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Etude de quelques équations différentielles fractionnaires par l'application de la fonction de Lyapunov

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Par
BELBALI Hadjer

Devant le jury composé de :

Mr Guerbati Kaddour	**Pr.**	**Univ. Ghardaïa**	**Président**
Mr Benbachir Maamar	**Pr.**	**Univ. Blida 1**	**Directeur de thèse**
Mr Benchohra Mouffak	**Pr.**	**Univ. Sidi Bel Abbes**	**Examinateur**
Mr Hachama Mohammed	**Pr.**	**Univ. Blida 1**	**Examinateur**
Mr Yadi Karim	**Pr.**	**Univ. Tlemcen**	**Examinateur**

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Dedications

I dedicate this modest work :
to my Father who passed away,
to my mother, my husband and my son, my sisters and my brothers , and all my Friends .

BELBALI Hadjer

Publications

1. **H. Belbali**, M. Benbachir, Existence results and Ulam-Hyers stability to impulsive coupled system fractional differential equations, Turkish Journal of Mathematics(2021), 45 : 1368-1385.
2. **H. Belbali**, M. Benbachir, Stability for coupled systems on networks with Caputo-Hadamard fractional derivative. Journal of Mathematical Modeling 2020 ; 9 (1) : 107-118.
3. **H. Belbali**, M. Benbachir, Stability results for linear fractional order systems with Caputo-Hadamard derivative.(submitted).
4. **H. Belbali**, M. Benbachir, S. Etemad and S. Rezapour, Existence Theory and Generalized Mittag-Leffler Stability for a Nonlinear Caputo-Hadamard FIVP via Lyapunov Method, AIMS Mathematics, (2022) 7(8) : 14419-14433 .

Abstract

In this thesis, we present some results on existence, uniqueness, and stability for a class of initial value problems and impulsive coupled system fractional differential equations involving Caputo-Hadamard, we also discuss stability for some coupled systems on networks and linear system fractional differential equations with Caputo-Hadamard derivative. Our results are based on some standard fixed point theorems, we also establish the Ulam-Hyers and Mittag-Leffler stability results for some addressed problems. We have also provided an illustrative example of each of our considered problems to show the validity of the conditions and justify the efficiency of our established results.

Keywords : Caputo-Hadamard derivative, fixed point theorems, impulsive coupled system, coupled system on networks, Ulam stability, Lyapunov function, Mittag-Leffler function, class-$\mathcal{K}$ functions, Laplace transform.

Table des matières

List of symbols

We use the following notations throughout this thesis

Sets

- ✓ $\mathbb{N}$: Set of natural numbers.
- ✓ $\mathbb{R}$: Set of real numbers.
- ✓ $\mathbb{C}$: Set of complex numbers.
- ✓ $\mathbb{R}^+$: Set of non-negative real numbers $\mathbb{R}^+ = [0, +\infty)$.
- ✓ $\mathbb{R}^*$: Set of non-zero real numbers .
- ✓ $\mathbb{R}^n$: Space of n-dimensional real vectors.
- ✓ $\mathbb{C}^n$: Space of n-dimensional complex vectors.
- ✓ $\mathbb{R}^{n\times m}$: Set of real matrices of dimension $n \times m$.
- ✓ $\mathbb{C}^{n\times m}$: Set of complex matrices of dimension $n \times m$.
- ✓ J : Be a finite interval on the half-axis $\mathbb{R}^+$.
- ✓ $B(x_0, r)$: Open ball of center x_0, radius $r > 0$.
- ✓ $\overline{B(x_0, r)}$: Closed ball of center x_0, radius $r > 0$.

Functions and subspaces of functions

- ✓ $\mathcal{L}$: Laplace transform.
- ✓ s : Variable of the Laplace transform of a continuous signal $s \in \mathbb{C}$.
- ✓ $(f(.))_i$: The ith component of the vector $f(.)$.
- ✓ $\mathcal{K}$: Class $\mathcal{K}$ function.
- ✓ $\Gamma(\cdot)$: Gamma function.
- ✓ $\mathrm{B}(\cdot, \cdot)$: Beta function.
- ✓ ${}^H I^{\alpha}_{a^+}$: The Hadamard fractional integral of order $\alpha > 0$.
- ✓ ${}^H D^{\alpha}_{a^+}$: The Hadamard fractional derivative of orde $\alpha > 0$.

