solution $x(t) \in J$. And the upper right-hand derivative

$$D^+y(t) = \limsup_{h \longrightarrow 0^+} \frac{y(t+h) - y(t)}{h}.$$

Let $y(t)$ be a continuous function that satisfies

$$\begin{aligned} D^+y(t) &\leqslant f(t,y), \\ y(t_0) &\leqslant x_0, \end{aligned}$$

with $y(t) \in J$ for all $t \in [t_0, T)$, then

$$y(t) \leqslant x(t), \qquad [t_0, T).$$

1.2 Equilibrium points and stability

1.2.1 Equilibrium points

The general n-dimensional autonomuous of the system (1.1), if t does not appear explicitly can be written as

$$\dot{x} = f(x) \tag{1.4}$$

and has the form

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1(t), x_2(t), ..., x_n(t)), \\ \frac{dx_2}{dt} &= f_2(x_1(t), x_2(t), ..., x_n(t)), \\ \vdots &= \vdots \\ \frac{dx_n}{dt} &= f_n(x_1(t), x_2(t), ..., x_n(t)), \end{aligned}$$

where $f_i, 0 \leqslant i \leqslant n$ are the functions of $x_1, x_2, ..., x_n$. The solutions $f(x) = 0_{\mathbb{R}^n}$ are called equilibrium points of the system (1.4).

Consider two-dimensional systems of the form

$$\begin{cases} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y). \end{cases} \tag{1.5}$$

The intersection point (x^*, y^*) of the curves

$$\begin{cases} f(x,y) &= 0, \\ g(x,y) &= 0, \end{cases}$$

is the equilibrium point of the system (1.5). Then the constant functions

$$x(t) = x^*, \qquad y(t) = y^*,$$

are a solutions of (1.5).

Definition 1.7. *(Liapunov stability)[15] Let $x^*(t)$ be the solution of the system (1.1) wich starts in $x^*(t_0) = x_0^*$. If for every $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0)$ such that*

$$\| x_0 - x_0^* \| < \delta \Rightarrow \| x(t) - x^*(t) \| < \varepsilon \quad \textit{for all} \quad t > t_0,$$

then the solution $x(t)$ is called Liapunov stable for $t \geq t_0$. Otherwise it is called Liapunov unstable.

Definition 1.8. *(Uniform stability)[15] If the solution is stable for $t \geq t_0$ and the δ is independent of t_0, then it is uniformly stable.*

Definition 1.9. *(Asymptotic stability)[15] If the solution is stable for $t \geq t_0$ and*

$$\lim_{t \longrightarrow \infty} \| x(t) - x^*(t) \| = 0,$$

then it is called asymptotically stable

1.2.2 Types of Equilibrium Points

Definition 1.10. *[15] Let C be a path of the system (1.5) and the solution $(x(t), y(t))$ of (1.5) which represents C parametrically, and the critical point (x^*, y^*) of this system. Thus we have the path C approaches the (x^*, y^*) as $t \longrightarrow +\infty$ if*

$$\lim_{t \longrightarrow +\infty} x(t) = x^*, \quad \lim_{t \longrightarrow +\infty} y(t) = y^*.$$

Definition 1.11. *[15] Let C be a path which approaches the equilibrium point (x^*, y^*) of the system (1.5) as $t \longrightarrow +\infty$, and the solution $(x(t), y(t))$ of this system represents C. We say that C enters the critical point (x^*, y^*) as $t \longrightarrow +\infty$ if*

$$\lim_{t \longrightarrow +\infty} \frac{y(t)}{x(t)}$$

exists.

Definition 1.12. *(Isolated equilibrium Point)[14] A equilibrium point* (x^*, y^*) *of (1.5) is called an isolated equilibrium point if there exists a neighborhood of* (x^*, y^*) *containing no other equilibrium points.*

Definition 1.13. *(Center)[15] The isolated equilibrium point* (x^*, y^*) *is called a center if there exists a neighborhood of* (x^*, y^*) *which contains a countably infinite number of closed paths each of which contains* (x^*, y^*) *in its interior.*

Definition 1.14. *(Saddle Point)[15] The isolated equilibrium point* (x^*, y^*) *is called a saddle point if there exists a neighborhood of* (x^*, y^*) *in which the following two conditions holds:*

1. *There exist two paths which approach and enter* (x^*, y^*) *from a pair of opposite directions as* $t \longrightarrow +\infty$ *and there exist two paths which approach and enter* (x^*, y^*) *from a different pair of opposite directions as* $t \longrightarrow -\infty$.

2. *In each of the four domains between any two of the four directions in* 1, *there are infinitely many paths which are arbitrarily close to* (x^*, y^*) *but do not approach* (x^*, y^*) *either as* $t \longrightarrow +\infty$ *or as* $t \longrightarrow -\infty$.

Definition 1.15. *(Spiral)[15] The isolated equilibrium point* (x^*, y^*) *is called a spiral point (or focus) if there exists a neighborhood of* (x^*, y^*) *such that every path* P *in this neighborhood has the following properties:*

1. P *is defined for all* $t > t_0$ *(or for all* $t < t_0$*) for some number* t_0.

2. P *approaches* (x^*, y^*) *as* $t \longrightarrow +\infty$ *(or as* $t \longrightarrow -\infty$*).*

3. P *approaches* (x^*, y^*) *in a spiral-like manner, winding around* (x^*, y^*) *an infinite number of times as* $t \longrightarrow +\infty$ *(or as* $t \longrightarrow -\infty$*).*

Definition 1.16. *(Node)[15] The isolated equilibrium point* (x^*, y^*) *is called a node if there exists a neighborhood of* (x^*, y^*) *such that every path* P *in this neighborhood has the following properties:*

1. P *is defined for all* $t > t_0$ *(or for all* $t < t_0$*) for some number* t_0.

2. P *approaches* (x^*, y^*) *as* $t \longrightarrow +\infty$ *(or as* $t \longrightarrow -\infty$*).*

3. P *enters* (x^*, y^*) *as* $t \longrightarrow +\infty$ *(or as* $t \longrightarrow -\infty$*).*

1.2.3 Stability

In order to study the stability of eqilibria of problem (1.1). It suffices to study stability of system

$$\begin{aligned} \dot{x} &= f(t,x) + A(t)x - A(t)x \\ &= A(t)x + (f(t,x) - A(t)x) \end{aligned}$$

and hence

$$\dot{x} = Ax + g(t,x), \tag{1.6}$$

assume that the Jacobian matrix $A = \dfrac{\partial g}{\partial x}(t,x)$, is an $n \times n$ constant matrix and $g(t,x) = (g_1(t,x), g_2(t,x), ..., g_n(t,x)$ satisfies

(i) $g(t,x)$ is continuos for $\parallel x \parallel < a, 0 \leqslant t < \infty$,

(ii) $\lim_{\parallel x \parallel \longrightarrow 0} \parallel g(t,x) \parallel / \parallel x \parallel = 0$ uniformly with respect to t, that is $\parallel g(t,x) \parallel = o(\parallel x \parallel)$ uniformly in t as $\parallel x \parallel$ approaches as zero.

Theorem 1.2. *If A is an $n \times n$ constant matrix whose characteristic matrix polynomial has all its roots have negatives real part, and function $g(t,x)$ satisfies conditions (i) and (ii) above, then the solution $x(t) \equiv 0$ of the system (1.6) is asymptotically stable.*

Definition 1.17. *(Stability of Non-linear systems) Consider the system (1.6) such that*

$$\frac{dx_i}{dt} = f(x_i) \approx f(x_i^*) + \frac{df}{dx}(x_i^*)(x_i - x_i^*) + o(h(x_i)).$$

Thus

$$\frac{dx_i}{dt} \approx \frac{df}{dx}(x_i^*)(x_i - x_i^*). \tag{1.7}$$

Jacobien matrix of f at x^* is given by

$$J = \left(\frac{\partial f_i}{\partial x_j}(x^*) \right).$$

Then

$$\frac{dx}{dt} = J(x^*)(x - x^*),$$

is the linearized system

Remark 1.2. *Suppose that the system (1.1) has an equilibrium point $x^* \in \mathcal{D}$, i.e., $f(x^*) = 0$. We would like to characterize if the equilibrium point x^* is stable. We can always apply a change of variables to $\xi = x - x^*$ to obtain*

$$\dot{x} = \dot{\xi} = f(\xi + x^*)$$

and hence

$$\begin{aligned} \dot{\xi} &= f(\xi + x^*) \\ &= f(x^*) + J\xi + o(h(\xi)) \\ &= J\xi + o(h(\xi)) \end{aligned}$$

then study the stability of the new system with respect to $\xi = 0$, the origin.

Theorem 1.3. *(Stability of Linear systems)[14] Let $\dot{x} = Ax$ be an n-dimensional linear system and A is an $n \times n$ constant matrix. Suppose that $\lambda_i, i = 1, ..., n$ are the eigenvalues of A.*

(i) All solutions of the system are asymptotically stable if $Re(\lambda_i) < 0, i = 1, 2, ..., n$.

(ii) If all solutions of the system are stable, then $Re(\lambda_i) \leqslant 0, i = 1, 2, ..., n$.

(iii) The solution is unstable if $Re(\lambda_i) > 0, i = 1, 2, ..., n$.

Definition 1.18. *If A is an 2×2 constant matrix and λ is eigenvalues of A, then classification of equilibrium points as follow*

- *both eigenvalues are negative means the equilibrium points are stable,*
- *both eigenvalues are positive means the equilibrium points are unstable,*
- *eigenvalues have different sign means the equilibrium ponts are saddle.*

1.2.4 Lyapunov function

Definition 1.19. *Let $V : \mathcal{D} \longrightarrow \mathbb{R}$ be a continuously differentiable function defined on the domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin [1]. The rate of change of V along the trajectories of (1.1) is given by*

$$\begin{aligned} \dot{V}(x(t)) &= \textstyle\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{d}{dt}x_i \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \dot{x} = \frac{\partial V}{\partial x} f(x). \end{aligned} \tag{1.8}$$

Theorem 1.4 (Direct Method). *[1] Let the origin $x = 0 \in \mathcal{D} \subset \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathcal{D} \longrightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} V(0) &= 0 \quad \text{and} \quad V(x) > 0, \qquad \forall x \in \mathcal{D} \setminus \{0\} \\ \dot{V}(x) &\leqslant 0, \qquad \forall x \in \mathcal{D} \end{aligned} \tag{1.9}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0, \qquad \forall x \in \mathcal{D} \setminus \{0\}$$

then $x = 0$ is asymptotically stable.

