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Thème

Dynamics and stability for Katugampola random fractional differential equations with delay

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Dedication

I dedicate this work:

To Allah All powerful who inspired me Who guided me on the right path I owe you what I've become Praises and thanks

For your clemency and mercy To my DEAR parents: BELKACEM and Zohra For their unwavering support over these many years and for the trust they have given me, for all the love and affection you have for me and which make me forget the distance, and the expression of all my love. I can never thank you enough.

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ملخص

في هذه المذكرة وجود الحلول العشوائية و Ulam استقرارية
لفئة معادلات تفاضلية ذات المشتقات الكسرية من نوع Katugampola
مع تأخير في فضاء باناخ.
لقد استخدمنا نظرية النقطة الثابتة لوجود الحلول العشوائية،
وأثبتنا أن مشكلتنا Ulam -Hyers-Rassias مستقرة عموما .مع تقديم مثال
توضيحي.
الكلمات المفتاحية : المعادلة التفاضلية,التكامل الكسري لفئة
Katugampola, الاشتقاق الكسري لفئة Katugampola, حل عشوائي, فضاء باناخ
, استقرار Ulam, نقطة ثابتة, تأخير.

Abstract

This thesis deals with some existence of Random solutions and the Ulam stability for a class of Katugampola Random fractional differential equations in Banach spaces with delay. We use the Random fixed point theorem for the existence of Random solutions, and we prove that our problem is generalized Ulam-Hyers-Rassias stable. An illustrative example is presented .
Keywords: differential equation, Katugampola fractional integral, Katugampola fractional derivative, Random solution, Banach space, Ulam stability, fixed point,delay.

Résumé

Ce mémoire traite de l'existence de solutions aléatoire leur ou sens et de la stabilité de Ulam pour une classe d'équations différentielles fractionnaires avec dérivé de Katugampola avec retard dans des espaces de Banach. on utilise le théorème de point fixe pour l'existence de solutions aléatoire , et nous prouvons que notre problème est d'Ulam-Hyers-Rassias généralement stable. Un exemple illustratif est présenté .
Mot clé: équation différentielle,Intégrale fractionnaire de katugampola,dérivé fractionnaire de Katugampola, solution aléatoire ,espace banach,Ulam stabilité, point fixe ,retard.

RATING

Rating	Definition
$\mathbb{N}$	Set of natural numbers.
$\mathbb{R}$	Set of real numbers .
$n!$	Factorial (n) , $n \in \mathbb{N}$.
$\Gamma(.)$:	Gamma function .
$\beta(.)$:	Beta function.
${}^{RL}I_a^{\alpha}$:	The Riemann-Liouville fractional integral of order $\alpha > 0$.
${}^{RL}D_a^{\alpha}$:	The Riemann-Liouville fractional derivative of order
${}^{c}D_a^{\alpha}$:	The Caputo fractional derivative of order $\alpha > 0$..
${}^{H}I_a^{\alpha}$:	The hadmarad fractional integral of order $\alpha > 0$.
${}^{H}D_a^{\alpha}$:	The hadmarad fractional derivative of order $\alpha > 0$.
${}^{\rho}I_a^{\alpha}$:	The Katugampola fractional integral of order $\alpha > 0$.
${}^{\rho}D_a^{\alpha}$:	The Katugampola fractional derivative of order $\alpha > 0$.
$L^1(I, E)$:	the Banach space of measurable function
$C_{\rho\zeta}(I)$:	space of continuous functions .

CONTENTS

INTRODUCTION

The history of fractional calculus dates back to the 17th century. So many mathematicians define the most used fractional derivatives, Riemann-Liouville in 1832, Hadamard in 1891 and Caputo in 1997 [24, 27, 31] . Fractional calculus plays a very important role in several fields such as physics, chemical technology, economics, biology; see [2, 24] and the references therein. In 2011, Katugampola introduced a derivative that is a generalization of the Riemann-Liouville fractional operators and the fractional integral of Hadamard in a single form [21, 22, 29].

Historically, differential equations went through the beginning of development From analysis, generally when mechanical problems or Engineering from [2, 9, 24].

Classical theory of mathematical modeling of transformation Physics is the assumption that the future behavior of a system can be summed up, In the inevitable framework, by its present alone, without relying on its development frontal. This assumption leads to modeling in the form of fractional differential equations. But there are many cases where this theory is true It is put in a state of deficiency, after which it is necessary to take into account other phenomena, This then increases the complexity of the system analysis. Among these reasons is the phenomenon of delay or heredity that characterizes the effect That the condition of the operation is exercised on its current behavior. Thus, fractional differential equations with delays can no longer be To be written mathematically in the form of a fractional differential, But it is described by genetic equations that becomes its theoretical dimension No final.

Fractional differential equations arise late in the formulation many dynamic phenomena in which some effects are not immediate, But it intervenes with a delay, in other words when the situation is at a certain moment a function of his past. It can be found in many areas of application, particularly in economics, Physics, medicine, biology, [8] The importance of being late Or such a model may be different: the time of gestation in biology, the time of driving reaction, the incubation period of an infectious disease, accumulation time. Thus, we find in the theory of these fractional differential equations with a delay from[8, 14, 28]

In this thesis we will focus on fractional differential equations With a delay and more accurately the results The solutions are unique. These results are based on the fixed point theory and the concept of Ulam-Hyers-Rassias stability [5, 6, 7, 11, 16, 17, 19, 20, 26, 28, 30]

The memory is organized as follows :

In the first chapter, we will present some preliminary notions about fractional calculation about fractional calculus (the Gamma function, the Beta function the derivatives and the fractional integrals of Caputo, Rimmann- Liouville ,Hadamard,Katugampola).

In the second chapter, we analyzed the article on dynamics and stability of Katugampola stochastic fractional differential Equations which is investigating the following class of Katugampola random fractional differential equation see [10]

$$^{\rho}D_0^{\zeta}u\left(t,w\right)=h\left(t,u_t\left(t,w\right),w\right),\ t\in I=\left[0,T\right],$$

with the terminal condition

$$u\left(T,w\right)=u_T\left(w\right).$$

where ρD_0^{ζ} is Katugampola operator of order ζ with $0<\zeta\leq 1$ and $\rho>0$, $T>0$, $x_T:\Omega\mapsto E$ is a measurable function , $h:I\times\mathcal{C}\times\Omega\mapsto E$ is a measurable function , and Ω is the sample space in a probability space, and $(E,\|.\|)$ is a Banach space.

In the third chapter, we study the existence and uniqueness of solutions for fractional differential equations with finite delay

$$^{\rho}D_0^{\zeta}u\left(t,w\right)=h\left(t,u_t\left(t,w\right),w\right),\ t\in I=\left[0,T\right],$$

and

$$u\left(t,w\right)=g\left(t,w\right),\qquad t\in\left[-\alpha,0\right].$$

where ρD_0^{ζ} is Katugampola operator of order ζ with $0<\zeta\leq 1$ and $\rho>0$, $T>0$, $g:[-\alpha,0]\times\Omega\mapsto E$ is a measurable continuous function in $[-\alpha,0]$ α is the delay $\alpha>0$, $h:I\times\mathcal{C}_\alpha\times\Omega\mapsto E$ is a measurable function , $\mathcal{C}_T=C_{\zeta\rho}\left(\left[-\alpha,T\right],E\right)$ the Banach space defined on $[-\alpha,T]$ and Ω is the sample space in a probability space, and $(E,\|.\|)$ is a Banach space.

$$u_t=u\left(t+\tau\right),\qquad \tau\in\left[-\alpha,0\right].$$

CHAPTER 1

PRELIMINARIES

1.1 Functional spaces

By $C(I) := C(I, \mathbb{E})$ we denote the Banach space of all continuous functions $x : I \longrightarrow \mathbb{E}$ with the norm

$$\|x\|_\infty = \sup_{x\in I} |x(\xi)|,$$

and $L^1(I;E)$ denotes the Banach space of measurable function $x : I \longrightarrow E$ with are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_I |x(\xi)|\, d\xi.$$

Let $\mathcal{C} = C_{\zeta\rho}(I)$ be the weighted space of continuous functions defined by

$$C_{\zeta\rho}(I) = \left\{x : I \to \mathbb{E} \quad : \xi^{\rho(1-\zeta)} x(\xi) \in C(I)\right\},$$

With the norm

$$\|x\|_{\mathcal{C}} := \sup_{\xi\in I} \|\xi^{\rho(1-\zeta)} x(\xi)\|_\infty.$$

Also Let $\mathcal{C}_\alpha = C([-\alpha; 0])$ be the weighted space of continuous functions

with the norm

$$\|g\|_{\mathcal{C}_\alpha} = \sup_{\tau\in[-\alpha;0]} \|g(\tau)\|,$$

and

$$\|x_\tau\|_{\mathcal{C}_\alpha} = \sup_{\sigma\in[-\alpha;0]} \|x(\tau+\sigma)\|.$$

Consider $\mathcal{C}_T = C_{\zeta\rho}\left(\left[-\alpha, T\right], E\right)$ the Banach space defined on $\left[-\alpha, T\right]$ with the norm

$$\|x\|_{\mathcal{C}_T} = \|x\|_{\mathcal{C}} + \|x\|_{\mathcal{C}_\alpha}.$$

1.2 Special Functions

1.2.1 Gamma function

The Euler Gamma function is an extension of the factorial function to real numbers and is considered the most important Eulerian function used in fractional calculus because it appears in almost every fractional integral and derivative definitions.

Definition 1.2.1 *[25]*

$$\Gamma\left(z\right) := \int_0^{+\infty} e^{-t}t^{z-1}dt, z > 0.$$

For positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ *and thus can be seen as an extension of the factorial function to real values. An important property of the gamma function is that it satisfies* $\Gamma(z+1) = z\Gamma(z)$ $a > 0$.

Example 1.2.1 *(Some particular values of* $\Gamma\left(z\right)$ *)*

1.