- ✓ ${}^{cH}D^{\alpha}_{a^+}$: The Caputo-Hadamard fractional derivative of orde $\alpha > 0$.
- ✓ $C(J,\mathbb{R})$: Space of continuous functions on J.
- ✓ $C^n(J,\mathbb{R})$: Space of n time continuously differentiable functions on J.
- ✓ $AC(J,\mathbb{R})$: Space of absolutely continuous functions on J.
- ✓ $L^1(J,\mathbb{R})$: Space of Lebesgue integrable functions on J.
- ✓ $L^p(J,\mathbb{R})$: Space of measurable functions u with $|u|^p$ belongs to $L^1(J,\mathbb{R})$.
- ✓ $L^{\infty}(J,\mathbb{R})$: Space of functions u that are essentially bounded on J .
- ✓ X^p_c : Space of complex-valued Lebesgue measurable functions J.
- ✓ $E_{\alpha}(.)$: One parameter Mittag-Leffler function.
- ✓ $E_{\alpha,\beta}(.)$: Two parameters Mittag-Leffler function.

Matrices, operations and matrix relations

- ✓ $\det(A)$: Determiner of $A \in \mathbb{R}^{n\times n}$.
- ✓ $\lambda(A)$: Eigenvalues of $A \in \mathbb{R}^{n\times n}$.
- ✓ $spec(A)$: Spectrum of the matrix A : set of eigenvalues of A.
- ✓ A^{-1} : Inverse of A; $\det(A) \neq 0$
- ✓ I : Identity matrix .
- ✓ $diag(A_1,\ldots,A_p)$: Diagonal matrix constituted with the elements of the diagonal of the matrices $A_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,p$.
- ✓ $||A||_M = \max\limits_{1\leq i,j\leq n}\{|a_{ij}|\}$: Matrix norm A, where M is the space of n-dimensional square matrices whose elements are complex numbers.

Other math operators

- ✓ $*$: Convolution product.
- ✓ $Re(z) > 0$: Real part of complex α.
- ✓ $\arg(.)$: Argument of a complex number.
- ✓ $|.|$: Module of a complex number.
- ✓ $\in$: Belongs to.
- ✓ sup : Supremum.
- ✓ max : Maximum.
- ✓ $n!$: Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.

Introduction

Fractional calculus is a recent field of mathematical analysis regarding fractional differential equations (FDEs) which have become the most important branch in applied analysis because of its extensive applications in a vast range of applied sciences [14, 44, 53, 75, 86, 117]. Up to now, there exist several kinds of fractional integrals and derivatives, like Riemann-Liouville, Caputo, Hadamard integrals and derivatives. The Hadamard fractional derivative is a kind of fractional derivative due to Hadamard in 1892[43], this fractional derivative differs from the Riemann-Liouville and Caputo fractional derivatives in the sense that the kernel of the integral appearing in the Hadamard derivative is the power of $\ln(\frac{t}{w})$, but the kernel takes the power of $(t-w)$ in the Riemann-Liouville. On the other hand, the Hadamard derivative is viewed as a generalization of the operator $(t\frac{d}{dt}^{n})$, while the Riemann-Liouville derivative is considered as an extension of the classical operator $(\frac{d}{dt})^n$. There are several articles describing the properties and applications of the Hadamard derivative [8, 9, 13, 19, 21, 27, 28, 29, 70, 71, 82].
One of the fractional derivatives that is defined by the combination of the properties of the Caputo and Hadamard operators is the Caputo-Hadamard fractional derivative[46]. There are limited fractional models and problems designed by this operator. Examples can be seen in [5, 9, 12, 25, 39, 60, 61].

Hence, as we see, the existence and uniqueness problems for FDEs have many forms according to the shape of the differential model and of course the form of the initial or boundary conditions. The Banach fixed point theorem [20] is essential for the uniqueness of the solution, but it needs a strong assumption to be applied, namely, the contraction principle (see monographs [53, 76, 86]). It is very popular in the literature to suggest a solution to fractional differential equations by adding various forms of fractional derivatives (see[10, 11, 12, 21, 23, 33, 50, 51, 94]).
In 1964 Perov[83] formulated a fixed point theorem that extends the well-known contraction mapping principle for the case when the metric d takes values in $\mathbb{R}^m$, that is, in the case when we have a generalized metric space. There are more findings concerned with the issues of Perov fixed point theorem [24, 26, 37, 38, 49, 81, 85, 103].