Remark 1.3. *If $V(x) > 0, \forall x \in \mathcal{D} \setminus \{0\}$, then V is called locally positive definite. If $V(x) \geq 0, \forall x \in \mathcal{D} \setminus \{0\}$, then V is locally positive semi-definite. If the conditions (1.9) are met, then V is called a Lyapunov function for the system $\dot{x} = f(x)$.*

Exemple 1. *[1] Consider the nonlinear system*

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^2 x_2 \\ -x_2 \end{bmatrix}$$

and the candidate Lyapunov function

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

with $\lambda_1, \lambda_2 > 0$. The derivative of the Lyapunov function candidate was given by

$$\dot{V}(x) = 2\lambda_1 x_1(-x_1 + 2x_1^2 x_2) + 2\lambda_2 x_2(-x_2) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$

For simplicity, assume that $\lambda_1 = \lambda_2 = 1$. Then

$$\dot{V}(x) = -2x_2^2 - 2x_1^2 g(x) < 0$$

where $g(x) = -1 + 2x_1 x_2$. $\dot{V} < 0$ will be invariant, or equivalently when $g(x) > 0$, i.e., when $x_1 x_2 < 1/2$. So we conclude that the origin is locally asymptotically stable.

Theorem 1.5. *Let $x = 0$ be an equilibrium point of the system (1.1).*
Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \qquad \forall x \neq 0 \tag{1.10}$$

$$\| x \| \longrightarrow \infty \Longrightarrow V(x) \longrightarrow \infty, \tag{1.11}$$

$$\dot{V}(x) < 0, \qquad \forall x \neq 0, \tag{1.12}$$

then the origin is globally asymptotically stable.

Remark 1.4. *If the function V satisfies the condition (1.11), then it is said to be radially unbounded.*

1.3 Cardan's method

Cardan's method [13] used for solve a cubic equation. Suppose this equation

$$x^3 + ax^2 + bx + c = 0,$$

with $a, b, c \in \mathbb{R}$. Making the substitution $y = x + \frac{a}{3}$, the equation becomes:

$$y^3 + (b - \frac{a^2}{3})y + c - \frac{ab}{3} + \frac{2a^3}{27} = 0,$$

which is of the form

$$y^3 + py + q = 0. \tag{1.13}$$

Letting $y = u + v$, *we obtain*

$$u^3 + v^3 + (u + v)(3uv + p) + q = 0.$$

Supposing

$$\begin{cases} 3uv + p = 0, \\ u^3 + v^3 + q = 0. \end{cases}$$

Then

$$\begin{cases} uv = \dfrac{-p}{3}, \\ u^3 + v^3 = -q. \end{cases}$$

Hence

$$\begin{cases} u^3 v^3 = -\dfrac{p^3}{27}, \\ u^3 + v^3 = -q, \end{cases}$$

placing $h = u^3, z = v^3$, *we get*

$$\begin{cases} hz &= -\dfrac{p^3}{27}, \\ h + z &= -q, \end{cases}$$

we put

$$\begin{cases} hz &= P, \\ h + z &= S, \end{cases}$$

replacing the value of $h = S - z$, *we obtain*

$$\begin{aligned} z(S - z) &= P, \\ Sz - z^2 &= P, \\ z^2 - Sz + P &= 0. \end{aligned}$$

Hence,

$$z^2 + qz + \left(-\frac{p}{3}\right)^3 = 0. \tag{1.14}$$

Similarly, replacing the value of $z = S - h$, *then*

$$h^2 + qh + \left(-\frac{p}{3}\right)^3 = 0.$$

which implies that u^3 *and* v^3 *are both the roots of the same equation for which the discriminant is* $\Delta = \dfrac{27q^2 + 4p^3}{27}$.

Lemma 3. *[13] The equation (1.13) admits a unique real root if and only if we have* $4p^3 + 27q^2 > 0$.

Proof. [13] We study the function $f(y) = y^3 + py + q$, and after following the steps of the Carden method we find $\Delta = 4p^3 + 27q^2$, derivative of f is $f'(y) = 3y^2 + p$. If p is positive. we consider $x_1 = -\sqrt{-\frac{p}{3}}, x_2 = \sqrt{-\frac{p}{3}}$,

1. if $x \in]-\infty, x_1] \bigcup [x_2, +\infty[$, $f(x)$ increasing,
2. if $x \in]x_1, x_2[$, $f(x)$ decreasing.

She admits a maximum relative M in x_1 and a relative minimum m in x_2. Say that the equation has only one real root means that m and M have the same sign, so that the product $mM > 0$. Or, we have:

$$\begin{aligned}
M &= f(x_1) = q - \frac{2p}{3}\sqrt{-\frac{p}{3}} \\
m &= f(x_2) = q + \frac{2p}{3}\sqrt{-\frac{p}{3}} \\
mM &= \left(q + \frac{2p}{3}\sqrt{-\frac{p}{3}}\right)\left(q - \frac{2p}{3}\sqrt{-\frac{p}{3}}\right) = \frac{4p^3 + 27q^2}{27} = \Delta,
\end{aligned}$$

and hence $\Delta > 0$. □

Exemple 2. *[13] We solve the equation* $x^3 - 2x - 5 = 0$.
The equation that gives u^3, v^3 *is then* $X^2 - 5X + \frac{8}{27} = 0$.
$\Delta = 25 - \frac{32}{27} = \frac{643}{27}$ *so,*

$$u^3 = \frac{5}{2} + \frac{\sqrt{643}}{6\sqrt{3}}, \quad v^3 = \frac{5}{2} - \frac{\sqrt{643}}{6\sqrt{3}}.$$

We deduce the unique root of the given equation:

$$x = u + v = \sqrt[3]{\frac{5}{2} + \frac{\sqrt{643}}{6\sqrt{3}}} + \sqrt[3]{\frac{5}{2} - \frac{\sqrt{643}}{6\sqrt{3}}}.$$

Lemma 4. *[13]*
When Δ *is negative, we find first the two roots* u^3, v^3 *of (1.3), which are complex conjugates, then extract their cubic roots* u, v. *The first is really find those roots. We return to it below. The second is does* u^3 *admit three cubic roots:* u, ju, j^2u *and the same for* v.

$$u_1 = \sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}} \qquad v_1 = \sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}}$$
$$u_2 = j\sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}} \qquad v_2 = j\sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}}$$
$$u_3 = j^2\sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}} \qquad v_3 = j^2\sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 + \left(-\frac{p}{3}\right)^3}}.$$

with j and j^2 cubic roots of 1.

Proof. We know
$$1 = e^0 = e^{i0} = \cos 0 + i \sin 0,$$
let z be a complex number such that
$$z^3 = 1 \Leftrightarrow r^3 e^{3i\theta} = e^0.$$
Hence,
$$\begin{cases} r^3 &= 1, \\ 3\theta &= 0 + 2k\pi. \end{cases}$$
Therefore,
$$\begin{cases} r &= 1, \\ \theta &= \dfrac{2k\pi}{3} \end{cases} \qquad \text{with} \qquad k \in \mathbb{Z}$$
so we search $\theta \in [0; 2\pi[$
$$0 \leqslant \frac{2k\pi}{3} < 2\pi \Leftrightarrow 0 \leqslant \frac{2k}{3} < 2 \Leftrightarrow 0 \leqslant k < 3.$$
So $k \in [0; 3[\cap \mathbb{Z}$ and $k \in [0; 1; 2]$
If we calculate all the possible sums $u + v$ of these roots, we will find 9 values for y, which is too much for an equation of degree 3.
$$\begin{aligned} z_1 &= e^0 = 1, \\ z_2 &= e^{\frac{2i\pi}{3}} = j, \\ z_3 &= e^{\frac{4i\pi}{3}} = \left(e^{\frac{2i\pi}{3}}\right)^2 = z_2^2 = j^2. \end{aligned}$$

So the possible sums $u + v$ of these roots is 9 values for y, which is too much for an equation of degree 3. we imposed the relationship
$$uv = -\frac{p}{3}.$$

So that if we perform one of the three possible choises for u, the other value v is well defined, so also x. Precisely, we have $v = \bar{u}$. Indeed, $uv = -p/3$ is real so

$$v = -\frac{p}{3u} = -\frac{p}{3u}.\frac{\bar{u}}{\bar{u}} = -\frac{p\bar{u}}{3|u|^2} = c\bar{u} \qquad \text{with} \qquad c \in \mathbb{R}.$$

But as v^3 is conjugated from u^3, we have $v^3 = c^3\bar{u}^3 = \bar{u}^3$, and $c = 1$, because

$$v^3 = -\frac{q}{2} \pm i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3},$$

involved

$$|u| = |v| = \sqrt[6]{\left(\frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3} = \left(-\frac{p}{3}\right)^{\frac{3}{6}} = \sqrt{-\frac{p}{3}},$$

to give

$$|u|^2 = -\frac{p}{3}, \quad \text{and} \quad c = 1.$$

we finally deduce the three real solutions:

$$\begin{aligned}
y_1 &= u_1 + v_1 = \sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}}, \\
y_2 &= u_2 + v_3 = j\sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}} + j^2\sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}}, \\
y_3 &= u_3 + v_2 = j^2\sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}} + j\sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}}.
\end{aligned} \tag{1.15}$$

We admit that:

$$\sqrt[3]{-\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}} = a+ib \quad \text{and} \quad \sqrt[3]{-\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(-\frac{p}{3}\right)^3}} = a-ib \quad \text{with} \quad a, b \in \mathbb{R}.$$

So

$$\begin{aligned}
y_1 &= a + ib + a - ib = 2a \Longrightarrow y_1 \in \mathbb{R}, \\
y_2 &= -(a + b\sqrt{3}) \Longrightarrow y_2 \in \mathbb{R}, \\
y_3 &= b\sqrt{3} - a \Longrightarrow y_3 \in \mathbb{R}.
\end{aligned} \tag{1.16}$$

□

Exemple 3. *[13]*
let's consider the equation $x^3 - 7x + 6 = 0$. and hence equation

$$X^2 + 6X + \frac{343}{27} = 0 \tag{1.17}$$

$\Delta = -\dfrac{400}{27}$. *The roots of (1.17) are*

$$\begin{cases} u^3 &= \dfrac{-6 + i\sqrt{\dfrac{400}{27}}}{2} = -3 + i\dfrac{10}{3\sqrt{3}}, \\ v^3 &= \dfrac{-6 - i\sqrt{\dfrac{400}{27}}}{2} = -3 - i\dfrac{10}{3\sqrt{3}}. \end{cases}$$

We find that the three cubic roots are

$$u_1 = 1 + \frac{2i\sqrt{3}}{3}, \quad u_2 = ju_1 = -\frac{3}{2} + \frac{i\sqrt{3}}{6}, \quad u_3 = j^2u_1 = \frac{1}{2} - \frac{5i\sqrt{3}}{6}.$$

To find the values of v, *we use the relation*

$$uv = \frac{7}{3},$$

and hence,

$$v = \frac{7\bar{u}}{3|u|^2},$$

and $|u| = \sqrt{1^2 + \dfrac{2i\sqrt{3}}{3}^2} = \dfrac{7}{3}$, *so* u_i, v_i *are conjugated*

$$v_1 = \bar{u_1} = 1 - \frac{2i\sqrt{3}}{3} \Longrightarrow x_1 = u_1 + v_1 = 2, \quad v_2 = \bar{u_2} \Longrightarrow x_2 = -3, \quad v_3 = \bar{u_3} \Longrightarrow x_3 = 1.$$

1.4 Sylvester's criteria

[16] Let $A = [a_{ij}]$ *be an* $n \times n$ *real symmetric matrix, and let* Δ_k *be the* k^{th} *principal minor of* A *for* $0 \leqslant k \leqslant n$. *Then:*

(a) *A is positive definite if and only if* $\Delta_k > 0$ *for* $k = 1, 2, ..., n$

(b) *A is negative definite if and only if* $(-1)^k\Delta_k > 0$ *for* $k = 1, 2, ..., n$.