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} e^{-t}t^{\frac{1}{2}}dt, \qquad let \qquad t = y^2, dt = 2ydy \\
&= \int_0^{+\infty} e^{-y^2}y^{-1}2ydy \\
&= 2\int_0^{+\infty} e^{-y^2}dy \\
&= 2\frac{\sqrt{\pi}}{2} \\
&= \sqrt{\pi}.
\end{aligned}$$

2.

$$\begin{aligned}
\Gamma\left(1\right) &= \int_0^{+\infty} e^{-t}t^{1-1}dt \\
&= \left[-e^{-t}\right]_0^{\infty} = 1.
\end{aligned}$$

1.2.2 Beta Function

Definition 1.2.2 *[25]: The Beta function, or the first order Euler function, can be defined as*

$$B\left(m,n\right)=\int_0^1 t^{m-1}(1-t)^{n-1}dt, n>0, m>0. \tag{1.1}$$

Proposition 1.2.1 *[25]:*

We use the following formula which expresses the beta function in terms of the gamma function . $\forall n>0, m>0$

- *(1)*

$$B(m,n)=B(n,m) \tag{1.2}$$

- *(2)*

$$B(m,n)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}. \tag{1.3}$$

Proof.Property (1):

we have $B\left(m,n\right)=\int_0^1 t^{m-1}(1-t)^{n-1}dt$, $n>0, m>0$.

Using the change of variable $t=1-x$, we get

$B\left(m,n\right)=\int_0^1 (1-x)^{m-1}t^{n-1}dt=B(n,m)$.

Property (2):

Letting :

$$\Gamma\left(s_1\right)=\int_0^{+\infty} x^{s_1-1}e^{-x}dx, \tag{1.4}$$

and and

$$\Gamma\left(s_2\right)=\int_0^{+\infty} y^{s_2-1}e^{-y}dy, \tag{1.5}$$

From (1.4) and (1.5) , we have

$$\Gamma\left(s_1\right)\Gamma\left(s_2\right)=\int_0^{+\infty} x^{s_1-1}e^{-x}dx, \int_0^{+\infty} y^{s_2-1}e^{-y}dy$$

Let' s put

$$x=u^2 \to dx=2udu,$$

$$y=v^2 \to dy=2vdv.$$

We have

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_0^{+\infty} u^{2s_1-2}e^{-u^2}2udu.\int_0^{+\infty} v^{2s_2-2}e^{-v^2}2vdv\\ &= 4\int_0^{+\infty} u^{2s_1-1}e^{-u^2}du.\int_0^{+\infty} v^{2s_2-1}e^{-v^2}dv\\ &= 4\int_0^{+\infty}\left(\int_0^{+\infty} v^{2s_2-1}e^{-v^2}dv\right)u^{2s_1-1}e^{-u^2}du\\ &= 4\int_0^{+\infty}\int_0^{+\infty} v^{2s_2-1}e^{-v^2}u^{2s_1-1}e^{-u^2}dvdu\\ &= 4\int_0^{+\infty}\int_0^{+\infty} v^{2s_2-1}u^{2s_1-1}e^{-\left(u^2+v^2\right)}dvdu\end{aligned}$$

Using the following change of variable

$$\begin{cases} v = r\cos(\alpha) \\ u = r\sin(\alpha)\end{cases} \Rightarrow dvdu = rd\alpha dr\ ,$$

So :

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} r^{2s_2-1}\cos(\alpha)^{2s_2-1}r^{2s_1-1}\sin(\alpha)^{2s_1-1}e^{-r^2}rd\alpha dr\\ &= 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} r^{2s_2-1}r^{2s_1-1}e^{-r^2}r\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha dr\\ &= 4\int_0^{+\infty}\left(\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha\right)r^{2s_1+2s_2-2}e^{-r^2}rdr\\ &= 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} r^{2s_1+2s_2-2}e^{-r^2}rdr\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha\\ &= \int_0^{+\infty} r^{2s_1+2s_2-2}e^{-r^2}2rdr.2\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha\\ &= \int_0^{+\infty}\left(r^2\right)^{s_1+s_2-1}e^{-r^2}2rdr.2\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha\end{aligned}$$

Let' s

$$r^2 = u \Rrightarrow 2rdr = du,\qquad \text{we have :}$$

$$\int_0^{+\infty} u^{s_1+s_2-1}e^{-u}udu.2\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha$$

$$\Gamma(s_1+s_2).2\int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-1}\sin(\alpha)^{2s_1-1}d\alpha$$

Thus :

$$\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^{\frac{\pi}{2}}\cos(\alpha)^{2s_2-2}\sin(\alpha)^{2s_1-2}2\cos(\alpha)\sin(\alpha)d\alpha$$

Let' s :

$$u = (\sin(\alpha))^2 \Rrightarrow du = 2\sin(\alpha)\,cos(\alpha)\,d\alpha\text{we obtain , :}$$

$$\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}\,du.$$

Consequently

$$\beta(s_1,s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}\,du.$$

■

1.3 Elements From Fractional Calculus Theory

In this section, we recall some definitions of fractional integral and fractional differential operators that include all we use throughout this theme

1.3.1 The Riemann-Liouville

Definition 1.3.1 *[2]: The Riemann-Liouville fractional integral operator of the function $h\in L1(I;E)$ of order $\zeta \in R_+$ is defined by*

$$^{RL}I_0^{\zeta}h(t) = \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}h(s)ds. \tag{1.6}$$

Example 1.3.1 *[24]: Let $a>0$, and $b>-1$. Then*

$$I_0^a t^b = \frac{\Gamma(b+1)}{\Gamma(a+b+1)}t^{a+b}.$$

Lemma 1.3.1 *[24]: The following basic properties of the Riemann-Liouville integrals hold :*

1. *The integral operator I_0^a is linear.*

2. *The semigroup property of the fractional integration operator I_0^a is given by the following result*

$$I_0^a\left(I_0^b h(t)\right) = I_0^{a+b}h(t), a,b>0, \tag{1.7}$$

holds at every point if $f \in C([0,T])$ and holds almost every where if $f\in L^1([0,T])$,

3. *Commutativity*

$$I_0^a\left(I_0^b h(t)\right) = I_0^b\left(I_0^a h(t)\right), a,b>0,$$

Definition 1.3.2 *[2]: The Riemann-Liouville fractional derivative of order $\zeta\in R_+$ is defined by*

$$^{RL}D_0^{\zeta}h(t)=\frac{1}{\Gamma(n-\zeta)}\left(\frac{d}{d\zeta}\right)^n\int_0^t(t-s)^{n-\zeta-1}h(s)ds,=\left(\frac{d}{dt}\right)^n\left[I^{n-a}f(t)\right]. \tag{1.8}$$

Example 1.3.2 *[24, 25]:*

Let $a>0$, $\beta>-1$ and $f(t)=(t-a)^{\beta}$ then

$$\left(^{RL}I_a^{\alpha}f\right)(t)=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}\ a>0,\beta>-1, \tag{1.9}$$

$$\left(^{RL}D_a^{\alpha}f\right)(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}\ a>0,\beta>-1. \tag{1.10}$$

This because

$$\left(^{RL}I_a^{\alpha}\right)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}(t-a)^{\beta}ds.$$

Performing the change of variable

$$t=a+\tau(t-a),0\leq\tau\leq 1. \tag{1.11}$$

So (1.7) becomes

$$\left(^{RL}I_a^{\alpha}\right)=\frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0 1\tau^{\beta}(1-\tau)^{\alpha-1}d\tau.$$

By using (1.1) and the property (1.3) we have

$$\begin{aligned}\left(^{RL}I_a^{\alpha}\right)&=\frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)}(t-a)^{\alpha+\beta}\\&=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\alpha+\beta}.\end{aligned}$$

In particular if $\beta=0$ and $\alpha>0$ then the fractional The Riemann-Liouville derivative of a constant is generally null indeed , we have

$$\left(^{RL}D_a^{\alpha}C\right)(t)=\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}\ \alpha>0.$$

It was necessary to establish the following result.

Proposition 1.3.1 *[24, 25] Let $m>\alpha>m-1$ and then*

$$^{RL}D_a^{\alpha}f(t)=0\ \Rightarrow\ f(t)=\sum_{j=0}^{m-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-m)}(t-a)^{j+\alpha-j},\quad\forall c_1....c_n\in\mathbb{R}. \tag{1.12}$$

Particular if $1>\alpha>0$ then

$$^{RL}D_a^{\alpha}f(t)=0\quad\Rightarrow f(t)=c(t-a)^{\alpha-1},\quad\forall c\in\mathbb{R}.$$

Proposition 1.3.2 *[24, 25]:*

The Riemann -liouville derivation operator has the following properties boast :

1. *It is linear operator.*
2. $\lim_{\alpha \to m-1} \left({}^{RL}D_a^{\alpha} f\right) = f^{(m-1)}$ *and* $\lim_{\alpha \to m} \left({}^{RL}D_a^{\alpha} f\right) = f^{(m)}$.
3. ${}^{RL}D_a^{\alpha} I_a^{\alpha} = id.$

1.3.2 Caputo fractional derivative

Definition 1.3.3 *[24]:*

Let $\alpha \in]m-1, m[$ *and* $f \in C^m([a, b])$ *we call the derivative of* f *in the sense of caputo the function defined is defined by*

$$\begin{aligned} \left({}^{c}D_a^{\alpha} f\right)(t) &= \left(I_a^{m-\alpha} f^{(m)}\right)(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-x)^{m-\alpha-1} f^{(m)}(x)\, dx. \end{aligned} \tag{1.13}$$

Lemma 1.3.2 *[24]:*

the Riemann derivative and the caputo derivative are related by the formula

$$\left({}^{c}D_a^{\alpha} f\right)(t) = {}^{RL}D_a^{\alpha} \left[f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(t-a)^j}{j} \right],$$

you can also write

$$\left({}^{c}D_a^{\alpha} f\right)(t) = \left({}^{RL}D_a^{\alpha} f\right)(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(t-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}. \tag{1.14}$$

Proof.we have by definition

$$\begin{aligned} \left({}^{c}D_a^{\alpha} f\right)(t) = \left(I_a^{m-\alpha} f^{(m)}\right)(t) &= \left(\frac{d}{dt}\right)^n I_a^m I_a^{m-\alpha} f^{(m)}(t) \\ &= \left(\frac{d}{dt}\right)^n I_a^{m-\alpha} I_a^m f^{(m)}(t) \\ &= \left(\frac{d}{dt}\right)^n I_a^{m-\alpha} \left[f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{j!} \right] \\ &= {}^{RL}D_a^{\alpha} \left[f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{j!} \right] \\ &= \left({}^{RL}D_a^{\alpha} f\right)(t) - \sum_{j=0}^{m-1} \frac{f^j(a)(x-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}. \end{aligned}$$