On the other hand, the stability of fractional order systems is one of the important problems in the theory of fractional calculus and its application in fractional control theory [99, 72, 74]. Many criteria for the stability of fractional order systems have been proposed by researchers. In 1996, Matignon [74] firstly studied the stability of linear fractional differential systems with the Caputo derivative. Since then, many researchers have implemented further investigations into the stability of such linear fractional

systems[15, 22, 34, 35, 41, 44, 62, 68, 69, 79, 80, 87, 114]. In regard to the nonlinear fractional systems, the stability criterion is much more difficult. The direct method attributed to Lyapunov gives a way to study a special type of stability termed the Mittag-Leffler stability for a given fractional nonlinear system without solving it explicitly(see [22, 91, 115]). Such a direct method due to Lyapunov is a sufficient condition to confirm the stability of the nonlinear systems ; in other words, the given systems may still be stable even if we cannot choose a Lyapunov mapping to fulfill the stability property for the mentioned system.
Another important aspect of the research that attracted the researcher's attention is Ulam stability and its various types. The above-mentioned stability was first introduced by Ulam [101] in 1940 and then was confirmed by Hyers in 1941 [102]. Rassias generalized the Ulam-Hyers stability by considering variables. Thereafter, mathematicians extended the work mentioned above to functional, differential, integrals, and FDEs. Wang [105] was the first mathematician who investigated the Ulam-Hyers stability for the impulsive ordinary differential equations in 2012. In the same line, he also obtained the aforesaid stability for the evolution equations [106]. For more details on the recent advances in the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of differential equations, one can see the monographs [30, 47] and the research papers [45, 48, 57, 65, 89, 109, 111]. We also note that Ulam stability has excellent applications in numerical analysis, optimization, economic, physics, biochemistry, and biological phenomena, and it does provide an effective way to seek the exact solution for the original equation.

Let us now briefly describe the organization of this thesis :

In **Chapter 1**, we provides the notation and preliminary results, descriptions, theorems and other auxiliary results that will be used throughout this thesis.

In **Chapter 2**, we study the existence, uniqueness and Ulam's type stability of a impulsive coupled system of fractional differential equations of the form :

$$\begin{cases} ({}^{cH}D_a^{\alpha}x)(t) &= f_1(t,x(t),y(t) \quad t\in[a,T], \quad t\neq t_k, k=1,...,m, \\ \left({}^{cH}D_a^{\beta}y\right)(t) &= f_2(t,x(t),y(t)) \quad t\in[a,T], \quad t\neq t_k, k=1,...,m, \\ \Delta x(t_k) &= I_k(x(t_k^-),y(t_k^-)), \qquad k=1,...,m, \\ \Delta y(t_k) &= \overline{I}_k(x(t_k^-),y(t_k^-)), \qquad k=1,...,m, \\ x(a)=x_a, \\ y(a)=y_a, \end{cases}$$

where $0<\alpha,\beta<1$, $a>0$. Here $a=t_0\leq t_1\leq\cdots\leq t_m\leq t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim\limits_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim\limits_{h\to 0}x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively. $x_a,y_a\in\mathbb{R}$, $f_1,f_2:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and I_k , $\overline{I}_k\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are a given functions.

In **Chapter 3**, we establish stability and uniform asymptotic stability of the trivial solution of the following coupled systems of fractional differential equations on networks

$$\begin{cases} {}^{cH}D^{\alpha}x_i=f_i(t,x_i)+\sum\limits_{j=1}^{n}g_{ij}(t,x_i,x_j), \quad t>t_0, \\ x_i(t_0)=x_{i0}, \end{cases}$$

where $1<\alpha\leq 2$, $i=1,2,\ldots,n$, and $f_i:\mathbb{R}_+\times\mathbb{R}^{m_i}\to\mathbb{R}^{m_i}$, $g_{ij}:\mathbb{R}_+\times\mathbb{R}^{m_i}\times\mathbb{R}^{m_j}\to\mathbb{R}^{m_i}$ are given functions.