Note that A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

The descriminant is defined as the form

$$\Delta_1 = a_{11}, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, ..., \Delta_n = det(A).$$

Exemple 4.

$$A = \begin{bmatrix} 10 & -1 & -3 \\ -1 & 4 & -1 \\ -3 & -1 & 5 \end{bmatrix}$$

We calculate the principal minors of A. We find that

$$\Delta_1 = 10, \quad \Delta_2 = 39, \quad \Delta_3 = 143.$$

Since all principal minors of A are positive, we conclude that A is positive definite.

Chapter 2

Mathematical Modeling

2.1 Single species population models

We present two simple models to describe the evolution of a population in time t.

2.1.1 Exponential growth model(Malthusian growth)

Exponential growth model [3] allows to describe the dynamics of one population density (or biomass) of a single species. If the population density at time t is represented by a real function $x : t \longrightarrow x(t)$, it is reasonable to assume that the function $x(t)$ is everywhere differentiable, denote by b the per capita birth rate and by d the per capita death rate, $(b-d)$ the difference between the birth and death rates is the per capita growth rate.
This model states that there is no migration and the rate of growth (with respect to time) is proportional to the population size. Then change in size from time t to $(t+\Delta t)$ is

$$\begin{aligned} x(t+\Delta t) - x(t) &= (b-d).x(t).\Delta t \\ \frac{x(t+\Delta t) - x(t)}{\Delta t} &= (b-d).x(t) \\ \lim_{\Delta t \longrightarrow 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} &= (b-d).x(t). \end{aligned}$$

Letting the net growth rate $r = (b-d),$, we obtain the Malthusian growth model

$$\frac{dx}{dt} = r.x(t) \tag{2.1}$$

This can be integrated between $[0, t]$

$$\begin{aligned}
\int_0^t \frac{dx}{dt} dt &= \int_0^t rx(t)dt \\
\int_{x(0)}^{x(t)} \frac{dx}{x} &= \int_0^t rdt \\
\ln x(t) - \ln x(0) &= rt \\
\ln \frac{x(t)}{x(0)} &= rt
\end{aligned}$$

hence, the solution is

$$x(t) = x(0)e^{rt}$$

We can infer three cases.

1. *If* $r > 0$, *the population will grow at exponential.*
2. *If* $r < 0$, *the population will diminish at exponential which tends towards extinction.*
3. *If* $r = 0$, *the population remains constant and equal to its initial value.*

2.1.2 The Logistic model

The logistic model [3] is given by

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}),$$

where K *is the carrying capacity of the population, it is the maximum number of individuals a particular environment. we divide both parts by* $x(K-x)$ *then multiply both parts by* dt *to obtain*

$$\frac{dx}{x(K-x)} = \frac{r}{K} dt.$$

Use partial fraction on the left hand side and get

$$\frac{dx}{x(K-x)} = \frac{1}{K}\left(\frac{1}{x} + \frac{1}{K-x}\right) dx,$$

so our equation takes the form

$$\frac{1}{K}\left(\int \frac{dx}{x} + \frac{1}{K}\int \frac{dx}{K-x}\right) = \frac{r}{K}\int dt.$$

Now we can integrate both parts and obtain

$$\frac{1}{K}\left(\ln(x) + \ln(K-x)\right) = \frac{rt}{K} + c.$$

- *If $x < 0$ or $x > K$ so logarithm of a negative number is undefined which means that the solution does not exist.*
- *If $x = 0$ or $x = K$ and hence $\frac{dx}{dt} = 0$ so the solution is constant.*
- *Assuming $0 < x < K$ so that we can drop absolute value signs from logarithms. If the population size at time $t = 0$ is x_0 we will find the value of*

$$c = \frac{1}{K}\left(\ln(x_0) + \ln(K - x_0)\right).$$

So now we have

$$\begin{aligned}
\frac{1}{K}\left(\ln(x) + \ln(K - x)\right) &= \frac{rt}{K} + \frac{1}{K}\left(\ln(x_0) + \ln(K - x_0)\right) \\
\ln\left(\frac{x}{K - x}\right) &= rt + \ln\left(\frac{x_0}{K - x_0}\right) \\
\ln\left(\frac{x(K - x_0)}{x_0(K - x)}\right) &= rt \\
\left(\frac{x(K - x_0)}{x_0(K - x)}\right) &= e^{rt}.
\end{aligned}$$

Now we solve the equation for x, it gives

$$\begin{aligned}
x(K - x_0) = x_0(K - x)e^{rt} &= Kx_0e^{rt} - xx_0e^{rt} \\
x(K - x_0 + x_0e^{rt}) &= Kx_0e^{rt}
\end{aligned}$$

and hence,

$$x(t) = \frac{Kx_0e^{rt}}{K - x_0 + x_0e^{rt}} = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}}$$

and when $t \longrightarrow \infty, x(t) \longrightarrow K$, we can discuss two cases:

1. *If $x(t) < K$, a rapid growth of the population.*
2. *If $x(t) \longrightarrow K$, a decrease growth of the population.*

2.2 Models for interacting populations

We consider a community of two population in this case the dynamics of each population is affected according to the type of interaction

2.2.1 Predator-prey

Lotka-Voltera [10] where the first is to suggest independently a predator-prey model .

$$\begin{cases} \dfrac{dx}{dt} &= x(a-by), \\ \dfrac{dy}{dt} &= y(cx-d), \end{cases} \tag{2.2}$$

where $x(t)$ represent the density of prey population and $y(t)$ is that of predator at time t. and a, b, c and d are positive parameters.
The model is based on the following assumptions:

(i) The prey grow exponentially with a the per capita growth rate in the absence of predation.

(ii) The effect of predation decreases the prey's per capita growth rate by $-by$.

(iii) The predator diminishes exponentially in the absence of any prey, hence d is the per capita rate death rate of predator in absence of the prey.

(iv) The prey's contribution to the predator's growth rate is cxy.

A general predator-prey model givin

$$\begin{cases} \dfrac{dX}{d\tau} &= f(X)X - g(X,Y)Y, \\ \dfrac{dY}{d\tau} &= h(X,Y)Y, \end{cases} \tag{2.3}$$

in this model $X(\tau)$ and $Y(\tau)$ represent respectively the population densities (or biomasses) of the prey and predator at the τ moment of time. $f(X)$ is the per capita net prey production in the absence of predation and $g(X,Y)$ the number of preys eaten per predator per unit time and $h(X,Y)$ measures the growth rate of predators. This model is based on the assumption functions: $\dfrac{\partial g}{\partial Y} \geq 0, \dfrac{\partial h}{\partial X} \geq 0$.
According [2] early population predator-prey are built on a basic assumption implicitely: the number of prey attacked to be proportional to the density of prey and the density of predator. While two experiments show that this assumption is false. the author in [2], suggeste the following model

$$\begin{cases} \dfrac{dX}{d\tau} &= \left(a_1 - b_1 X - \dfrac{m_1 Y}{\alpha_1 X + \beta_1 Y + \gamma_1}\right) X, \\ \dfrac{dY}{d\tau} &= \left(a_2 - \dfrac{m_2 Y}{X + k_1}\right) Y, \end{cases} \tag{2.4}$$

the model under name Beddington-DeAngelis functional response with predator-prey model, with the initial values $X(0) > 0$ and $Y(0) > 0$. The constants $a_1, a_2, b_1, m_1, m_2, \alpha_1, \beta_1, \gamma_1,$

and k_1 are the parameters of model and are assumed to be positive.

We have by placing $\tau = a_1 t, X(\tau) = x(t)/(b_1/a_1), Y(\tau) = y(t)/(m_2 b_1/a_1 a_2), a = (m_1 b_1)/a_1, b = a_2/a_1, \alpha = \alpha_1(a_1 m_2)/a_2), \beta = \beta_1(a_1 a_2)/m_2, \gamma = \gamma_1((b_1/a_1)^2(m_2 a_1/a_2),$ *and* $k = k_1(b_1/a_1).$

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & x(1-x) - \dfrac{axy}{\alpha x + \beta y + \gamma}, \\ \dfrac{dy}{dt} & = & b\left(1 - \dfrac{y}{x+k}\right) y, \end{array} \right. \tag{2.5}$$

Chapter 3

Bondedness of Solutions and Existence of an Attracting Set

We are interested in the existence and uniqueness of solution $(x(t), y(t))$ of the Beddington-DeAngelis with predator-prey model

$$\begin{cases} \frac{dx}{dt} &= x(1-x) - \dfrac{axy}{\alpha x + \beta y + \gamma}, \\ \frac{dy}{dt} &= b\left(1 - \dfrac{y}{x+k}\right) y. \end{cases} \tag{3.1}$$

We show that the positive quadrant is positively invariant for (3.1) and the solution $(x(t), y(t))$ satisfy the initial conditions $(x(0), y(0))$ is attracted by a bounded set $Int(\mathbb{R}^2_+)$.

3.1 Positively invariant quadrant

Definition 3.1. *The set $A \subseteq \mathbb{R}^n$ is called an invariant set of system (1.1) if*

$$x(0) \in A \Longrightarrow x(t) \in A, \quad \forall t \in \mathbb{R}.$$

Exemple 5. *any equilibrium point i.e., $\bar{x}$, where $f(\bar{x}) = 0$*

Definition 3.2. *A set A is a positively invariant set of system (1.1) if*

$$x(0) \in A \Longrightarrow x(t) \in A, \quad \forall t \geq 0.$$

Denote by $\mathbb{R}^2_+$ the positive quadrant of $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_i \geq 0, i = 1, 2\right\}$$

And denote by $Int(\mathbb{R}^2_+)$ the interior positive quadrant of $\mathbb{R}^2$, i.e.,

$$Int(\mathbb{R}^2_+) = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_i > 0, i = 1, 2\right\}$$

Lemma 5. *Positive quadrant $Int(\mathbb{R}_+^2)$ is positively invariant for system(3.1).*

Proof:
We show that the positive x-axis and y-axis are invariant for this system:

Consider an initial value on y-axis, that is $x(0) = 0, y(0) = y_0 \geq 0$, let $x(t) = 0$ for $t \geq 0$, and $y(t)$ the solution of

$$\frac{dy}{dt} = b\left(1 - \frac{y}{k}\right)y,$$

the $(0, y)$, is solution of system (3.1) for the initial value $(0, y_0)$ arbitrary in $\mathbb{R}^+$, this makes the positive part of y-axis positively invariant.