■

Example 1.3.3 *[24]:*

We deduce that if $f^{(m)} = 0$ *for* $j = 0.1.2..m-1$, *we have* ${}^{c}D_{a}^{\alpha} = {}^{RL} D_{a}^{\alpha}$ *concerning the derivative of function* $t \mapsto (t-\alpha)^{\beta}$, *we have*

$$\begin{aligned}
\left({}^{c}D_{a}^{\alpha} f\right)(t) &= {}^{RL}I_{a}^{m-\alpha}\left[\left(\frac{d}{dt}\right)^{n}(t-\alpha)^{\beta}\right] \\
&= {}^{RL}I_{a}^{m-\alpha}\left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)}\right)(t-\alpha)^{\beta-m} \\
&= \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)}\right)^{RL} I_{a}^{m-\alpha}(t-\alpha)^{\beta-m} \\
&= \frac{\Gamma(\beta+1-m)}{\Gamma(\beta+1-\alpha)}(t-\alpha)^{\beta-\alpha} \\
&\Longrightarrow \left({}^{c}D_{a}^{\alpha} f\right)(t) = \frac{\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1-\alpha)}(t-\alpha)^{\beta-\alpha}. \qquad (1.15)
\end{aligned}$$

The derivative of a constant function in the caputo sense is zero , according to (??)

$$ {}^{c}D_{a}^{\alpha} C = {}^{RL} I_{a}^{m-\alpha}(0) = 0. \qquad (1.16)$$

Proposition 1.3.3 *[25]:*

Let $\alpha \in]m-1, m[$ *and* $f \in C^{m}([a,b])$ *We have*

1. ${}^{c}D_{a}^{\alpha}\left[{}^{RL}I_{a}^{\alpha} f\right] = f.$

2. ${}^{c}D_{a}^{\alpha}\left[{}^{RL}I_{a}^{\alpha} f\right] = 0$ *then* $f(t) = \sum_{j=0}^{m-1} c_{j}(t-a)^{j}.$

3. ${}^{c}I_{a}^{\alpha}\left[{}^{c}D_{a}^{\alpha} f\right](t) = f(t) + \sum_{j=0}^{m-1} \frac{(t-a)^{j}}{j!} f^{(j)}(a).$

4. *If* $0 \le \alpha$ *and* $\beta \le 1$ *with* $\alpha + \beta \le 1$ *and of class* C^{1}

$$\left({}^{c}D_{a}^{\alpha} \circ {}^{c} D_{a}^{\beta}\right) f = {}^{c} D_{a}^{\alpha+\beta} = \left({}^{c}D_{a}^{\beta} \circ {}^{c} D_{a}^{\alpha}\right) f.$$

1.3.3 The Hadamard fractional

Definition 1.3.4 *[3] The Hadamard fractional integral of order* ζ *is defined as*

$$\left({}^{H}I_{a}^{\zeta} h\right)(t) := \frac{1}{\Gamma(\zeta)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\zeta-1} h(s) \frac{ds}{t}, \ \zeta > 0. \qquad (1.17)$$

provided that the left-hand side is well defined for almost every $t \in (0,T)$

Example 1.3.4 *[3]:*

Let f be the function defuned by :

$$f(t) = \log^{\beta}\left(\frac{t}{a}\right), \beta > -1,$$

we have :

$$I^{\alpha}\log^{\beta}\left(\frac{t}{a}\right) = \frac{1}{\Gamma(\alpha)}\int_a^t \log^{\alpha-1}\left(\frac{t}{\tau}\right)\log^{\beta}\left(\frac{\tau}{a}\right)\frac{d\tau}{\tau}.$$

By the change of variable $x = \dfrac{\log(\tau)-\log(a)}{\log(t)-\log(a)}$ *and beta function it follows that*

$$\begin{aligned} I^{\alpha}\log^{\beta}\left(\frac{t}{a}\right) &= \frac{1}{\Gamma(\alpha)}\int_0^1\left[\log\left(\frac{t}{a}\right) - x\log\left(\frac{t}{a}\right)\right]^{\alpha-1} x\log^{\beta}\left(\frac{t}{a}\right)\log\left(\frac{t}{a}\right)dx \\ &= \frac{1}{\Gamma(\alpha)}\log^{\alpha+\beta}\left(\frac{t}{a}\right)\int_0^1 (1-x)^{\alpha-1}x^{\beta}dx \\ &= \frac{1}{\Gamma(\alpha)}\log^{\alpha+\beta}\left(\frac{t}{a}\right)B(\alpha,\beta+1) \\ &= \frac{1}{\Gamma(\alpha)}\log^{\alpha+\beta}\left(\frac{t}{a}\right)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\log^{\alpha+\beta}\left(\frac{t}{a}\right) \end{aligned}$$

for $\alpha = 1$ *and* $\beta = 1$ *we obtain* $I^1\log\left(\frac{t}{a}\right) = \frac{\Gamma(2)}{\Gamma(3)}\log^2\left(\frac{t}{a}\right) = \frac{1}{2}\log^2\left(\frac{t}{a}\right)$,
If $\alpha = \frac{1}{2}$, *we have*

$$I^{\frac{1}{2}}\log^{\beta}\left(\frac{t}{a}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{3}{2})}\log^{\beta+\frac{1}{2}}\left(\frac{t}{a}\right).$$

Proposition 1.3.4 *[3]:*
let $f,g \in C([a,b])$ *for* $\alpha \geq 0$, $\beta \geq 0$ *and* $A,B \in \mathbb{R}$ *we have*

1. ${}^{H}I_a^{\alpha}\left({}^{H}I_a^{\beta}f\right)(t) = {}^{H}I_a^{\beta}\left({}^{H}I_a^{\alpha}f\right)(t) = {}^{H}I_a^{\alpha+\beta}f(t)$

2. ${}^{H}I_a^{\alpha}[Af(t) + Bg(t)] = A\,{}^{H}I_a^{\alpha}f(t) + B\,{}^{H}I_a^{\alpha}g(t)$

Definition 1.3.5 *[3]*

The Hadamard fractional derivative of order α *is defined as*

$$\begin{aligned} \left({}^{H}D_0^{\alpha}h\right)(t) \quad : &= \left(t\frac{d}{dt}\right)^n\left(I_a^{n-\alpha}f\right)(t) \qquad (1.18) \\ &= \frac{a}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}f(s)\frac{ds}{t}ds,\ n\in\mathbb{N}^*, n-1<\alpha\leq n, t>a. \end{aligned}$$

Such as $\delta = t\frac{d}{dt}$.

provided that the left-hand side is well defined for almost every $t \in (0.T)$.

Example 1.3.5 *[24]:*

We consider the function defined by

$$f : t \longrightarrow \left(\log\frac{t}{a}\right)^{\beta},$$

We have

$$^{H}D_{a}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\left(\log\frac{t}{a}\right)^{\beta}\frac{ds}{t}ds$$

By change of variable $x = \frac{\log\frac{s}{a}}{\log\frac{t}{a}}$, *we obtain*

$$\begin{aligned} ^{H}D_{a}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} &= \frac{\left(t\frac{d}{dt}\right)\left(\log\frac{t}{s}\right)^{n-\alpha+\beta}}{\Gamma(n-\alpha)}\int_{0}^{1}\mu^{\beta}(1-\mu)^{n-\alpha-1}d\mu \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}\left(t\frac{d}{dt}\right)^{n}\left(\log\frac{t}{s}\right)^{n-\alpha+\beta}. \end{aligned}$$

If we take $\beta = \frac{5}{2}$, $\alpha = \frac{1}{2}$ *then*

$$\begin{aligned} ^{H}D_{a}^{\frac{1}{2}}\left(\log\frac{t}{a}\right)^{\frac{5}{2}} &= \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)}\left(t\frac{d}{dt}\right)^{1}\left(\log\frac{t}{s}\right)^{3} \\ &= \frac{15\sqrt{\pi}}{64}\left(\log\frac{t}{s}\right)^{2}. \end{aligned}$$

If we take $\beta = 0$, $\alpha = \frac{5}{2}$ *then*

$$\begin{aligned} ^{H}D_{a}^{\frac{1}{2}}\left(\log\frac{t}{a}\right)^{0} &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}+1\right)}\left(t\frac{d}{dt}\right)^{1}\left(\log\frac{t}{s}\right)^{\frac{1}{2}} \\ &= \frac{4}{\sqrt{\pi}}\left(\log\frac{t}{s}\right)^{-\frac{1}{2}} \end{aligned}$$

Is not nulle.

Proposition 1.3.5 *[24] Let* $\alpha \leq \beta \leq 0$ *.If* $f \in C([a,b])$ *then :*

$$^{H}D_{a}^{\alpha}\left(^{H}I_{a}^{\beta}f\right)(t) =^{H} I_{a}^{\alpha-\beta}f(t).$$

In particular if $\alpha = \beta$ *then*

$$^{H}D_{a}^{\alpha}\left(^{H}I_{a}^{\alpha}f\right)(t) = f(t).$$

1.3.4 The Katugampola fractional

Definition 1.3.6 *[2] The Katugampola fractional integrals of order* $(\zeta > 0)$ *is defined by*

$$^{\rho}I_{0}^{\zeta}h(t) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_{0}^{t} \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\zeta}} h(s)\, ds, \tag{1.19}$$

for $\rho > 0$.

Definition 1.3.7 *[2] The Katugampola fractional derivative of order* $(\zeta > 0)$ *is defined by*

$$\begin{aligned} ^{\rho}D_{0}^{\zeta}h(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^{n} \left(^{\rho}I_{0}^{\zeta}h\right)(t) \\ &= \frac{\rho^{\zeta-n+1}}{\Gamma(n-\zeta)} \left(t^{1-\rho}\frac{d}{dt}\right)^{n} \int_{0}^{t} \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{\zeta-n+1}} h(s)\, ds, \end{aligned} \tag{1.20}$$

for $\rho > 0$ *, provided that the left-hand side is well defined for almost every* $t \in (0,T)$.

We present in the following theorem some properties of Katugampola fractional integrals and derivatives.