Chapter 4, This chapter contain four sections. After the introduction section, in Section 4.2, we establish the stability of linear autonomous fractional differential system with Caputo-Hadamard derivative

$$\begin{cases} {}^{cH}D_a^{\alpha}x(t) = Ax(t), \quad t > a > 0,\ 0 < \alpha < 1 \\ x(a) = x_0, \end{cases}$$

where $x(t) \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$ and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

In section 4.3 , we examine the stability of perturbed fractional differential system

$$\begin{cases} {}^{cH}D_a^{\alpha}x(t) = Ax(t) + B(t)x(t), \quad t > a > 0,\ 0 < \alpha < 1 \\ x(a) = x_0, \end{cases}$$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $B(t) : [a,\infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

Finally, an example is also constructed to illustrate our results.

In **Chapter** 5, we discusses the existence, uniqueness and stability for a nonlinear fractional differential system consisting of nonlinear Caputo-Hadamard FIVP

$$\begin{cases} {}^{cH}D_c^{\ell}\phi(t) = A\phi(t) + \psi(t, \phi(t), {}^{cH}D_c^{\beta}\phi(t)), \quad t > c > 0, \\ \\ \Theta^k\phi(t)\mid_{t=c} = \phi_k, k = 0, 1, \end{cases}$$

where $1 < \ell < 2, 0 < \beta < \ell - 1$, $\phi_0, \phi_1 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\Theta = t\frac{d}{dt}$ and $\psi : [c,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

Chapitre 1

Preliminaries and Background Materials

The aim of this chapter is to introduce some basic concepts, notation and elementary results that are used throughout this thesis.

1.1 Functional spaces

Let $\mathbb{R} = (-\infty, +\infty)$ and let $J := [a, b]$ the compact interval of $\mathbb{R}$. we present the following functional spaces :

Definition 1.1. [52, 53] Let $C(J, \mathbb{R})$ is the Banach space of continuous functions $u : J \longrightarrow \mathbb{R}$ have the valued in $\mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup_{t \in J} |u(t)|.$$

Analogoustly, $C^n(J, \mathbb{R})$ the Banach space of functions $u : J \longrightarrow \mathbb{R}$ where u is n time continuously differentiable on J.

Denote by $L^1(J, \mathbb{R})$ the Banach space of functions u Lebesgues integrable with the norm

$$\|u\|_{L^1} = \int_a^b |u(t)| \mathrm{dt},$$

and we denote $L^p(J, \mathbb{R})$ the space of Lebesgue integrable functions on J where $|u|^p$ belongs to $L^1(J, \mathbb{R})$, endowed with the norm

$$\|u\|_{L^p} = \left[\int_a^b |u(t)|^p \mathrm{dt} \right]^{\frac{1}{p}}.$$

In particular, if $p = \infty$, $L^\infty(J, \mathbb{R})$ is the space of all functions u that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty} = \operatorname{ess\,sup}_{t \in J} |u(t)| = \inf\{c \geq 0 : |u(t)| \leq c, \text{ for a.e. } t\}.$$

We denote by $X^p_c(a,b)(c \in \mathbb{R}; 1 \leq p \leq \infty)$, consists of those complex-valued measurable functions f on (a,b) for which $||f||_{X^p_c} \leq \infty$, with the norm

$$||f||_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \ (1 \leq p < \infty).$$

For the case $p = \infty$, we note

$$||f||_{X^p_c} = ess \sup_{a \leq t \leq b} [t^c f(t)].$$

In particular, when $c = 1/p$, the space X^p_c coincides with the $L^p(a,b)-$space : $X^p_{1/p} = L^p(a,b)$.