Similarly, suppose $x(0) = x_0 \geq 0, y(0) = 0$ and $(x(t), y(t))$, let $y(t) = 0$ for $t \geq 0$, and $x(t)$ the salution of

$$\frac{dx}{dt} = x(1 - x),$$

we can easily see that $(x, 0)$ is the solution of (3.1) for $(x_0, 0)$ as an initial value, therefor the positive part x-axis is also positivaly invariant. Now it remains to show that if $x(0) > 0$ and $y(0) > 0$ then the trajectory of the solution of (3.1) does not leave the positive quadrant. Suppose this result is not true, and proof that it leads to some contradiction. This means that we assume that there exists t_0 such that $x(t_0) < 0$ or $y(t_0) < 0$. if $x(t_0) < 0$ we have, $x(0) > 0$ accorrding to the Intermediate Value Theorem it exists $t_1 \in [0, t_0]$ such that $x(t_1) = 0$.
Then, the curve of the solution intersects the positive part of the axis $'Oy'$ which is another trajectory and therfor contradicts the uniqueness of the solution.
If $y(t_0) < 0$, the same reasoning applies and leads the contradiction. Therefore, densities $x(t)$ and $y(t)$ are positive for all $t \geq 0$ if $x(0) > 0$ and $y(0) > 0$.

3.2 Ultimate Boundedness and Attracting Set

Definition 3.3. *A solution $\phi(t, t_0, x_0, y_0)$ of system(3.1) is said to be ultimately bounded with respect to $\mathbb{R}_+^2$ if there exists a compact region $A \in \mathbb{R}_+^2$ and a finite time T $\quad (T = T(t_0, x_0, y_0))$ such that, for any $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+^2$,*

$$\phi(t, t_0, x_0, y_0) \in A, \qquad t > T.$$

Definition 3.4. *[5] A closed invariant set $\gamma \subset \mathbb{R}^n$ is called an attracting set if there exists some open neighborhood A of γ such that, for all $x_0 \in A, \phi(t, x_0) \in A$ for all $t \geq 0$ and $\phi(t, x_0) \longrightarrow \gamma$ as $t \longrightarrow \infty$.*

Theorem 3.1. *Let A be the defined by*

$$A = \left\{(x, y) \in (\mathbb{R}_+^2) : 0 \leqslant x \leqslant 1, 0 \leqslant x + y \leqslant L_1\right\},$$

where
$$L_1 = \frac{1}{4b}\left(5b + (1+b)^2(1+k)\right).$$
then

1. *A is positively invariant*
2. *all solutions of (3.1) initiating in $\mathbb{R}^2_+$ are ultimately bounded with respect to $\mathbb{R}^2_+$ and eventually enter the attracting set A.*

Proof: *Let $(x(0), y(0)) \in A$, we will show that $(x(t), y(t)) \in A$ for all $t \geq 0$: from Lemma 5. as, $(x(0)), y(0)) \in A, (x(t), y(t))$ is in Int($\mathbb{R}^2_+$). Then, we have show to that for all $t \geq 0, 0 \leqslant x(t) \leqslant 1$, and $0 \leqslant x(t) + y(t) \leqslant L_1$.*

1. *(a) First, We prove that for all $t \geq 0, 0 \leqslant x(t) \leqslant 1$.*
   We have $x > 0$ and $y > 0$ in Int($\mathbb{R}^2_+$); then every solution $\phi(t) = (x(t), y(t))$ of system(3.1), which starts in Int($\mathbb{R}^2_+$), satisfies the differential inequality $\frac{dx}{dt} \leqslant (1 - x(t))x(t)$. Thus, $x(t)$ may be compared with solutions of
   $$\begin{cases} \frac{du}{dt} &= (1-u(t))u(t) \\ u(0) &= x(0) > 0 \end{cases}$$
   so we have,
   $$u'(t) = u(t) - (u(t))^2$$
   and
   $$\frac{u'(t)}{[u(t)]^2} = \frac{1}{u(t)} - 1$$
   set
   $$z(t) = \frac{1}{u(t)}$$
   then
   $$z'(t) = -\frac{u'(t)}{[u(t)]^2}$$
   and $z(t)$ satisfies the equation
   $$-z'(t) = z(t) - 1$$
   hence
   $$z'(t) + z(t) = 1$$

all the solutions of the homegenous differential equation $z'(t) + z(t) = 0$, are of the form:

$$z_h(t) = ce^{-t}$$

particular solution of $z'(t)+z(t) = 1$ is obtained by constant variation method

$$\begin{aligned} z'(t) &= c'e^{-t} - ce^{-t} \\ c'e^{-t} - ce^{-t} + ce^{-t} &= 1 \\ c' &= e^t \\ c &= e^t \\ z_p(t) &= 1 \end{aligned}$$

Then the solution is:

$$\begin{aligned} u(t) &= \frac{1}{z_h(t) + z_p(t)} \\ u(t) &= \frac{1}{ce^{-t} + 1}. \end{aligned}$$

Since $u(0) = x(0) = \dfrac{1}{ce^{-0} + 1}$

$$\begin{aligned} x(0) &= \frac{1}{c+1} \\ c+1 &= \frac{1}{x(0)} \\ c &= \frac{1}{x(0)} - 1 \end{aligned}$$

with $x(0) \leqslant 1$, which implies that $c = \dfrac{1}{x(0)} - 1 \geq 0$. From the fact that $0 < e^{-t} \leqslant 1$ for $t \geq 0$, and we have $c \geq 0$ therefor $x(t) \leqslant 1$.
It follows that every nonnegative solution $\phi(t) = (x(t), y(t))$ of (3.1) satisfies

$$x(t) \leqslant 1 \qquad \forall t \geq 0.$$

(b) For all $t \geq 0$, and $(x(t), y(t)) \in \mathbb{R}^2, 0 \leqslant x + y \leqslant L_1$.
We define the function $\sigma(t) = x(t) + y(t)$;

$$\frac{d\sigma}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \left(1 - x - \frac{ay}{\alpha x + \beta y + \gamma}\right) x + b\left(1 - \frac{y}{x+k}\right) y,$$

then

$$\frac{d\sigma}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \leqslant (1-x)x + b\left(1 - \frac{y}{x+k}\right) y$$

Thus, as

$$\max_{[0,1]}(1-x)x = \frac{1}{4},$$

we have

$$\frac{d\sigma}{dt} \leqslant \frac{1}{4} + b\left(1 - \frac{y}{x+k}\right)y;$$

then

$$\frac{d\sigma}{dt} \leqslant \frac{1}{4} + b\left(1 - \frac{y}{x+k}\right)y + \sigma(t) - \sigma(t),$$

$\sigma(t) = x(t) + y(t)$ *which implies that*

$$\frac{d\sigma}{dt} + \sigma(t) \leqslant \frac{1}{4} + \left(b - \frac{by}{x+k}\right)y + x + y$$

$$\frac{d\sigma}{dt} + \sigma(t) \leqslant \frac{1}{4} + x + \left(b + 1 - \frac{by}{x+k}\right)y,$$

Since in $(1)(a)$*,* $x(t) \leqslant 1$ *for all* $t \geq 0$*, we obtain*

$$\frac{d\sigma}{dt} + \sigma(t) \leqslant \frac{5}{4} + \left(b + 1 - \frac{by}{1+k}\right)y.$$

Moreover, set

$$g(y) \quad = \quad (b+1)y - \frac{by^2}{1+k}, \qquad y \in \mathbb{R}_+$$

with

$$g'(y) = \frac{-2by}{1+k} + (b+1),$$

which give

$$(b+1)(1+k) \quad = \quad 2by$$

so,

$$y = \frac{(b+1)(1+k)}{2b}$$

Thus,

$$\max_{\mathbb{R}_+}[g(y)] = \max_{\mathbb{R}_+}\left[\left(b + 1 - \frac{by}{1+k}\right)y\right] = \frac{1}{4b}(1+b)^2(1+k).$$

Consequently

$$\frac{d\sigma}{dt}+\sigma(t)\leqslant L_1,$$

By applying Lemma 1. *then we get*

$$\forall t\geq \bar{T}\geq 0:\sigma(t)\leqslant L_1-\left(L_1-\sigma\left(\bar{T}\right)\right)e^{-(t-\bar{T})}.$$

Then, if $\bar{T}=0$, *for* $(x(0),y(0))\in A$,

$$\sigma(t)=x(t)+y(t)\leqslant L_1.$$

Then

$$(x(t),y(t))\in A,\qquad \forall t\geq 0.$$

2. *We have to prove that, for* $(x(0),y(0))\in\mathbb{R}_+^2$, $(x(t),y(t))\longrightarrow A$ *when* $t\longrightarrow +\infty$. *We will show that* $\overline{\lim}_{t\longrightarrow+\infty}x(t)\leqslant 1$ *and* $\overline{\lim}_{t\longrightarrow+\infty}(x(t)+y(t))\leqslant L_1$.

   (a) *For,* $\overline{\lim}_{t\longrightarrow+\infty}x(t)\leqslant 1$, *we have from the system*

   $$\frac{dx}{dt}\leqslant(1-x(t))x(t)$$

   and using lemma 2 *with* $u(t)$ *from* $1,a)$ *we obtain*

   $$x(t)\quad\leqslant\quad u(t)$$

   therefor
   $\overline{\lim}_{t\longrightarrow+\infty}x(t)\leqslant\lim_{t\longrightarrow+\infty}u(t)=1.$

   (b) *Let* $\varepsilon>0$, *and* $T_1>0$ *exists, such that*

   $$x(t)\leqslant 1+\frac{\varepsilon}{2}\qquad\forall t\geq T_1.$$

   From (1) with $\bar{T}=T_1$, *we get for all* $t\geq T_1$,

   $$\begin{aligned}\sigma(t)&=x(t)+y(t)\\&\leqslant L_1-(L_1-\sigma(T_1))e^{-(t-T_1)}\\&\leqslant L_1-\left\{L_1e^{T_1}-(x(T_1)+y(T_1))e^{T_1}\right\}e^{-t}\\&\leqslant L_1-\left\{L_1-(x(T_1)+y(T_1))e^{T_1}\right\}e^{-t}\end{aligned}$$

   $$\sigma(t)=x(t)+y(t)\leqslant\left(L_1+\frac{\varepsilon}{2}\right)-\left\{\left(L_1+\frac{\varepsilon}{2}\right)-(x(T_1)+y(T_1))e^{T_1}\right\}e^{-t},\qquad t\geq T_1\geq 0.$$

   Let $T_2\geq T_1$

$$\begin{array}{rcl}\sigma(t) & = & x(t)+y(t) \\ & \leqslant & \left(L_1+\frac{\varepsilon}{2}\right)-\left\{\left(L_1+\frac{\varepsilon}{2}\right)-(x(T_1)+y(T_1))e^{T_1}\right\}e^{-t},\end{array}$$

because $t \geq T_1$ *and* $T_2 \geq T_1$ *so,*

$$\left|\left(L_1+\frac{\varepsilon}{2}\right)-(x(T_1)+y(T_1))e^{T_1}\right|e^{-t} \leqslant \left|\left(L_1+\frac{\varepsilon}{2}\right)-L_1\right| \leqslant \frac{\varepsilon}{2} \qquad \forall t \geq T_2.$$

Then

$$x(t)+y(t) \leqslant L_1+\varepsilon \qquad \forall t \geq T_2.$$

Hence,

$$\overline{\lim_{t\longrightarrow+\infty}}(x(t)+y(t)) \leqslant L_1.$$

In the latter we deduce that system (3.1) is dissipative (solutions are bounded) in $\mathbb{R}_+^2$.