Example 1.3.6 ***[22]*** *We find the generalized derivative of the function* $f(x) = x^{v}$ *where* $v \in \mathbb{R}$ *. The formula (1.20) yields*

$$^{\rho}D_{0}^{\alpha}x^{v} = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right) \int_{0}^{x} \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{\alpha}} t^{v}\, dt. \tag{1.21}$$

To evaluate the inner integral, use the substitution $u = \frac{t^{\rho}}{x^{\rho}}$ *to obtain* $\left(du = \frac{1}{\rho}t^{1-\alpha}du \quad t^{v} = u^{\frac{v}{\rho}}.x^{v}\right)$

$$\begin{aligned} \int_{0}^{x} \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{\alpha}} t^{v}\, dt &= \frac{x^{v+\rho(1-\alpha)}}{\rho} \int_{0}^{1} \frac{u^{\frac{v}{\rho}}}{(1-u)^{\alpha}} du \\ &= \frac{x^{v+\rho(1-\alpha)}}{\rho} \beta\left(1-\alpha, 1+\frac{v}{\rho}\right), \end{aligned}$$

where $\beta(.,.)$ *is the Beta function. Thus, we obtain,*

$$^{\rho}D_{0}^{\alpha}x^{v} = \frac{\Gamma\left(1+\frac{v}{\rho}\right)\rho^{\alpha-1}}{\Gamma\left(1+\frac{v}{\rho}-\alpha\right)} x^{v-\alpha.\rho},$$

Theorem 1.3.1 ***[21, 22]:*** *Let* $0 < Re(\zeta) < 1$ *and* $0 < Re(\eta) < 1$ *and* $\rho > 0$, *for* $a > 0$:

- *Index property*

$$\left(^{\rho}D_{a}^{\zeta}\right)\left(^{\rho}D_{a}^{\eta}h\right)(t) =^{\rho} D_{a}^{\zeta+\eta}h(t),$$

$$\left({}^{\rho}I_{a}^{\zeta}\right)\left({}^{\rho}I_{a}^{\eta}h\right)(t) =^{\rho} I_{a}^{\zeta+\eta}h\left(t\right).$$

- *Linearity property*

$${}^{\rho}D_{a}^{\zeta}\left(h+g\right)(t) =^{\rho} D_{a}^{\zeta}h\left(t\right) +^{\rho} D_{a}^{\zeta}g\left(t\right),$$

$${}^{\rho}I_{a}^{\zeta}\left(h+g\right)(t) =^{\rho} I_{a}^{\zeta}h\left(t\right) +^{\rho} I_{a}^{\zeta}g\left(t\right).$$

Proof.:

- Using Fubinis Theorem, for "sufficiently good" function h, we have

$$\begin{aligned}
\left({}^{\rho}I_{a}^{\zeta}\right)\left({}^{\rho}I_{a}^{\eta}h\right)(t) &= \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_{a}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}\,{}^{\rho}I_{0}^{\eta}\left(h\left(s\right)\right)ds \\
&= \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_{a}^{t}s^{\zeta-1}\left(t^{\rho}-s^{\rho}\right)^{\zeta-1}\frac{\rho^{1-\eta}}{\Gamma(\eta)}\int_{a}^{s}x^{\rho-1}\left(s^{\rho}-x^{\rho}\right)^{\eta-1}h\left(x\right)dxds \\
&= \frac{\rho^{1-\zeta-\eta}}{\Gamma(\zeta)\Gamma(\eta)}\int_{a}^{t}h\left(x\right)x^{\rho-1}dx\int_{x}^{t}\rho\left(t^{\rho}-s^{\rho-1}\right)s^{\rho-1} \\
&\quad .\left(s^{\rho}-x^{\rho}\right)^{\eta-1}ds. \qquad (1.22)
\end{aligned}$$

Let $\left(s^{\rho}-x^{\rho}\right)=u\left(t^{\rho}-x^{\rho}\right)$, we get $\rho s^{\rho-1}ds=\left(t^{\rho}-x^{\rho}\right)du$ then

$$\begin{aligned}
\int_{x}^{t}\rho\left(t^{\rho}-s^{\rho}\right)^{\zeta-1}\left(s^{\rho}-x^{\rho}\right)^{\eta-1}s^{\rho-1}ds &= \int_{0}^{1}u^{\eta-1}\left(t^{\rho}-x^{\rho}\right)^{\eta-1}\left(t^{\rho}-x^{\rho}\right)du \\
&= \int_{0}^{1}\left(t^{\rho}-x^{\rho}\right)^{\zeta-1}u^{1-\zeta}u^{\eta-1}\left(t^{\rho}-x^{\rho}\right)^{\eta-1}\left(t^{\rho}-x^{\rho}\right)du \\
&= \left(t^{\rho}-x^{\rho}\right)^{\zeta+\eta-1}B\left(\zeta,\eta\right) \\
&= \left(t^{\rho}-x^{\rho}\right)^{\zeta+\eta-1}.\frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)}. \qquad (1.23)
\end{aligned}$$

according to the known formulae for the beta function Substituting (1.23) into (1.22), we obtain

$$\begin{aligned}
\left({}^{\rho}I_{a}^{\zeta}\right)\left({}^{\rho}I_{a}^{\eta}h\right)(t) &= \frac{\rho^{1-\zeta-\eta}}{\Gamma}(\zeta)\Gamma(\eta)\int_{a}^{t}h\left(x\right)x^{\rho-1}\left(t^{\rho}-x^{\rho}\right)^{\zeta+\eta-1}.\frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)}dx \\
&= \frac{\rho^{1-\zeta-\eta}}{\Gamma}(\zeta+\eta)\int at\frac{x^{\rho-1}}{\left(t^{\rho}-x^{\rho}\right)^{1-\zeta-\eta}}h\left(x\right)dx \\
&= {}^{\rho}I_{0}^{\zeta+\eta}h\left(t\right).
\end{aligned}$$

- The results follow from direct integration and derivatives for katugampola fractional integral of order $\zeta \in R_+$ defined by (1.19), we obtain

$$\begin{aligned} {}^{\rho}I_0^{\zeta}(h+g) &= \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\zeta}}(h+g)(s)\,ds \\ &= \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\zeta}}h(s)\,ds + \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\zeta}}g(s)\,ds \\ &= {}^{\rho}I_0^{\zeta}h(t) +^{\rho} I_0^{\zeta}g(t). \end{aligned}$$

for katugampola fractional derivatives of order $\zeta \in R_+$ defined by (1.20), we get

$$\begin{aligned} {}^{\rho}D_0^{\zeta}(h+g) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}I_0^{\zeta}(h+g)\right)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}I_0^{\zeta}h\right)(t) + \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}I_0^{\zeta}g\right)(t) \\ &= {}^{\rho}I_0^{\zeta}h(t) +^{\rho} I_0^{\zeta}g(t). \end{aligned}$$

■

Theorem 1.3.2 *[10, 21, 22]:*

Let ρ be a complex number, $Re(\rho) > 0$,$n = [Re(\rho)]$ and $\rho > 0$: Then, for $t > a$,

1.
$$\lim_{\rho\to 1}\left({}^{\rho}I_a^{\zeta}h\right)(t) = \frac{1}{\Gamma(\zeta)}\int_a^t (t-s)^{\zeta-1}h(s)\,ds \tag{1.24}$$

2.
$$\lim_{\rho\to 0}\left({}^{\rho}I_a^{\zeta}h\right) = \frac{1}{\Gamma(\zeta)}\int_a^t \left(\log\frac{t}{s}\right)^{\zeta-1}h(s)\frac{ds}{t} \tag{1.25}$$

3.
$$\lim_{\rho\to 1}\left({}^{\rho}D_a^{\zeta}h\right) = \frac{1}{\Gamma(n-\zeta)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\zeta-1}f(s)\,ds \tag{1.26}$$

4.
$$\lim_{\rho\to 0}\left({}^{\rho}D_a^{\zeta}h\right) = \frac{1}{\Gamma(n-\zeta)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{s}\right)^{n-\zeta-1}h(s)\frac{ds}{t}ds \tag{1.27}$$

Proof. The equations (1.24) and (1.26) follow from taking limits when $\rho \to 1$, while (3.2) follows from the L'Hospital rule by noticing that

$$\begin{aligned} \lim_{\rho\to 0}\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_a^t \frac{h(s)\,s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\zeta}} &= \frac{1}{\Gamma(\zeta)}\int_a^t \lim_{\rho\to 0}h(s)\,s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\zeta-1}ds \\ &= \frac{1}{\Gamma(\zeta)}\int_a^t \left(\frac{t}{s}\right)^{\zeta-1}h(s)\frac{ds}{s}. \end{aligned}$$

The proof of (1.27)

$$\begin{aligned}\lim_{\rho\to 0}\frac{\rho^{\zeta-n+1}}{\Gamma(n-\zeta)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t\frac{h(s)\,s^{\rho-1}}{(t^\rho-s^\rho)^{\zeta-n+1}} &= \frac{1}{\Gamma(n-\zeta)}\lim_{\rho\to 0}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t h(s)\,s^{\rho-1}\\ &\quad .\left(\frac{t^\rho-s^\rho}{\rho}\right)^{n-\zeta+1}ds\\ &= \frac{1}{\Gamma(n-\zeta)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-\zeta-1}h(s)\,\frac{ds}{t}ds.\end{aligned}$$

■

Remark 1.3.1 *:[10]*

1. $\lim_{\rho\to 1}\left({}^rI_a^\zeta h\right)=\left({}^{RL}I_a^\zeta h\right)(t)$.
2. $\lim_{\rho\to 0}\left({}^rI_a^\zeta h\right)=\left({}^hI_a^\zeta h\right)(t)$.
3. $\lim_{\rho\to 1}\left({}^rD_a^\zeta h\right)=\left({}^{RL}D_a^\zeta h\right)(t)$.
4. $\lim_{\rho\to 0}\left({}^rD_a^\zeta h\right)=\left({}^hI_a^\zeta h\right)(t)$.

Lemma 1.3.3 *[10]:*
Let $0<\zeta<1$. The fractional equation $\left({}^\rho D_0^\zeta v\right)(t)=0$, , has as solution

$$v(t)=ct^{\rho(\zeta-1)}, \tag{1.28}$$

with $c\in\mathbb{R}$.

Lemma 1.3.4 *[10]:*
Let $0<\zeta<1$. Then

$${}^\rho I_0^\zeta\left({}^\rho D_0^\zeta h\right)(t)=h(t)+ct^{\rho(\zeta-1)}. \tag{1.29}$$

Proof.