Chapter 4

Stability

4.1 Linear stability

4.1.1 Equilibrium points

we find equilibrium points of the system in order to study their stability.

$$\begin{cases} \frac{dx}{dt} &= x(1-x) - \dfrac{axy}{\alpha x + \beta y + \gamma}, \\ \frac{dy}{dt} &= b\left(1 - \dfrac{y}{x+k}\right)y. \end{cases}$$

We solve

$$\begin{cases} x(1-x) - \dfrac{axy}{\alpha x + \beta y + \gamma} &= 0, \\ b\left(1 - \dfrac{y}{x+k}\right)y &= 0, \end{cases}$$

which implies that $by = 0$ *and hence* $y = 0$ *or*

$$1 - \frac{y}{x+k} = 0$$

therefore

$$y = x + k.$$

Hence, if $y = 0$:

$$x(1-x) = 0 \quad \textit{means} \quad x = 0 \quad \textit{or} \quad x = 1.$$

Or, if

$$y \;=\; x+k,$$

by replacing y in the first equation we get

$$x(1-x) - \frac{ax(x+k)}{\alpha x + \beta(x+k) + \gamma} = 0$$

then we get

$$(x-x^2)(\alpha x + \beta(x+k) + \gamma) - ax^2 + axk \;=\; 0,$$

hence

$$\alpha x^2 + \beta x^2 + \beta xk + \gamma x - \alpha x^3 - \beta x^3 - \beta k x^2 - ax^2 + axk \;=\; 0,$$

then

$$-x^3(\alpha+\beta) + x^2(\alpha+\beta-\beta k - a) + x(\beta k + \gamma + ak) \;=\; 0,$$

therefore

$$-x(x^2(\alpha+\beta) + x(\alpha+\beta-\beta k-a) + (\beta k+\gamma+ak)) \;=\; 0$$

which implies that

$$x = 0 \quad or \quad x = \frac{-(\alpha+\beta-\beta k-a) \pm \sqrt{((\alpha+\beta-\beta k-a)^2 - 4(\alpha+\beta)(\beta k+\gamma+ak))}}{2(\alpha+\beta)}$$

so,

$$\begin{cases} y &= 0 \\ x &= 0 \end{cases} \vee \begin{cases} y &= 0 \\ x &= 1 \end{cases} \vee \begin{cases} y &= x+k \\ x &= 0. \end{cases}$$

Then we conclude that the trivial equilibria are

$$P_1(0,0), \qquad P_2(0,k), \qquad P_3(1,0).$$

The other equilibria are defined by the system

$$\begin{cases} \dfrac{ay}{\alpha x + \beta y + \gamma} &= 1-x, \\ y &= x+k. \end{cases} \tag{4.1}$$

Proposition 4.1. *The system (3.1) has a unique interior equilibria* $P^*(x^*, y^*)$ *(i.e.,* $x^* > 0$ *and* $y^* > 0$*) if the following condition is verified:*

$$k(a-\beta) \leqslant \gamma. \tag{4.2}$$

Proof. *We remplace the value of the unknown* y *in the second equation of (3.1) in the first one, then*

$$a(x+k) = (1-x)((\alpha+\beta)x + (\beta k + \gamma)),$$

and we obtain

$$(\alpha+\beta)x^2 + (\beta k + \gamma + a - \alpha - \beta)x + ak - \beta k - \gamma = 0. \tag{4.3}$$

(4.3) is a second degree equation, so in order to solve it we need to determine the sign of the discriminant

$$\begin{array}{rcl} \Delta & = & (\beta k + \gamma + a - \alpha - \beta)^2 - 4(ak - \beta k - \gamma)(\alpha+\beta) \\ & = & ((\beta k + \gamma + a) - (\alpha+\beta))^2 + 4((\beta k + \gamma) - ak)(\alpha+\beta) \\ & = & (\beta k + \gamma + a)^2 + (\alpha+\beta)^2 - 2(\beta k + \gamma + a)(\alpha+\beta) \\ & + & 4(\beta k + \gamma + a)(\alpha+\beta) - 4a(k+1)(\alpha+\beta) \\ & = & ((\beta k + \gamma + a) + (\alpha+\beta))^2 - 4a(1+k)(\alpha+\beta). \end{array}$$

Therefore, if (4.2) holds (i.e.,$\beta k + \gamma \geq ak$*), then*

$$\begin{array}{rcl} \Delta & = & ((\beta k + \gamma) + a + (\alpha+\beta))^2 - 4a(1+k)(\alpha+\beta) \\ & \geq & (a(k+1) + (\alpha+\beta))^2 - 4a(1+k)(\alpha+\beta) \\ & \geq & (a(k+1) + (\alpha+\beta))^2 \geq 0. \end{array} \tag{4.4}$$

Consequently, Δ *is positive, and the system (3.1) has two other equilibriums* $P_1^*(x_1, y_1)$ *and* $P_2^*(x_2, y_2)$*, where*

$$\begin{array}{rcl} x_{1,2} & = & -\dfrac{((\beta k + \gamma) - (\alpha+\beta) + a) \pm \sqrt{\Delta}}{2(\alpha+\beta)}, \\ y_{1,2} & = & (x_{1,2} + k). \end{array}$$

however one of these equilibriums is not in $(\mathbb{R}_+^2)$*. In deed, let*

$$x_2 = -\frac{((\beta k + \gamma + a) - (\alpha+\beta)) - \sqrt{\Delta}}{2(\alpha+\beta)},$$

then

$$2x_2 = 1 - \frac{((\beta k + \gamma + a) + \sqrt{\Delta})}{(\alpha+\beta)},$$

from (4.4) we have,

$$\sqrt{\Delta} \geq |a(k+1) - (\alpha+\beta)|,$$

and due to (4.2)

$$2x_2 \leqslant 1 - \frac{(a(k+1) + |a(k+1) - (\alpha+\beta)|)}{(\alpha+\beta)},$$

which implies that

(1) *if* $a(k+1) < (\alpha+\beta)$

$$2x_2 \leqslant 1 - \frac{a(k+1) - a(k+1) + (\alpha+\beta)}{(\alpha+\beta)} \leqslant 0;$$

(2) *if* $a(k+1) > (\alpha+\beta)$,

$$2x_2 \leqslant 1 - \frac{a(k+1) + a(k+1) - (\alpha+\beta)}{(\alpha+\beta)} \leqslant 2 - 2\frac{a(k+1)}{\alpha+\beta} \leqslant 0.$$

it results that $P_2^*(x_2, y_2)$ *is not in* $(\mathbb{R}_+^2)$, *remains to verify that* $x_1 > 0$ *in order to have (and only one) equilibrium point in the interior of the positive quadrant, we know that* x_1 *and* x_2 *satisfy*

$$x_1 x_2 = ak - \beta k - \gamma \leqslant 0;$$

then the first point $P_1^*(x_1, y_1)$ *is in* $(\mathbb{R}_+^2)$.

4.1.2 Stability of equilibria

The Jacobian matrix of the system (3.1) at equilibrium P_i *is given by*

$$J(P_i) = \begin{pmatrix} 1 - 2x - \dfrac{ay(\beta y+\gamma)}{(\alpha x+\beta y+\gamma)^2} & -\dfrac{ax(\alpha x+\gamma)}{(\alpha x+\beta y+\gamma)^2} \\ b(\dfrac{y}{x+k})^2 & b - \dfrac{2by}{x+k} \end{pmatrix}$$

(1) *At* $P_0(0,0)$,

$$J(P_0) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = 1, \qquad \lambda_2 = b.$$

Because, (λ_1, λ_2) *are positive, then* $P_0(0,0)$, *is an unstable node.*

(2) $P_1(0,k)$,

$$J(P_1) = \begin{pmatrix} 1 - \dfrac{ak}{\beta k + \gamma} & 0 \\ b & -b \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = 1 - \frac{ak}{\beta k + \gamma} = \frac{-k(a-\beta)+\gamma}{\beta k + \gamma}, \qquad \lambda_2 = -b < 0.$$

Then, we have

(a) if $k(a-\beta) > \gamma$, $P_1(0,k)$ *is stable node;*

(b) if $k(a-\beta) \leqslant \gamma$, $P_1(0,k)$ *is unstable point.*

(3) At $P_2(1,0)$,

$$J(P_2) = \begin{pmatrix} -1 & -\dfrac{a}{\alpha + \gamma} \\ 0 & b \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = -1 < 0, \qquad \lambda_2 = b > 0.$$

Then the equilibrium $P_2(1,0)$ *is a saddle point.*

Around $P^*(x^*, y^*)$*, the Jacobien matrix takes the form*

$$J(P^*) = \begin{pmatrix} 1 - 2x^* - \dfrac{ay^*(\beta y^* + \gamma)}{(\alpha x^* + \beta y^* + \gamma)^2} & -\dfrac{ax^*(\alpha x^* + \gamma)}{(\alpha x^* + \beta y^* + \gamma)^2} \\ b & -b \end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - trJ(P^*)\lambda + \det J(P^*) = 0,$$

where

$$\begin{aligned} \det J(P^*) &= -b\left(1 - 2x^* - \frac{ay^*(\beta y^* + \gamma)}{(\alpha x^* + \beta y^* + \gamma)^2}\right) + ab\frac{x^*(\alpha x^* + \gamma)}{(\alpha x^* + \beta y^* + \gamma)^2} \\ &= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{ay^*(\beta y^* + \gamma) \\ &+ (2x^* - 1)(\alpha x^* + \beta y^* + \gamma)^2 + ax^*(\alpha x^* + \gamma)\} \\ &= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{(x^* - 1)(\alpha x^* + \beta y^* + \gamma)^2 \\ &+ x^*(\alpha x^* + \beta y^* + \gamma)^2 + ay^*(\beta y^* + \gamma) + ax^*(\alpha x^* + \gamma)\}. \end{aligned}$$

From (4.1) we get

$$\begin{aligned}
\det J(P^*) &= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{-ay^*(\alpha x^* + \beta y^* + \gamma) + x^*(\alpha x^* + \beta y^* + \gamma)^2 \\
&+ ay^*(\beta y^* + \gamma) + ax^*(\alpha x^* + \gamma)\} \\
&= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{-a\alpha x^* y^* + x^*(\alpha x^* + \beta y^* + \gamma)^2 + ax^*(\alpha x^* + \gamma)\} \\
&= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{x^*(\alpha x^* + \beta y^* + \gamma)^2 + ax^*(\alpha x^* - \alpha y^* + \gamma)\} \\
&= \frac{b}{(\alpha x^* + \beta y^* + \gamma)^2}\{x^*(\alpha x^* + \beta y^* + \gamma)^2 - ax^*(\alpha k - \gamma)\};
\end{aligned}$$

then

$$\det J(P^*) = \frac{bx^*}{(\alpha x^* + \beta y^* + \gamma)^2}\left\{(\alpha x^* + \beta y^* + \gamma)^2 - a(\alpha k - \gamma)\right\}. \tag{4.5}$$

If $\det J(P^*)$ *is positive, then the stability of the interior equilibrium* P^* *is determined by the sign of* $tr J(P^*)$. *We observe that* $\det J(P^*)$ *is positive if*

$$\left\{(\alpha x^* + \beta y^* + \gamma)^2 - a(\alpha k - \gamma)\right\} > 0.$$

To simplify, we developed $\det J(P^*)$ *respecting one variable, from 4.1; then*

$$\begin{aligned}
(\alpha x^* + \beta y^* + \gamma)^2 - a(\alpha k - \gamma) &= \left(\frac{a(x^* + k)}{1 - x^*}\right)^2 - a(\alpha k - \gamma) \\
&= \frac{a^2(x^* + k)^2}{(1 - x^*)^2} - a(\alpha k - \gamma),
\end{aligned}$$

which implies that $\det J(P^*)$ *has the same sign of*

$$a(x^* + k)^2 + (1 - x^*)(\gamma - \alpha k).$$

We rewrite

$$a(x^*)^2 + 2kax^* + ak^2 + (\gamma - \alpha k)\left((x^*)^2 - 2x^* + 1\right).$$

Let

$$f(x) = (a + \gamma - \alpha k)x^2 + 2(ak + \alpha k - \gamma)x + (\gamma - \alpha k + ak^2).$$

The discriminant is

$$\begin{array}{rcl}
\Delta' & = & (ak+\alpha k-\gamma)^2 - 4(a+\gamma-\alpha k)(\gamma-\alpha k+ak^2)\\
& = & (a+\alpha)^2k^2 - 2\gamma k(a+\alpha)+\gamma^2-(a+\gamma)(ak^2+\gamma)+(a+\gamma)\alpha k+\alpha k(ak^2+\gamma-\alpha k)\\
& = & (a+\alpha)^2k^2 - 2\gamma k(a+\alpha)+\gamma^2-(a+\gamma)(ak^2+\gamma)+(ak^2+a+2\gamma)\alpha k-(\alpha k)^2\\
& = & (a^2+2a\alpha+\alpha^2)k^2-2\gamma k(a+\alpha)+\gamma^2\\
& - & (a^2k^2+a(1+k^2)\gamma+\gamma^2)+a\alpha k^3+\gamma+(a+2\gamma)\alpha k-\alpha^2k^2,\\
& = & 2a\alpha k^2-2a\gamma k-a\gamma k^2-a\gamma+a\alpha k^3+a\alpha k\\
& = & a\left\{\alpha k^3+(2\alpha-\gamma)k^2+(\alpha-\gamma)k-\gamma\right\}\\
& = & a\left\{\alpha k(k^2+2k+1)-\gamma(k^2+2k+1)\right\}\\
& = & a(\alpha k-\gamma)(k+1)^2.
\end{array}$$

We get three cases.

(1) If $\alpha k<\gamma, \Delta'$ is negative, $f(x)$ has the same sign of $(a+\gamma-\alpha k)$, and we have $(a+\gamma-\alpha k)>a>0$.

   Then, $\det J(P^*)$ *is positive.*

(2) If $\alpha k>\gamma, \Delta'$ is positive and $\det J(P^*)$ *has at least two solutions x_1 and x_2, then*
   - *(a) if $x\in]-\infty,x_1]\bigcup[x_2,+\infty[, \det J(P^*)$ has the same sign of $(a+\gamma-\alpha k)$;*
   - *(b) if $x\in]x_1,x_2[$ it has sign of $-(a+\gamma-\alpha k)$.*

(3) If $\alpha k=\gamma$, then

$$f(x)=ax^2+2akx+ak^2=a(x+k)^2>0.$$

   Then, $\det J(P^*)$ *is positive.*

Remark 4.1. *From the expression (4.5), we find that* $\det J(P^*)$ *is positive, if $\alpha k\leqslant\gamma$, hence the eigenvalues associated to P^* have the same sign.*

To determine the sign of these eigenvalues, it suffices to determine the sign of $trJ(P^*)$,

$$\begin{array}{rcl}
trJ(P^*) & = & 1-2x^*-\dfrac{ay^*(\beta y^*+\gamma)}{(\alpha x^*+\beta y^*+\gamma)^2}-b\\
& = & \dfrac{1}{(\alpha x^*+\beta y^*+\gamma)^2}\left\{(1-2x^*-b)(\alpha x^*+\beta y^*+\gamma)^2-ay^*(\beta y^*+\gamma)\right\}\\
& = & \dfrac{1}{(\alpha x^*+\beta y^*+\gamma)^2}\left\{(1-x^*)(\alpha x^*+\beta y^*+\gamma)^2-(x^*+b)(\alpha x^*+\beta y^*+\gamma)^2\right.\\
& - & \left. ay^*(\beta y^*+\gamma)\right\}.
\end{array}$$

From (4.1), we get

$$
\begin{aligned}
trJ(P^*) &= \frac{1}{(\alpha x^* + \beta y^* + \gamma)^2}\left\{ay^*(\alpha x^* + \beta y^* + \gamma)\right. \\
&- \left.(x^* + b)(\alpha x^* + \beta y^* + \gamma)^2 - ay^*(\beta y^* + \gamma)\right\} \\
&= \frac{1}{(\alpha x^* + \beta y^* + \gamma)^2}\left\{a\alpha x^* y^* - (x^* + b)(\alpha x^* + \beta y^* + \gamma)^2\right\} \\
&= \alpha x^* \frac{ay^*}{(\alpha x^* + \beta y^* + \gamma)^2} - (x^* + b) \\
&= \alpha x^* \frac{ay^*(1 - x^*)^2}{a^2(y^*)^2} - (x^* + b) \\
&= \alpha x^* \frac{(1 - x^*)^2}{ay^*} - (x^* + b) \\
&= \frac{1}{a(x^* + k)}\left\{\alpha x^*(1 - x^*)^2 - a(x^* + b)(x^* + k)\right\} \\
&= \frac{1}{a(x^* + k)}\left\{\alpha(x^*)^3 - (a + 2\alpha)(x^*)^2 - (a(b + k) - \alpha)x^* - abk\right\}.
\end{aligned}
$$

Let

$$P_3(x) = \alpha x^3 - (a + 2\alpha)x^2 - (a(b + k) - \alpha)x - abk.$$

$\alpha k < \gamma$ ensures that the determinant is positive, so the stability of P^* is related to sign of $trJ(P^*) = \dfrac{1}{a(x^* + k)}P_3(x)$.

Lemma 6. *If $\alpha k < \gamma$ is verified, the interior equilibrium $P^*(x^*, y^*)$ is locally asymptotically stable if $P_3(x^*) < 0$ and it is unstable if $P_3(x^*) > 0$.*

We use the Cadran's methode [13] to solve the cubic equation $P_3(x^*) = 0$. Then we consider the equation

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{4.6}$$

with $a_3 = \alpha, a_2 = -(a + 2\alpha), a_1 = -(a(b + k) - \alpha), a_0 = -abk$. Making the substitution $y = a_3x + a_2/3$ reduces the equation to the standard from

$$y^3 - py - q = 0, \tag{4.7}$$

where p and q depend on a_3, a_2, a_1, a_0

$$
\begin{aligned}
p &= -a_1a_3 + \frac{a_2^2}{3}, \\
q &= -a_0a_3^2 - 2\left(\frac{a_2}{3}\right)^3 + \frac{a_3a_2a_1}{3}.
\end{aligned}
$$

Let

$$y = u + v.$$

$$y^3 - py - q = (u+v)^3 - 3uv(u+v) - (u^3 + v^3) = 0.$$

And from there

$$\begin{aligned} 3uv &= p, \\ u^3 + v^3 &= q. \end{aligned}$$

Then

$$\begin{aligned} u^3 v^3 &= \left(\frac{p}{3}\right)^3, \\ u^3 + v^3 &= q, \end{aligned}$$

and we obtain that u^3 *and* v^3 *are solutions of the quadratic equation*

$$z^2 - qz + \left(\frac{p}{3}\right)^3 = 0. \qquad (4.8)$$

Then we constitute three cases.

(1) if $27q^2 - 4p^3 > 0$*, then (4.8) admits two real roots* u^3, v^3 *such that*

$$\begin{aligned} u^3 &= \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}, \\ v^3 &= \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}, \end{aligned}$$

each admits a single real cubic root

$$\begin{aligned} u &= \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}, \\ v &= \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}. \end{aligned}$$

From lemma 3 *we deduce the unique real root of the equation (4.7) is*

$$y = \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}};$$

then (4.6) has one real root

$$r_0 = \frac{1}{a_3}\left(\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} - \frac{a_2}{3}\right).$$

So, we have $P_3(x) < 0$ *if* $0 < x < r_0$, *and* $P_3(x) > 0$ *if* $r_0 < x$.

(2) if $27q^2 - 4p^3 < 0$, *we find first both* u^3, v^3 *roots of (4.8), which are complex conjugates, then extract their cubic roots* u, v. *And* u^3 *admits three cubic roots:* u, ju, j^2u *and even for* v.

$$\begin{aligned} u_1 &= \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} & v_1 &= \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} \\ u_2 &= j\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} & v_2 &= j\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} \\ u_3 &= j^2\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} & v_3 &= j^2\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}. \end{aligned}$$

So the possible sums of these roots is 9 *values for* y, *which is too much for an equation of degree* 3. *we imposed the relationship*

$$uv = \frac{p}{3}.$$

So that if we perform one of the three possible choises for u, *the other value* v *is well defined, so also* x. *Precisely, we have* $v = \bar{u}$. *Indeed,* $uv = p/3$ *is real so*

$$v = \frac{p}{3u} = \frac{p}{3u}.\frac{\bar{u}}{\bar{u}} = \frac{p}{3|u|^2}\bar{u} = c\bar{u} \qquad \textit{with} \qquad c \in \mathbb{R}.$$

But as v^3 *is conjugated from* u^3, *we have* $v^3 = c^3\bar{u}^3 = \bar{u}^3$, *and* $c = 1$ *we finally deduce the three real solutions: (lemma* 4*)*

$$\begin{aligned} y_1 &= u_1 + v_1 = \sqrt[3]{\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}, \\ y_2 &= u_2 + v_3 = j\sqrt[3]{\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + j^2\sqrt[3]{\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}, \\ y_3 &= u_3 + v_2 = j^2\sqrt[3]{\frac{q}{2} + i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + j\sqrt[3]{\frac{q}{2} - i\sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}. \end{aligned}$$

Then there are three real roots: r_1, r_2 *and* r_3.
we know that $P_3(x) = (x - r_1)(x - r_2)(x - r_3)$, *and therefore,* $r_1r_2r_3 = abk > 0$ *then one of them is positive, then*

(a) if $r_2 < r_3 < 0 < r_1$, then $P_3(x) > 0$ if $0 < r_1 < x$, and $P_3(x) < 0$ if $0 < x < r_1$;

(b) if $0 < r_1 < r_2 < r_3$, then $P_3(x) > 0$ if $r_1 < x < r_2$, or $r_3 < x$, and $P_3(x) < 0$ if $x < r_1$, and $r_2 < x < r_3$.