Let n=1 and ${}^\rho I_0^\zeta h(t)=\left(t^{1-\rho}\frac{d}{dt}\right)^\rho I_0^{\zeta+1}h(t)$, we have

$$\begin{aligned}I_0^\zeta\left({}^\rho D_0^\zeta h\right)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)I_0^{\zeta+1}D_0^\zeta h(t)\\ &= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-\zeta}}{\Gamma(\zeta+1)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{-\zeta}}\left({}^\rho D_0^\zeta h(s)\right)ds\right)\\ &= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-\zeta}}{\Gamma(\zeta+1)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{-\zeta}}\left[\left(s^{1-\rho}\frac{d}{ds}\right)\left(I_0^{1-\zeta}h\right)(s)\right]ds\right)\\ &= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-\zeta}}{\Gamma(\zeta+1)}\int_0^t(t^\rho-s^\rho)^\zeta\left[\frac{d}{ds}\left(I_0^{1-\zeta}h\right)(s)\right]ds\right)\end{aligned}$$

Notice here that the integral is evaluated by

$$g = (t^{\rho} - s^{\rho}) \rightarrow g' = -\rho\zeta s^{\rho-1} (t^{\rho} - s^{\rho})^{\zeta-1},$$
$$f' = \frac{d}{ds}\left(I_0^{1-\zeta} h\right)(s) \rightarrow f = I_0^{1-\zeta}.$$

Then

$$I_1 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{\zeta}}{\Gamma(\zeta+1)}\left(\left[(t^{\rho}s^{\rho})^{\zeta} I_0^{1-\zeta}\right]_0^t\right),$$
$$I_2 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{-\zeta}}{\Gamma(\zeta+1)}\int_0^t \zeta\rho s^{\rho-1}(t^{\rho} - s^{\rho})^{\zeta-1} I_0^{1-\zeta} h(s)\, ds.$$

Hence , we get

$$I_1 = ct^{\rho(\zeta-1)}$$
$$\begin{aligned} I_2 &= \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t s^{\rho-1}(t^{\rho} - s^{\rho})^{\zeta-1} I_0^{1-\zeta} h(s)\, ds \\ &= \left(t^{1-\rho}\frac{d}{dt}\right) I_0^{\zeta}\left(I^{1-\zeta}\right) h(t) \\ &= \left(\frac{t^{1-\rho}}{dt}\right)\left(I_0^1 h\right)(t) \\ &= h(t). \end{aligned}$$

Finally, we obtain

$${}^{\rho}I_0^{\zeta}\left({}^{\rho}D_0^{\zeta} h\right)(t) = h(t) + ct^{\rho(\zeta-1)}.$$

■

Definition 1.3.8 *[10]:*

By a random solution of problem (3.1) and (??) we mean a measurable function $x(w,.) \in \mathcal{C}_{\mathcal{T}}$ which satisfies (3.1) and (??).

Lemma 1.3.5 *[3, 25] :*

Let $T : \Omega \times E \rightarrow E$ be a mapping such that $T(.,v)$ is measurable for all $v \subset E$, and $T(w,.\ )$ is continuous for all $w \in \Omega$ Then the map $(w,v) \rightarrow T(w,v)$ is jointly measurable.

Definition 1.3.9 *[10]:*

A function $h : I \times E \times \Omega \rightarrow E$ is called random Carathéodory if the following conditions are satisfied:

- *The map* $(s,w) \to h(s,x,w)$ *is jointly measurable for all* $x \in E$ *, and*
- *The map* $x \to h(s,x,w)$ *is continuous for almost all* $s \in I$ *and* $w \in \Omega$.

  Let $\epsilon > 0$ *and* $\phi : \Omega \times I \mapsto \mathbb{R}_+$ *be a jointly measurable function. We consider the following inequality*

$$\left\| {}^{\rho}D_0^{\zeta}x\left(s,w\right) - h\left(s,x\left(s,w\right),w\right)\right\| \leq \phi\left(s,w\right), for\ s \in I \quad and \quad w \in \Omega. \tag{1.30}$$

Definition 1.3.10 *[4]:*
The problem (3.1) *and* (**??**) *is generalized Ulam-Hyers-Rassias stable with respect to* ϕ *if there exists* $c_{h.\phi} > 0$ *such that for each solution* $x\left(.,w\right) \in \mathcal{C}_{\mathcal{T}}$ *of the inequality* (1.30) *there exists* $y\left(.,w\right) \in \mathcal{C}_{\mathcal{T}}$ *satisfies and* (3.1) *and* (**??**) *with*

$$\left\| t^{\rho(1-\zeta)}x\left(t,w\right) - t^{\rho(1-\zeta)}y\left(t,w\right)\right\| \leq c_{h.\phi}\phi\left(t,w\right), \qquad t \in I, w \in \Omega. \tag{1.31}$$

Theorem 1.3.3 *[18] :*
Let X *be a nonempty, closed convex bounded subset of the separable Banach space* E *and let* $G : \Omega \times X \mapsto X$ *be a compact and continuous random operator. Then the random equation* $G\left(w\right)u = u$ *has a random solution.*

1.4 Topics of functional analysis

[13]

Let E, F be two Banach spaces, and let $C(E,F)$ be the space of all continuous functions $f : E \to F$.

Definition 1.4.1 *Let* M *be a subset of* $C(E,F)$.
1. M *is said to be equicontinuous in* $u \in E$ *if for any* $\varepsilon > 0$, *there exists* $\eta > 0$ *such that :*

$$\| f(u) - f(v) \|_F < \varepsilon,$$

and this for all $f \in M$*and for all* $v \in E$ *verifying :*

$$\| u - v \|_E < \eta.$$

2. We say that M *is equicontinuous on* E*, if* M *is equicontinuous in any* $u \in E$*. In particular,* *if $E = [a, b]$ and $F = \mathbb{R}$. $M \in C(E, F)$ is said to be equicontinuous on E if and only if :*

$$\forall \varepsilon > 0, \exists \eta > 0, \forall f \in M, \forall x, y \in [a, b] : |x - y| < \eta \Longrightarrow | f(x) - f(y) | < \varepsilon.$$

Definition 1.4.2 *Let M be a subset of $C(E, F)$. We say that M is uniformly bounded, if there exists a constant $C > 0$ such that :*

$$\|f\| \leq C \quad \forall f \in M.$$

Theorem 1 *Let M be a part of $C([a, b])$ equipped with the uniforme convergence norm. M is relatively compact in $C([a, b])$ if and only if M is equicontinuous and uniformely bounded.*

Definition 1.4.3 *Let $A : E \to F$ be an operator. We say that :*
- A is compact if the image by A of any bounded of E is relatively compact in F.
- A is said to be completely continuous if it is continuous and compact.

Theorem 2 ***(Arzela-Ascoli theorem)***

Let E be a compact Banach space and F any Banach space. A part M of $C(E, F)$ is relatively compact if and only if:
1. M is equicontinuous on E.
2. For any $x \in E$, the space $M(x)$ defined by :

$$M(x) = \{f(x)/f \in M\}$$

is relatively compact in F.

CHAPTER 2

DYNAMICS AND STABILITY FOR KATUGAMPOLA RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS

In the article of Dynamics and stability for Katugampola random fractional differential equations we investigate the following class of Katugampola random fractional differential equation see[10]

$$\left({}^{\rho}D_0^{\zeta}u\left(t,w\right)\right)=h\left(t,u_t\left(t,w\right),w\right),\ t\in I=\left[0,T\right],\tag{2.1}$$

with the terminal condition

$$u\left(T,w\right)=u_T\left(w\right),\tag{2.2}$$

where ρD_0^{ζ} is Katugampola operator of order ζ with $0<\zeta\leq 1$ and $\rho>0$, $T>0$, $x_T:\Omega\mapsto E$ is a measurable function , $h:I\times\mathcal{C}\times\Omega\mapsto E$ is a measurable function , and Ω is the sample space in a probability space, and $(E,\|.\|)$ is a Banach space.

2.1 Preliminaries of article

Lemma 2.1.1 *[10] The problem*

$$\begin{cases}\left({}^{\rho}D_0^{\zeta}\left(t\right)\right)=h\left(t\right),t\in I:=\left[0,T\right],\\ u\left(T\right)=u_T,\end{cases}\tag{2.3}$$

has the following solution

$$u(t) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s)\, ds - \frac{t^{\rho(\zeta-1)}}{T^\rho(\zeta-1)} \left(\frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^T \frac{s^{\rho-1}}{(T^\rho - s^\rho)^{1-\zeta}} h(s)\, ds - u_T \right). \tag{2.4}$$

Proof. Solving the equation we use lemma (1.28),(1.29)

$$^\rho I_0^\zeta \left(^\rho D_0^\zeta (t) \right) = u(t) + Ct^{\rho(\zeta-1)} =^\rho I_0^\zeta h(t)$$
$$u(t) =^\rho I_0^\zeta h(t) - Ct^{\rho(\zeta-1)}$$

we get

$$u(t) =^\rho I_0^\zeta h(t) - Ct^{\rho(\zeta-1)}$$

From the condition, we get

$$u(T) =^\rho I_0^\zeta h(T) - CT^{\rho(\zeta-1)} = u_T$$
$$C = \frac{^\rho I_0^\zeta h(T) - u_T}{T^{\rho(\zeta-1)}}$$

hence, we obtain (2.3) ■

Definition 2.1.1 ***[10]:***

By a random solution of problem (2.1) *and* (2.2) *we mean a measurable function* $x(w,.) \in C_{\zeta\rho}(I)$ *which satisfies* (2.1) *and* (2.2)

Lemma 2.1.2 *u is a random solution* (2.1) *and* (2.2) *, if and only if it satisfies*

$$x(t,w) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s)\, ds - C(w)\, t^{\rho(\zeta-1)}, \tag{2.5}$$

where

$$C(w) = \frac{1}{T^\rho(\zeta-1)} \left(\frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^T \frac{s^{\rho-1}}{(T^\rho - s^\rho)^{1-\zeta}} h(T,x,w)\, ds - x_T(w) \right).$$

Lemma 2.1.3 ***[3, 25] :***

Let $T : \Omega \times E \to E$ *be a mapping such that* $T(.,v)$ *is measurable for all* $v \subset E$, *and* $T(w,.)$ *is continuous for all* $w \in \Omega$ *Then the map* $(w,v) \to T(w,v)$ *is jointly measurable.*

Definition 2.1.2 ***[4]:***

The problem (2.1) *and* (2.2) *is generalized Ulam-Hyers-Rassias stable with respect to* ϕ *if there* *exists* $c_{h.\phi} > 0$ *such that for each solution* $x(.,w) \in \mathcal{C}_{\mathcal{T}}$ *of the inequality* (1.30) *there exists* $y(.,w) \in \mathcal{C}_{\mathcal{T}}$ *satisfies and* (2.1) *and* (2.2) *with*

$$\left\| t^{\rho(1-\zeta)} x(t,w) - t^{\rho(1-\zeta)} y(t,w) \right\| \leq c_{h.\phi} \phi(t,w), \qquad t \in I, w \in \Omega. \tag{2.6}$$

2.2 Existence and Ulam stability results

We shall make use of the following hypothese

- (H_1) the function $h : I \times \mathcal{C} \to \mathbb{E}$ is a random Caratheodory function .
- (H_2) There exist measurable and essentially bounded functions.
  $l_i : \Omega \mapsto C(\mathbb{I})$; $i = 1, 2$ such that

  $$\|h(t,x,w)\| \leq l_1(t,w) + l_2(t,w) t^{\rho(1-r)} \|x\|_{\mathcal{C}_\alpha},$$

  for all $x \in \mathcal{C}$ and $t \in I$ with

  $$l_i^*(w) = \sup_{t \in I} l_i(t,w); i = 1, 2, \quad w \in \Omega.$$

Theorem 2.2.1 *If* (H_1) *and* (H_2) *hold , and*

$$\frac{\rho^{-\zeta} T^\rho}{\Gamma(\zeta+1)} l_2^*(w) < 1, \tag{2.7}$$

then there exists a random solution for (2.1) *and* (2.2) .