(3) if $27q^2 = 4p^3$, then

$$\begin{aligned} u^3 &= q/2, \\ v^3 &= q/2, \end{aligned}$$

then (4.7) admits

$$y_0 = 2\sqrt[3]{\frac{q}{2}},$$

and

$$y_{1,2} = j\sqrt[3]{\frac{q}{2}} + j^2\sqrt[3]{\frac{q}{2}},$$

and $j^2 + j + 1 = 0 \Leftrightarrow j^2 + j = -1$ so,

$$y_{1,2} = (j^2 + j)\sqrt[3]{\frac{q}{2}} = -\sqrt[3]{\frac{q}{2}}.$$

then (4.6) there are one real root positive $r_0 = \dfrac{1}{a_3}(y_0 - \dfrac{a_2}{3})$, and a double root $r_{1,2} = \dfrac{1}{a_3}(y_{1,2} - \dfrac{a_2}{3})$; we also have $P_3(x) < 0$ if $0 < x < r_0$, and $P_3(x) > 0$ if $r_0 < x$.

Remark 4.2. *In [11] we find:* (b) *if $0 < r_1 < r_2 < r_3$, then $P_3(x) > 0$ if $0 < r_1 < x$, and $P_3(x) < 0$ if $x < r_1$, which is different from the conclution we are lead to in* (b).

4.2 Global stability

Lapunov function is one of the methods in study the stability and in this chapter, through this function, we will prove globally asymptotically stable of system (3.1).

Theorem 4.1. *The interior equilibrium $P^*(x^*, y^*)$ is globally asymptotically stable if*

$$\begin{aligned} \beta &< \alpha, && (4.9) \\ 2aL_1 &< \gamma, && (4.10) \\ (\beta k + \gamma)(\alpha L_1 + \gamma)(1 + k) &< 4\beta k^2\gamma^2, && (4.11) \\ \alpha(1 + 2k) - \beta(1 + k) &< \gamma, && (4.12) \\ a - 4\beta L_1 &< 4\gamma. && (4.13) \end{aligned}$$

Proof. *Theorem 1.3 gives a result for solution $x = 0$ to be globaly asymptotically stable. In our cas we are studying the global stability at $(x^*, y^*) \neq 0$, by this a simple changes variables which*

$$x(t) = x'(t) + x^*, \quad \textit{and} \quad y(t) = y'(t) + y^*,$$

then

$$\begin{aligned}
\frac{dx'}{dt} &= (x' + x^*)(1 - (x' + x^*)) - \frac{a(x' + x^*)(y' + y^*)}{\alpha x + \beta y + \gamma}, \\
\frac{dy'}{dt} &= b\left(1 - \frac{y' + y^*}{(x' + x^*) + k}\right) y.
\end{aligned}$$

we choose a Liapunov function

$$V(x', y') = (\alpha x^* + \beta y^* + \gamma)\left(x' - x^* - x^* \ln\left(\frac{x'}{x^*}\right)\right) + \frac{a}{b}(x^* + k)\left(y' - y^* - y^* \ln\left(\frac{y'}{y^*}\right)\right).$$

where

$$\begin{aligned}
V_1(x', y') &= (\alpha x^* + \beta y^* + \gamma)\left(x' - x^* - x^* \ln\left(\frac{x'}{x^*}\right)\right), \\
V_2(x', y') &= \tfrac{a}{b}(x^* + k)\left(y' - y^* - y^* \ln\left(\frac{y'}{y^*}\right)\right).
\end{aligned} \tag{4.14}$$

we find that V has to satisfy

(a) $V(x^*, y^*) = 0$, *if* $x' = y' = 0$.

(b) $V(x', y') > 0$, *for all* $(x', y') \neq (0, 0)$, *we have that* $(\alpha, \beta, \gamma, a, b)$ *are positive and* $(x^*, y^*) > 0$, *under the condition (4.2), and hence*

we suppose

$$\begin{aligned}
f(x') &= x' - x^* - x^* \ln\left(\frac{x'}{x^*}\right) \\
&= x^*\left(\frac{x'}{x^*} - 1 - \ln\left(\frac{x'}{x^*}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
g(y') &= y' - y^* - y^* \ln\left(\frac{y'}{y^*}\right), \\
&= y^*\left(\frac{x'}{y^*} - 1 - \ln\left(\frac{y'}{y^*}\right)\right),
\end{aligned}$$

we making $\alpha = \dfrac{1}{x^*}, \beta = \dfrac{1}{y^*}$, *so*

$$\begin{aligned} f(\alpha) &= \frac{1}{\alpha}\left(\alpha x' - 1 - \ln(\alpha x')\right), \\ g(\beta) &= \frac{1}{\beta}\left(\beta y' - 1 - \ln(\beta y')\right), \end{aligned}$$

befor to study variation of this function f, g *we get that* $f(\alpha)$ *and* $g(\beta)$ *are positive for all* $(\alpha, \beta) \in \mathbb{R}^+$, *therefor* $V(x^*, y^*) > 0$.

(c) $\dfrac{dV(x', y')}{dt} < 0$, *for all* $x \neq 0$,
we know,

$$\frac{dV_i}{dt} = \frac{dV_i}{dx'}\dot{x}' + \frac{dV_i}{dy'}\dot{y}', \qquad i = 1, 2.$$

$$\begin{aligned} \frac{dV_1}{dt}(x', y') &= (\alpha x^* + \beta y^* + \gamma)\left(1 - \frac{x^*}{x'}\right)\dot{x}', \\ \frac{dV_2}{dt}(x', y') &= \tfrac{a}{b}(x^* + k)\left(1 - \frac{y^*}{y'}\right)\dot{y}', \end{aligned}$$

and using (4.1), we get

$$\begin{aligned} \frac{dV_1}{dt}(x', y') &= \frac{(\alpha x^* + \beta y^* + \gamma)(x' - x^*)}{x'}\left(1 - x' - \frac{ay'}{\alpha x' + \beta y' + \gamma}\right)x \\ &= (\alpha x^* + \beta y^* + \gamma)(x' - x^*)\left(\frac{ay^*}{\alpha x^* + \beta y^* + \gamma} + x^* - x' - \frac{ay'}{\alpha x' + \beta y' + \gamma}\right) \\ &= -(\alpha x^* + \beta y^* + \gamma)(x' - x^*)^2 \\ &+ (\alpha x^* + \beta y^* + \gamma)(x' - x^*)\left(\frac{ay^*(\alpha x' + \beta y' + \gamma) - ay'(\alpha x^* + \beta y^* + \gamma)}{(\alpha x^* + \beta y^* + \gamma)(\alpha x' + \beta y' + \gamma)}\right) \\ &= -(\alpha x^* + \beta y^* + \gamma)(x' - x^*)^2 \\ &- (x' - x^*)(y' - y^*)\left(\frac{a\alpha x' + a\gamma}{\alpha x' + \beta y' + \gamma}\right) + \left(\frac{a\alpha y'}{\alpha x' + \beta y' + \gamma}\right)(x' - x^*)^2, \end{aligned}$$

Similarly,

$$
\begin{aligned}
\frac{dV_2}{dt}(x',y') &= \frac{a}{b}(x^*+k)b\left(\frac{y'-y^*}{y'}\right)\left(1-\frac{y'}{x'+k}\right)y' \\
&= a(x^*+k)(y'-y^*)\left(\frac{y^*}{x^*+k}-\frac{y'}{x'+k}\right) \\
&= a(x^*+k)(y'-y^*)\left(\frac{y^*x'+y^*k-y'x^*-y'k+y'x'-y'x'}{(x^*+k)(x'+k)}\right) \\
&= a(y'-y^*)\frac{(-k(y'-y^*)+y'(x'-x^*)-x'(y'-y^*))}{(x'+k)} \\
&= a(y'-y^*)\frac{[-(y'-y^*)(x'+k)+y(x'-x^*)]}{(x'+k)} \\
&= -a(y'-y^*)^2+\frac{ay'}{(x'+k)}(x'-x^*)(y'-y^*).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{dV}{dt} &= \left(-(\alpha x^*+\beta y^*+\gamma)+\left(\frac{a\alpha y'}{\alpha x'+\beta y'+\gamma}\right)\right)(x'-x^*)^2 \\
&+ (x'-x^*)(y'-y^*)\left(-\left(\frac{a\alpha x'+a\gamma}{\alpha x'+\beta y'+\gamma}\right)+\frac{ay'}{(x'+k)}\right)-a(y'-y^*)^2.
\end{aligned}
$$

The above equation can be written as

$$
\frac{dV}{dt} = -(x'-x^*,y'-y^*)\begin{pmatrix} -g(x',y') & -h(x',y') \\ -h(x',y') & a \end{pmatrix}\begin{pmatrix} x'-x^* \\ y'-y^* \end{pmatrix},
$$

where

$$
\begin{aligned}
g(x',y') &= -(\alpha x^*+\beta y^*+\gamma)+\frac{a\alpha y'}{\alpha x'+\beta y'+\gamma}, \\
h(x',y') &= \frac{a}{2}\left(\frac{-\alpha x'-\gamma}{\alpha x'+\beta y'+\gamma}+\frac{y'}{(x'+k)}\right).
\end{aligned}
$$

' To prove $dV/dt < 0$ we follow the steps Sylvester's criteria that we mentioned in the chapter 1, since $a > 0$, if only if

(1) $g(x',y') < 0$;

(2) $\phi(x',y') = ag(x',y') + h^2(x',y') < 0$.

Proof. *It is of (1)*

$$
g(x',y') = -(\alpha x^*+\beta y^*+\gamma)+\frac{a\alpha y'}{\alpha x'+\beta y'+\gamma} < 0.
$$

So, as A is an attracting positively invariant set, where, all solutions satisfy $0 \leqslant x' \leqslant 1$ and $0 \leqslant x'+y' \leqslant L_1$, then

$$\begin{aligned}
g(x',y') &\leqslant -\alpha x^* + \frac{a\alpha y'}{\alpha x' + \beta y' + \gamma}, \\
&\leqslant \alpha\left(-1 + \frac{a y^*}{\alpha x^* + \beta y^* + \gamma} + \frac{a y'}{\alpha x' + \beta y' + \gamma}\right), \\
&\leqslant \alpha\left(-1 + \frac{a}{\gamma}(y' + y^*)\right) \\
&\leqslant \alpha\left(-1 + \frac{2aL_1}{\gamma}\right).
\end{aligned}$$

Therefore, if (4.10) holds, then

$$g(x',y') < 0, \qquad \forall (x',y') \in A.$$

Proof. *of (2)*

$$\phi(x',y') = -a(\alpha x^* + \beta y^* + \gamma) + \frac{a^2\alpha y'}{\alpha x' + \beta y' + \gamma} + \frac{a^2}{4}\left(-\frac{\alpha x' + \gamma}{\alpha x' + \beta y' + \gamma} + \frac{y'}{x' + k}\right)^2 < 0.$$