Proof. Let $N : \mathcal{C} \times \Omega \to \mathcal{C}$ be the operator defined by

$$(Nx)(t,w) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x_s(s,w), w) - C(w) t^{\rho(\zeta-1)}, \tag{2.8}$$

and

$$R(w) \geq \frac{\|C(w)\| + \frac{\rho^{-\zeta} T^\rho}{\Gamma(\zeta+1)} l_1^*(w)}{1 - \frac{\rho^{-\zeta} T^\rho}{\Gamma(\zeta+1)} l_2^*(w) R(w)}, \qquad w \in \Omega. \tag{2.9}$$

and define the ball

$$B_R = B(0, R(w)) := \{x \in \mathcal{C} \|x\|_{\mathcal{C}} \leq R(w)\}.$$

For any $w \in \Omega$, $x \in B_R$ and each $t \in I$ and we have

$$\begin{aligned}
\| \left(t^{\rho(1-\zeta)} N x\right)(t,w) \| &\leq \|C(w)\| + \|\frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x_s(s,w), w)\, ds\| \\
&\leq \|C(w)\| + \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} \|l_1(s,w)\| ds \\
&+ \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}T^{\rho(1-\zeta)}}{(t^\rho - s^\rho)^{1-\zeta}} \|s^{\rho(1-\zeta)} l_2(s,w)\, x(s,w)\| ds \\
&\leq \|C(w)\| + \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \frac{T^{\zeta\rho}}{\zeta\rho} l_1^*(w) \\
&+ \frac{l_2^*(w)\,\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} \|s^{\rho(1-\zeta)} x(s,w)\| ds \\
&\leq \|C(w)\| + \frac{\rho^{-\zeta}T^{\rho}}{\Gamma(\zeta+1)} l_1^*(w) + \frac{\rho^{-\zeta}T^{\rho}}{\Gamma(\zeta+1)} l_2^*(w) \|x(s,w)\|_C \\
&\leq \|C(w)\| + \frac{\rho^{-\zeta}T^{\zeta\rho}}{\Gamma(\zeta+1)} l_1^*(w) + \frac{\rho^{-\zeta}T^{\zeta\rho}}{\Gamma(\zeta+1)} l_2^*(w)\, R(w) \\
&\leq R(w).
\end{aligned}$$

Thus

$$\|N(w)\,x\| \leq R(w).$$

Hence

$N(w)(B_R) \subset B_R$ We shall prove that $N : \Omega \times B_R \mapsto B_R$ satisfies the assumptions of Theorem

- **step1**. $N(w)$ is a random operator

  From (H_1) the map $w \mapsto h(t, x_t, w)$ is measurable and further the integral is a limit of a finite sum of measurable functions therefore the map

$$w \longrightarrow \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x, w)\, ds,$$

  is measurable.

- **step2**. $N(w)$ is continuous.

  Consider the sequence $(x_n)_n$ such that $x_n \to u \in \mathcal{C}$.

  Set

  $v_n(t,w) = (Nx_n)(t,w)$ and $v(t,w) = (Nx)(t,w)$.

Then

$$\|t^{\rho(1-\zeta)}v_n(t,w) - t^{\rho(1-\zeta)}v(t,w)\|$$

$$\begin{aligned}
&\leq \|\frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}\left(h\left(s,(x)_n(s,w),w\right) - h\left(s,x(s,w),w\right)\right)ds\| \\
&\leq \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}\|\left(h\left(s,(x)_n(s,w),w\right) - h\left(s,x(s,w),w\right)\right)\|ds.
\end{aligned}$$

By (H_1) we obtain

$$\|t^{\rho(1-\zeta)}v_n(t,w) - t^{\rho(1-\zeta)}v(t,w)\|_{\mathcal{C}} \to 0 \text{ as } n \to \infty.$$

Consequently , $N(w) : B_R \subset B_R$ is continuous .

- **Step 3.** $N(w)\,B_R$ is is equicontinuous .

- **Case2.** For $0 \leq t_1 \leq t_2 \leq T$

$$\left\|t_2^{\rho(1-\zeta)}(Nx)(t_2,w) - t_1^{\rho(1-\zeta)}(Nx)(t_1,w)\right\|$$

$$
\begin{aligned}
&\leq \left\| \frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_2} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right. \\
&- \left. \frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_1} \frac{s^{\rho-1}}{\left(t_1^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right\| \\
&\leq \left\| \frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_1} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right. \\
&+ \frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \\
&- \left. \frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_1} \frac{s^{\rho-1}}{\left(t_1^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right\| \\
&\leq \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} \left\| h\left(s, \left(s, w\right), w\right) \right\| ds \\
&+ \left\| \frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_1} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right. \\
&- \left. \frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_0^{t_1} \frac{s^{\rho-1}}{\left(t_1^\rho - s^\rho\right)^{1-\zeta}} h\left(s, x\left(s, w\right), w\right) ds \right\| \\
&\leq \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} \left\| h\left(s, x\left(s, w\right), w\right) \right\| ds \\
&+ \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^{t_1} \left(\frac{t_2^{\rho(1-\zeta)}s^{\rho-1}}{\left(t_2^\rho - s^\rho\right)^{1-\zeta}} - \frac{t_1^{\rho(1-\zeta)}s^{\rho-1}}{\left(t_1^\rho - s^\rho\right)^{1-\zeta}} \right) \left\| h\left(s, x\left(s, w\right), w\right) \right\| ds \\
&\leq \frac{\left(t_2^\rho - t_1^\rho\right)^\zeta}{\Gamma(\zeta+1)\rho^\zeta} T^{\rho(1-\zeta)} \left(l_1^*(w) + l_2^*(w) R(w)\right) \\
&+ \frac{t_2^{\rho(1-\zeta)}\left(-\left(t_2^\rho - t_1^\rho\right)^\zeta + t_2^{\rho\zeta}\right) - t_1^{\rho(1-\zeta)}t_1^{\rho\zeta}}{\Gamma(\zeta+1)\rho^\zeta} \left(l_1^*(w) + l_2^*(w) R(w)\right) \\
&\leq \frac{t_2^\rho - t_2^{\rho(1-\zeta)}\left(t_2^\rho - t_1^\rho\right)^\zeta - t_1^\rho + T^{\rho(1-\zeta)}\left(t_2^\rho - t_1^\rho\right)^\zeta}{\Gamma(\zeta+1)\rho^\zeta} \left(l_1^*(w) + l_2^*(w) R(w)\right)
\end{aligned}
$$

$$\to 0 \text{ ;as } t_1 \to t_2.$$

Arzela-Ascoli theorem implies that $N : \Omega \times B_R \to B_R$ is continuous and compact. Hence; from Theorem (1.3.3) we deduce the existence of random solution to problem ■

2.2.1 Ulam stability results

- Now, we prove a result concerning the generalized Ulam-Hyers-Rassias stability of (2.1) and (2.2) .

We introduce the following additional hypotheses:

- (H_3) For any $\omega \in \Omega$,$\Phi(t, .) \subset L^1(0, \infty)$ and there exists a measurable and essentially bounded function q : $\Omega \mapsto C\left(I,[0,\ \infty)\right)$ such that

$$\left(1+\|x-y\|\right)\|h\left(t,x\left(t,w\right),w\right)-h\left(t,y\left(t,w\right),w\right)\|\leq q\left(t,w\right)\phi\left(t,w\right)t^{\rho(1-\zeta)}\|x-y\|.$$

- (H_4) There exists $\lambda_\phi>0$ such that

$$^{\rho}I_0^{\zeta}\phi\left(t,w\right)\leq\lambda_\phi\phi\left(t,w\right).$$

Remark 2.2.1 *: Hypothesis (H_3) implies (H_2) with*

$$l_1\left(t,w\right)=h\left(t,0,w\right)\ ,\ l_2\left(t,w\right)=q\left(t,w\right)\phi\left(t,w\right)\ .$$

Set

$$\phi^*\left(w\right)=\sup_{t\in I}\phi\left(t,w\right)\ ,\ q^*\left(w\right)=\sup_{t\in I}q\left(t,w\right)\ ,\ w\in\Omega\ .$$

Theorem 2.2.2 *If (H_1) , (H_3) (H_4)*

and

$$\frac{\rho^{-\zeta}T^{\rho}}{\Gamma\left(1+\zeta\right)}<1, \tag{2.10}$$

hold. Then the problem (2.1) and (2.2) has random solutions defined on I and and it is generalized Ulam- Hyers-Rassias stable.