Since (for x' fixed)

$$\begin{aligned}
\frac{\partial \phi}{\partial y'} &= \frac{a^2\alpha(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} \\
&+ \frac{a^2}{2}\left(-\frac{\alpha x' + \gamma}{\alpha x' + \beta y' + \gamma} + \frac{y'}{x' + k}\right)\left(\frac{\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} + \frac{1}{x' + k}\right),
\end{aligned}$$

then,

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial y'^2} &= \frac{-2a^2\beta\alpha(\alpha x + \gamma)(\alpha x' + \beta y' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} \\
&+ \frac{a^2}{2}\left[\left(\frac{\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} + \frac{1}{x' + k}\right)\left(\frac{\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} + \frac{1}{x' + k}\right)\right. \\
&+ \left.\left(-\frac{\alpha x' + \gamma}{\alpha x' + \beta y' + \gamma} + \frac{y'}{x' + k}\right)\left(\frac{-2\beta^2(\alpha x' + \gamma)(\alpha x' + \beta y' + \gamma)}{(\alpha x' + \beta y' + \gamma)^4}\right)\right] \\
&= -2a^2\frac{\alpha\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^3} + \frac{a^2}{2}\left(\frac{\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2} + \frac{1}{x' + k}\right)^2 \\
&+ \frac{a^2}{2}\left(\frac{-2\beta^2(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^3}\right)\left(-\frac{\alpha x' + \gamma}{\alpha x' + \beta y' + \gamma} + \frac{y'}{x' + k}\right) \\
&= -\frac{2a^2\alpha\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^3} + \frac{a^2\beta^2(\alpha x' + \gamma)^2}{(\alpha x' + \beta y' + \gamma)^4} + \frac{a^2}{2(x' + k)^2} \\
&+ \frac{a^2\beta(\alpha x' + \gamma)}{(\alpha x' + \beta y' + \gamma)^2(x' + k)} - \frac{a^2\beta^2 y'(\alpha x' + \gamma)}{2(\alpha x' + \beta y' + \gamma)^3(x' + k)}.
\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial^2\phi}{\partial y'^2} &\leqslant -\frac{2a^2\alpha\beta(\alpha x'+\gamma)}{(\alpha x'+\beta y'+\gamma)^3} - \frac{a^2\beta^2 y'(\alpha x'+\gamma)}{2(\alpha x'+\beta y'+\gamma)^3(x'+k)}\\ &+ \frac{a^2\beta^2}{(\alpha x'+\gamma)^2} + \frac{a^2}{2(x'+k)^2} + \frac{a^2\beta}{(\alpha x'+\gamma)(x'+k)}\\ &\leqslant -\frac{a^2\beta(\alpha x'+\gamma)}{(\alpha x'+\beta y'+\gamma)^3(x'+k)}(4\alpha(x'+k)+\beta y') + \frac{a^2\beta^2}{(\alpha x'+\gamma)^2}\\ &+ \frac{a^2}{2(x'+k)^2} + \frac{a^2\beta}{(\alpha x'+\gamma)(x'+k)}.\end{aligned}$$

We note, using (4.9), that for $(x',y')\in A$

$$\begin{aligned}\frac{1}{a^2}\frac{\partial^2\phi}{\partial y'^2} &\leqslant -\frac{\beta(\alpha+\gamma)}{2(\alpha L_1+\gamma)^3(1+k)}(4\alpha(1+k)+\beta y')\\ &+ \frac{\beta^2}{(\beta+\gamma)^2} + \frac{1}{2(1+k)^2} + \frac{\beta}{(\beta+\gamma)(1+k)}\end{aligned}$$

therefore,

$$\frac{1}{a^2}\frac{\partial^2\phi}{\partial y'^2} \leqslant -\frac{2k\beta^2}{(\alpha L_1+\gamma)^3(1+k)} + \frac{\beta^2}{\gamma^2} + \frac{1}{2k^2} + \frac{\beta}{k\gamma}.$$

If (4.11) holds, then

$$\frac{1}{a^2}\frac{\partial^2\phi}{\partial y'^2} \leqslant 0.$$

Hence, $\partial\phi/\partial y'$ *is strictly decreasing in* $\mathbb{R}_+$ *with respect to* y'.
Now,

$$\begin{aligned}\frac{\partial\phi}{\partial y'}|_{y'=0} &= \frac{a^2\alpha}{\alpha x'+\gamma} - \frac{a^2}{2}\left(\frac{\beta}{\alpha x'+\gamma} + \frac{1}{x'+k}\right)\\ &= \frac{a^2}{2(\alpha x'+\gamma)(x'+k)}((\alpha-\beta)x'+k(2\alpha-\beta)-\gamma).\end{aligned}$$

In, A, all solutions satisfy $0\leqslant x\leqslant 1$, *and from (4.9)*

$$(\alpha-\beta)x'+k(2\alpha-\beta)-\gamma \leqslant (\alpha-\beta)+k(2\alpha-\beta)-\gamma,$$

then, if (4.12) holds, $(\partial\phi/\partial y')|_{y'=0}\leqslant 0$ *in* $\mathbb{R}_+$. *Hence,* $\phi(x',y')$ *is strictly decreasing in* $\mathbb{R}_+$. *This yields* $\phi(x',y') < D(x',0)$ *for* $(x',y')\in A$; *that is, using (4.9), we get*

$$\begin{aligned}
\phi(x', y') &< -a(\alpha x'^{*} + \beta y'^{*} + \gamma) + \frac{a^2}{4} \\
&< -a\left(\beta(x'^{*} + y'^{*}) + \gamma - \frac{a}{4}\right) \\
&< a\left(\beta L_1 + \gamma - \frac{a}{4}\right).
\end{aligned}$$

Consequently, due to (4.13),

$$\phi(x', y') < 0, \qquad \forall (x', y') \in A.$$

Then $dV/dt < 0$ *along all trajectories in the first quadrant* (x^*, y^*); *so* $P^*(x^*, y^*)$ *is globally asymptotically stable.*

Chapter 5

Permanence

Permanence or uniform persistence of strictely positive equilibria means that the omega limit of the trivial equilibrium points cannot intersect of the positive cone. In this chapter, we prove under necessary condition the permanence of the system (3.1)

Definition 5.1. *Consider an ODE model for n interacting biological species*

$$\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_n), \qquad i = 1, 2, ..., n,$$

where $x_i(t)$ denotes the density of the ith species. Let $(x_1(t), x_2(t), ..., x_n(t))$ denote the solution of (5.1) with componentwise positive initial values. The system (5.1) is said to be weakly persistent if

$$\limsup_{t \longrightarrow +\infty} x_i(t) > 0, \qquad i = 1, 2, ..., n,$$

persistent if

$$\liminf_{t \longrightarrow +\infty} x_i(t) > 0, \qquad i = 1, 2, ..., n,$$

and uniformly persistent if there is an $\varepsilon_0 > 0$ such that

$$\liminf_{t \longrightarrow +\infty} x_i(t) > \varepsilon_0, \qquad i = 1, 2, ..., n.$$

The system (5.1) is said to be permanent if for each $i = 1, 2, ..., n$ there are constant ε_0 and M_i such that

$$0 < \varepsilon_0 < \liminf_{t \leqslant +\infty} x_i(t) < \limsup_{t \leqslant +\infty} x_i(t) \leqslant M_i.$$

Suppose that Υ is a complete metric space with $\Upsilon = \Upsilon_0 \bigcup \partial\Upsilon_0$ for an open set Υ_0. We choose Υ_0 to be the positive cone in $\mathbb{R}^2$.

Definition 5.2. *A flow or semiflow on $\Upsilon = \Upsilon_0 \bigcup \partial\Upsilon_0$ under which Υ_0 and $\partial\Upsilon_0$ are forward invariant is said to be permanent if it is dissipative and if there is a number $\varepsilon > 0$ such that any trajectory starting in Υ_0 will be at least a distance ε from $\partial\Upsilon_0$ for all sufficiently large t.*

Let $\omega(\partial\Upsilon_0) = \bigcup_{u\in\partial\Upsilon_0} \omega(u)$,such that $\omega(\partial\Upsilon_0) \in \partial\Upsilon_0$ denote the union of the sets $\omega(u)$ over $u \in \partial\Upsilon_0$.

Definition 5.3. *The ω-limit set $\omega(\partial\Upsilon_0)$ is said to be isolated if it has a covering $\Omega = \bigcup_{k=1}^{N} \Omega_k$ of pairwise disjoint sets Ω_k which are isolated and invariant with respect to the flow or the semiflow both on $\partial\Upsilon_0$ and on $\Upsilon = \Upsilon_0 \bigcup \partial\Upsilon_0$, ($\Omega$ is called an isolated covering). The set $\omega(\partial\Upsilon_0)$ is said to be acyclic if there exists an isolated covering $\bigcup_{k=1}^{N} \Omega_k$ such that no subset of Ω_k is a cycle (The chain is called a cycle if $\Omega_k = \Omega_1$).*

Theorem 5.1. *Suppose that a semiflow on Υ leaves both Υ_0 and $\partial\Upsilon_0$ forward invariant, maps bounded sets in Υ to precompact set for $t > 0$, and it is dissipative. If in addition*

(1) $\omega(\partial\Upsilon_0)$ is isolated and acyclic;

(2) $W^s(\Omega_k) \cap \Upsilon_0 = \emptyset$ for all K, where $\bigcup_{k=1}^{N} \Omega_k$ is the isolated covering used in the definition of acyclicity of $\partial\Upsilon_0$.

Then the semiflow is permanent.

Remark 5.1. *[9] The stable set of pairwise disjoint sets Ω_k is denoted by W^s (the stable manifold) and is defined as*

$$W^s(\Omega_k) = \{x | x \in \Upsilon, \omega(x) \neq \emptyset, \omega(x) \subset \Omega_k\}.$$

And, we have this theorem.

Theorem 5.2. *Let us assume the following condition:*

$$k(a-\beta) \leqslant \gamma. \tag{5.1}$$

Then, system (3.1) is permanent.

Proof. We take Υ the strictly positive quadrant of $\mathbb{R}^2$; then $\omega(\partial\Upsilon_0)$ consists of the equilibria $P_0(0,0), P_1(0,k),$ and $P_2(1,0)$.
$P_0(0,0)$ is an unstable node, $P_2(1,0)$ is saddle point, and its stable manifold

is x-axis.
If $ak \leqslant \beta k + \gamma$, $P_1(0,k)$ is a saddle point stable along the y-axis and unstable along the x-axis.
Then, all trajectories on the axis (ox) other than $P_0(0,0)$ approach the point $P_1(0,k)$. It follows from these structural features that the flow in $\partial\Upsilon_0$ is acyclic. So $\omega(\partial\Upsilon_0)$ is isolated and acyclic.
The stable manifold of $P_2(1,0)$ is the x-axis and the stable manifold of $P_1(0,k)$ is the y-axis, and we know, from Theorem 3.1, thet these stable manifolds cannot intersect the interior of Υ_0.
In this case, Theorem 5.1 implies permanence of the flow defined by (3.1).

Conclusion

In this memory, we have presented and analysed the mathimatical models describing the dynamics of the population of interacting species.This model takes into account the domain biological.

The main study out during this work is the treatment of the question of existence of an attracting set, boundedness of solutions and persistence for model (3.1) .

Under certain imposed conditions, we have formulated the result of existence, boundedness of solutions in the Theorem 3.1.

We have considered with a certain condition the local and global asymptotic stability of trivial and interior equilibrium .

We have completed this study by the permannence of system(3.1) .

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