Proof.:

From (H_1) , (H_3) and Remark (2.2.1) ; the problem (2.1) and (2.2) has at least one random solution y. Then, we have

$$y\left(t,w\right)=\frac{\rho^{1-\zeta}}{\Gamma\left(\zeta\right)}\int_0^t\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,y_s\left(s,w\right),w\right)ds.$$

Assume x be a random solution (1.30)

$$\|t^{\rho(1-\zeta)}x\left(t,w\right)-\frac{t^{\rho(1-\zeta)}\rho^{1-\zeta}}{\Gamma\left(\zeta\right)}\int_0^t\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,x_s\left(s,w\right),w\right)ds-C\left(w\right)t^{\rho(\zeta-1)}\|$$
$$\leq T^{\rho(1-\zeta)}\left({}^{\rho}I_0^{\zeta}\phi\right)\left(t,w\right).$$

From hypotheses (H_3) and (H_4) , we have

$$\|t^{\rho(1-\zeta)}x\left(t,w\right)-t^{\rho(1-\zeta)}y\left(t,w\right)\|$$

$$
\begin{aligned}
&\leq \|t^{\rho(1-\zeta)}x(t,w) - \frac{t^{\rho^{1-\zeta}\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, y(s,w), w)\, ds + C(w)\| \\
&+ \|\frac{\rho^{1-\zeta}t^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x(s,w), w)\, ds - C(w) \\
&- \frac{\rho^{1-\zeta}t^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, y(s,w), w)\, ds + C(w)\| \\
&\leq T^{\rho(1-\zeta)}\left({}^\rho I_0^\zeta \phi\right)(t,w) \\
&+ \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} \|h(s, x(s,w), w) - h(s, y(s,w), w)\| ds \\
&\leq T^{\rho(1-\zeta)}\left({}^\rho I_0^\zeta \phi\right)(t,w) \\
&+ \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} q^*(w)\phi(s,w)\, s^{\rho(1-\zeta)}\frac{\|x-y\|}{1-\|x-y\|} ds \\
&\leq T^{\rho(1-\zeta)}\lambda_\phi \phi(t,w) + T^{2\rho(1-\zeta)}\lambda_\phi \phi(t,w)\, q^*(w).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\|t^{\rho(1-\zeta)}x(t,w) - t^{\rho(1-\zeta)}y(t,w)\| &\leq \left(1 + T^{\rho(1-\zeta)}q^*(w)\right) T^{\rho(1-\zeta)}\lambda_\phi \phi(t,w) \\
&:= c_\phi \phi(t,w).
\end{aligned}
$$

Hence, problem (2.1) and (2.2) is generalized Ulam-Hyers-Rassias stable.

■

CHAPTER 3

DYNAMICS AND STABILITY FOR KATUGAMPOLA RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

In this chapter we change Equation (2.1) and (2.2) in the article(Dynamics and stability for Katugampola random fractional differential equations) with the delay and we use [1, 12, 15, 23]

3.1 Position of the problem

we consider the following initial value problem :

$$\begin{cases} {}^{\rho}D_0^{\zeta}u\left(t,w\right)=h\left(t,u_t,w\right), & t\in I=\left[0,T\right], \\ u\left(t,w\right)=g\left(t,w\right), & t\in\left[-\alpha,0\right], \end{cases} \tag{3.1}$$

where in this section ρD_0^{ζ} is Katugampola operator of order ζ with $0<\zeta\leq 1$ and $\rho>0$g is a measurable continuous function in $[-\alpha,0]$ α is the delay $\alpha>0$ h is a measurable function in $[0,T]$, $\mathcal{C}_T=C_{\zeta\rho}\left(\left[-\alpha,T\right],E\right)$ the Banach space defined on $[-\alpha,T]$.

$$u_t\left(w\right)=u\left(t+\tau,w\right).$$

Proposition 3.1.1 *Let $\zeta\in[0,1]$ and $g:\Omega\times[-\alpha,0]\to E$ is is a continuous function with $g\left(0\right)=0$. Then the problem*

$$\begin{cases} \left({}^{\rho}D_0^{\zeta}u\right)\left(t\right)=h\left(t,u_t\right), & t\in I=\left[0,T\right], \\ u\left(t\right)=g\left(t\right), & t\in\left[-\alpha,0\right], \end{cases} \tag{3.2}$$

has a solution defined by

$$\begin{cases} u(t) = {}^{\rho} I_0^{\zeta} h(t, u_t), & t \in I = [0, T], \\ u(t) = g(t), & t \in [-\alpha, 0], \end{cases} \tag{3.3}$$

Proof. Solving the equation

$$\left({}^{\rho} D_0^{\zeta} u\right)(t) = h(t, u_t)$$

We get

$$\begin{aligned} u(t) &= {}^{\rho} I_0^{\zeta} h(t, u_t) - c t^{\rho(\zeta - 1)} \\ u(t) &= \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\zeta}} h(s, u_s)\, ds - c t^{\rho(\zeta-1)} \end{aligned}$$

From the condition, we get c=0

hence, we obtain (3.3) .

■

3.2 Main results

3.2.1 Existence of solution

Lemma 3.2.1 *u is a random solution of* (3.1) *if and only if it satisfies*

$$\begin{cases} u(t, w) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\zeta}} h(s, u_s(s, w), w)\, ds, & t \in I = [0, T], \\ u(t, w) = g(t, w), & t \in [-\alpha, 0], \end{cases} \tag{3.4}$$

We shall make use of the following hypothese

- (H_1) the function $h : I \times \mathcal{C}_\alpha \to \mathcal{E}$ is a random Caratheodory function.
- (H_2) There exist measurable and essentially bounded functions $l_i : \Omega \mapsto C(E)$, $i = 1, 2$ such that

  $$\|h(t, x, w)\| \leq l_1(t, w) + l_2(t, w)\, t^{\rho(1-r)} \|x\|_{\mathcal{C}_\alpha},$$

  for all $x \in \mathcal{C}_\alpha$ and $t \in I$ with

  $$l_i^*(w) = \sup_{t \in I} l_i(t, w);\, i = 1, 2, \quad w \in \Omega.$$

Theorem 3.2.1 *If (H_1) and (H_2) hold , and*

$$\frac{\rho^{-\zeta}T^{\rho}}{\Gamma(\zeta+1)}l_2^*(w) < 1, \tag{3.5}$$

then there exists a random solution for (3.1) .

Proof. Let $N : \mathcal{C}_T \times \Omega \to \mathcal{C}_T$ be the operator defined by

$$(Nx)(t,w) := \begin{cases} \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x_s(s,w), w), & t \in I = [0,T], \\ g(t,w), & t \in J = [-\alpha, 0], \end{cases} \tag{3.6}$$

and

$$R(w) \geq \frac{\frac{\rho^{-\zeta}T^{\rho}}{\Gamma(\zeta+1)}l_1^*(w)}{1 - \frac{\rho^{-\zeta}T^{\rho}}{\Gamma(\zeta+1)}l_2^*(w)R(w)}, \qquad w \in \Omega, \tag{3.7}$$

Where

$$B_R = B(0, R(w)) := \{x \in \mathcal{C}_T \|x\|_{\mathcal{C}_T} \leq R(w)\}.$$

- **case1**:

  For any $w \in \Omega$, $x \in B_R$ and each $t \in [-\alpha, 0]$, we have

$$\|(Nx)(t,w)\| \leq \|g(t,w)\| \leq \|g(w)\|_{\mathcal{C}_\alpha} \leq \|g(w)\|_{\mathcal{C}_T} \leq R(w).$$

- **case2**: For any $w \in \Omega$, $x \in B_R$ and each $t \in I$, we have

$$\begin{aligned} \|\left(t^{\rho(1-\zeta)}Nx\right)(t,w)\| &\leq \|\frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, x_s(s,w), w)\, ds\| \\ &\leq \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}\|l_1(s,w)\| ds \\ &+ \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}T^{\rho(1-\zeta)}}{(t^\rho - s^\rho)^{1-\zeta}}\|s^{\rho(1-\zeta)}l_2(s,w)\, x_s(s,w)\| ds \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\frac{T^{\zeta\rho}}{\zeta\rho}l_1^*(w) \\
&+ \frac{l_2^*(w)\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}\|s^{\rho(1-\zeta)}x_s(s,w)\|ds \\
&\le \frac{\rho^{-\zeta}T^\rho}{\Gamma(\zeta+1)}l_1^*(w) + \frac{\rho^{-\zeta}T^\rho}{\Gamma(\zeta+1)}l_2^*(w)\|x(s,w)\|_{C_T} \\
&\le \frac{\rho^{-\zeta}T^{\zeta\rho}}{\Gamma(\zeta+1)}l_1^*(w) + \frac{\rho^{-\zeta}T^{\zeta\rho}}{\Gamma(\zeta+1)}l_2^*(w)R(w) \\
&\le R(w).
\end{aligned}
$$

Thus

$$\|N(w)x\| \le R(w)$$

Hence $N(w)(B_R) \subset B_R$. We shall prove that $N : \Omega \times B_R \mapsto B_R$. satisfies the assumptions of Theorem (1.3.3)

- **step1**. $N(w)$ is a random operator
  - **case1** For each $t \in [-\alpha, 0]$, we have the map $w \mapsto g(t,w)$ is measurable.
  - **case2** For each $t \in I$, we have from H_1 the map $w \mapsto h(t, x_t, w)$ is measurable and further the integral is a limit of a finite sum of measurable functions therefore the map

    $$w \longrightarrow \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}h(s, x_s, w)\,ds.$$

    is measurable.

- **step2**. $N(w)$ is continuous.
  - **Case1**: each $t \in [-\alpha, 0]$, $g(t)$ continuous then $N(w)$ is continuous
  - **Case2** : each $t \in I$, Consider the sequence $(x_n)_n$ such that $x_n \to u \in \mathcal{C}_T$.

    Set $v_n(t,w) = (Nx_n)(t,w)$ and $v(t,w) = (Nx)(t,w)$.

    Then

    $$\|t^{\rho(1-\zeta)}v_n(t,w) - t^{\rho(1-\zeta)}v(t,w)\|$$

$$
\begin{aligned}
&\le \|\frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}(h(s,(x_s)_n(s,w),w) - h(s,x_s(s,w),w))\,ds\| \\
&\le \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}}\|(h(s,(x_s)_n(s,w),w) - h(s,x_s(s,w),w))\|ds.
\end{aligned}
$$

By (H_1), we obtain

$$\|t^{\rho(1-\zeta)}v_n(t,w)-t^{\rho(1-\zeta)}v(t,w)\|_{\mathcal{C}_T}\to 0 \text{ as } n\to\infty.$$

Consequently , $N(w): B_R \subset B_R$ is continuous .

- **Step 3.** $N(w)B_R$ is is equicontinuous .
  - **Case1.** For $-\alpha\le t_1\le t_2\le 0$, we have

$$\|(Nx)(t_1,w)-(Nx)(t_2,w)\| = \|g(t_1,w)-g(t_2,w)\|\to 0 \qquad \text{as } t_1\to t_2.$$

  - **Case2.** For $0\le t_1\le t_2\le T$, we have

$$\left\|t_2^{\rho(1-\zeta)}(Nx)(t_2,w)-t_1^{\rho(1-\zeta)}(Nx)(t_1,w)\right\|$$

$$\begin{aligned}
\le\ & \left\|\frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right.\\
-\ & \left.\frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_1}\frac{s^{\rho-1}}{(t_1^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right\|\\
\le\ & \left\|\frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_1}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right.\\
+\ & \frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\\
-\ & \left.\frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_1}\frac{s^{\rho-1}}{(t_1^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right\|\\
\le\ & \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}\|h(s,x_s(s,w),w)\|\,ds\\
+\ & \left\|\frac{\rho^{1-\zeta}t_2^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_1}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right.\\
-\ & \left.\frac{\rho^{1-\zeta}t_1^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_0^{t_1}\frac{s^{\rho-1}}{(t_1^\rho-s^\rho)^{1-\zeta}}h(s,x_s(s,w),w)\,ds\right\|\\
\le\ & \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma(\zeta)}\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}\|h(s,x_s(s,w),w)\|\,ds\\
+\ & \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^{t_1}\left(\frac{t_2^{\rho(1-\zeta)}s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\zeta}}-\frac{t_1^{\rho(1-\zeta)}s^{\rho-1}}{(t_1^\rho-s^\rho)^{1-\zeta}}\right)\|h(s,x_s(s,w),w)\|\,ds\\
\le\ & \frac{(t_2^\rho-t_1^\rho)^\zeta}{\Gamma(\zeta+1)\rho^\zeta}T^{\rho(1-\zeta)}(l_1^*(w)+l_2^*(w)R(w))\\
+\ & \frac{t_2^{\rho(1-\zeta)}\left(-(t_2^\rho-t_1^\rho)^\zeta+t_2^{\rho\zeta}\right)-t_1^{\rho(1-\zeta)}t_1^{\rho\zeta}}{\Gamma(\zeta+1)\rho^\zeta}(l_1^*(w)+l_2^*(w)R(w))\\
\le\ & \frac{t_2^\rho-t_2^{\rho(1-\zeta)}(t_2^\rho-t_1^\rho)^\zeta-t_1^\rho+T^{\rho(1-\zeta)}(t_2^\rho-t_1^\rho)^\zeta}{\Gamma(\zeta+1)\rho^\zeta}(l_1^*(w)+l_2^*(w)R(w)).
\end{aligned}$$

$$\to 0 \text{ as } t_1 \to t_2$$

Arzela-Ascoli theorem implies that $N : \Omega \times B_R \to B_R$ is continuous and compact. Hence; from Theorem (1.3.3) we deduce the existence of random solution to problem ■

3.2.2 Ulam stability results

- Now, we prove a result concerning the generalized Ulam-Hyers-Rassias stability of (3.1) . We introduce the following additional hypotheses:

- (H_3) For any $\omega \in \Omega$,$\Phi(t,.) \subset L^1(0,\infty)$ and there exists a measurable and essentially bounded function q : $\Omega \mapsto C(I,[0,\ \infty))$. such that

$$(1+\|x-y\|)\,\|h(t,x(t,w),w)-h(t,y(t,w),w)\| \leq q(t,w)\,\phi(t,w)\,t^{\rho(1-\zeta)}\|x-y\|$$

- (H_4) There exists $\lambda_\phi > 0$ such that

$${}^{\rho}I_0^{\zeta}\phi(t,w) \leq \lambda_\phi \phi(t,w)\,.$$

Remark 3.2.1 *: Hypothesis (H_3) implies (H_2) with*

$$l_1(t,w) = h(t,0,w)\ ,\ l_2(t,w) = q(t,w)\,\phi(t,w)\ .$$

Set

$$\phi^*(w) = \sup_{t\in I}\phi(t,w)\ ,\ q^*(w) = \sup_{t\in I} q(t,w)\ ,\ w \in \Omega\ .$$

Theorem 3.2.2 *If (H_1) , (H_3) (H_4)*

and

$$\frac{\rho^{-\zeta}T^{\rho}}{\Gamma(1+\zeta)} < 1, \tag{3.8}$$

hold. Then the problem (3.1) has random solutions defined on I and J, and it is generalized Ulam- Hyers-Rassias stable.

Proof.:

- **Case1** $t \in [-\alpha, 0]$, g is unique (obvious)

- **Case2** From (H_1) , (H_3) and Remark (3.2.1) ; the problem (3.1) has at least one random solution y. Then, we have

$$y(t,w) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\zeta}} h(s, y_s(s,w), w)\,ds.$$

Assume x be a random solution of (1.30)

$$\|t^{\rho(1-\zeta)}x\left(t,w\right)-\frac{t^{\rho(1-\zeta)}\rho^{1-\zeta}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,x_{s}\left(s,w\right),w\right)ds\| \leq T^{\rho(1-\zeta)}\left({}^{\rho}I_{0}^{\zeta}\phi\right)\left(t,w\right).$$

From hypotheses (H_3) and (H_4) , we have

$$\|t^{\rho(1-\zeta)}x\left(t,w\right)-t^{\rho(1-\zeta)}y\left(t,w\right)\|$$

$$\begin{aligned}
\leq\ & \|t^{\rho(1-\zeta)}x\left(t,w\right)-\frac{t^{\rho^{1-\zeta}\rho(1-\zeta)}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,y_{s}\left(s,w\right),w\right)ds\| \\
+\ & \|\frac{\rho^{1-\zeta}t^{\rho(1-\zeta)}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,x_{s}\left(s,w\right),w\right)ds \\
-\ & \frac{\rho^{1-\zeta}t^{\rho(1-\zeta)}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}h\left(s,y_{s}\left(s,w\right),w\right)ds\| \\
\leq\ & T^{\rho(1-\zeta)}\left({}^{\rho}I_{0}^{\zeta}\phi\right)\left(t,w\right) \\
+\ & \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}\|h\left(s,x_{s}\left(s,w\right),w\right)-h\left(s,y_{s}\left(s,w\right),w\right)\|ds \\
\leq\ & T^{\rho(1-\zeta)}\left({}^{\rho}I_{0}^{\zeta}\phi\right)\left(t,w\right) \\
+\ & \frac{\rho^{1-\zeta}T^{\rho(1-\zeta)}}{\Gamma\left(\zeta\right)}\int_{0}^{t}\frac{s^{\rho-1}}{\left(t^{\rho}-s^{\rho}\right)^{1-\zeta}}q^{*}\left(w\right)\phi\left(s,w\right)s^{\rho(1-\zeta)}\frac{\|x_{s}-y_{s}\|}{1-\|x_{s}-y_{s}\|}ds \\
\leq\ & T^{\rho(1-\zeta)}\lambda_{\phi}\phi\left(t,w\right)+T^{2\rho(1-\zeta)}\lambda_{\phi}\phi\left(t,w\right)q^{*}\left(w\right).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|t^{\rho(1-\zeta)}x\left(t,w\right)-t^{\rho(1-\zeta)}y\left(t,w\right)\| &\leq \left(1+T^{\rho(1-\zeta)}q^{*}\left(w\right)\right)T^{\rho(1-\zeta)}\lambda_{\phi}\phi\left(t,w\right) \\
&:= c_{h\phi}\phi\left(t,w\right).
\end{aligned}$$

Hence, problem (3.1) is generalized Ulam-Hyers-Rassias stable.

■

3.3 An example

Consider the Katugampola random fractional differential equation with delay of the form

$$\begin{cases} \left({}^{\rho}D_0^{\zeta}u\left(t\right)\right) = \frac{t^{\rho(1-\zeta)}\exp(-t)|u_t|}{(9+\exp(t))(1+|u_t|)}, & t \in I = [0,1]\,, \qquad 0<\zeta<1, \\ u\left(t\right) = g\left(t\right) = \frac{\exp(-t)-1}{2}, & t \in [-\alpha, 0]\,. \end{cases} \tag{3.9}$$

Let us take, $\zeta = \frac{1}{2}, \rho = 1$, α a non negative constant . $u_t\left(\theta\right) = \left(\theta + t\right)$, for $-\alpha \leq \zeta \leq 0$ and $0 \leq t \leq 1$. Set
$f\left(t,u\right) = \frac{t^{\rho(1-\zeta)}\exp(-t)u_t}{(9+\exp(t))(1+u_t)}$ for $(t,z) \in [0,1] \times [0,\ +\infty)$.
We have

$$u\left(0\right) = g\left(0\right) = 0.$$

And f is a measurable,continous function.

Now, we can see that

$$\begin{aligned} \left|f\left(t,x_t\right) - f\left(t,y_t\right)\right| &= \left|\frac{t^{\frac{1}{2}}\exp\left(-t\right)\left|x_t\right|}{\left(9+\exp\left(t\right)\right)\left(1+\left|x_t\right|\right)} - \frac{t^{\frac{1}{2}}\exp\left(-t\right)\left|y_t\right|}{\left(9+\exp\left(t\right)\right)\left(1+\left|y_t\right|\right)}\right| \\ &\leq \frac{t^{\frac{1}{2}}\exp\left(-t\right)}{\left(9+\exp\left(t\right)\right)}\frac{\left|x_t-y_t\right|}{\left(1+\left|x_t\right|\right)\left(1+\left|y_t\right|\right)} \\ &\leq \frac{\exp\left(-t\right)}{\left(9+\exp\left(t\right)\right)}\frac{t^{\frac{1}{2}}\left|x_t-y_t\right|}{\left|x_t-y_t\right|+1}. \end{aligned}$$

Hence, hypotheses (H_3) and (H_4) are satisfied ((H_3) implies (H_2)) with

$$q\left(t\right) = exp\left(-t\right), \; \phi\left(t\right) = \frac{1}{9+\exp(t)}.$$

Hence by theorems (3.2.1) and (3.2.2) problem (3.9) admits a random solution, and is generalized Ulam-Hyers-Rassias stable.

CONCLUSION

In this thesis, we provided some suffcient conditions ensuring the existence of random solutions and the Ulam stability for a class of fractional differential equations involving the Katugampola fractional derivative with delay in Banach spaces. The techniques used are the random fixed point theory and the notion of Ulam-Hyers-Rassias stability.

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