1.1.1 Spaces of Absolutely Continuous Functions

We denote by $AC(J,\mathbb{R})$ (or $AC^1(J,\mathbb{R})$) the space of all absolutely continuous functions defined on J. It is known that $AC(J,\mathbb{R})$ coincides with the space of primitives of Lebesgue summable functions :

$$u \in AC(J,\mathbb{R}) \Leftrightarrow u(t) = c + \int_a^t \psi(s)\mathrm{ds}, \quad \psi \in L^1(J,\mathbb{R}), \tag{1.1}$$

and therefore an absolutely continuous function u has a summable derivative $u'(t) = \psi(t)$ almost everywhere on J. Thus (1.1) yields

$$u'(t) = \psi(t) \text{ and } \ c = u(a).$$

For $n \in \mathbb{N}^*$ we denote by $AC^n(J,\mathbb{R})$ the space of functions $u : J \longrightarrow \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$ belongs to $AC(J,\mathbb{R})$:

$$\begin{aligned} AC^n(J,\mathbb{R}) &= \left\{ u \in C^{n-1}(J,\mathbb{R}) : (D^{(n-1)}u)(x) \in AC(J,\mathbb{R}) \ \ (D = \frac{d}{dx}) \right\} \\ &= \left\{ u \in C^{n-1}(J,\mathbb{R}) : (D^{(n)}u)(x) \in L^1(J,\mathbb{R}) \ \ (D = \frac{d}{dx}) \right\}. \end{aligned}$$

The space $AC^n(J,\mathbb{R})$ consists of those and only those functions u which can be represented in the form

$$u(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \psi(s)\mathrm{ds} + \sum_{k=0}^{n-1} c_k t^k, \tag{1.2}$$

where $\psi \in L^1(J,\mathbb{R}), c_k \ (k = 1, \ldots, n-1) \in \mathbb{R}$.
It follows from (1.2) that

$$\psi(t) = u^{(n)}(t) \text{ and } c_k = \frac{u^{(k)}(a)}{k!}, \ (k = 1, \ldots, n-1).$$

We denote by $AC^n_{\delta,\mu}[a,b](n\in\mathbb{N};\mu\in\mathbb{R})$, involves the complex-valued Lebesgue measurable functions g on (a,b) such that $x^\mu g(x)$ has $\delta-$derivatives up to order $n-1$ on $[a,b]$ and $\delta^{n-1}[x^\mu g(x)]$ is absolutely continuous on $[a,b]$

$$AC^n_{\delta,\mu}[a,b]=\{g:[a,b]\to\mathbb{C}:\delta^{n-1}[x^\mu g(x)]\in AC[a,b],\ \mu\in\mathbb{R},\delta=x\frac{d}{dx}\}.$$

In particular, when $\mu=0$, the space $AC^n_{\delta}[a,b]:=AC^n_{\delta,0}[a,b]$ is defined by

$$AC^n_{\delta}(J)=\{g:[a,b]\to\mathbb{C}:\delta^{n-1}g(x)\in AC(J)\quad\delta=x\frac{d}{dx}\}.$$

If $\mu=0$ and $n=0$ the space $AC^1_\delta[a,b]$ coincides with $AC[a,b]$. For more details see [52, 53, 54] .

Theorem 1.2. *[19] The space $AC^n_{\delta,\mu}[a,b]$ consists of those and only those functions $g(x)$ which are represented in the form*

$$g(x)=x^{-\mu}\left[\frac{1}{(n-1)!}\int_a^x\left(\ln\frac{x}{t}\right)^{n-1}\varphi(t)\mathrm{dt}+\sum_{k=0}^{n-1}c_k\left(\ln\frac{x}{a}\right)^k\right],\tag{1.3}$$

where $\varphi(t)\in L^1[a,b]$ and $c_k(k=0,1,\dots,n-1)$ are arbitrary constants.

1.2 Special Functions of the Fractional Calculus

Before introducing the basic facts on fractional operators, we recall three types of functions that are important in Fractional Calculus : the Gamma, Beta, and Mittag-Leffler functions. Some properties of these functions are also recalled. More details about these functions can be found in [36, 40, 86].

1.2.1 Gamma function

Undoubtedly, one of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.3. [86] The Gamma function $\Gamma(z)$ is defined by the integral :

$$\Gamma(z)=\int_0^{+\infty}t^{z-1}e^{-t}\mathrm{dt},\tag{1.4}$$

which converges in the right half of the complex plane $Re(z)>0$
For positive integer values n, the Gamma function becomes $\Gamma(n)=(n-1)!$ and thus can be seen as an extension of the factorial function to real values.
One of the basic properties of the gamma function is that it satisfies the following functional equation :

$$\Gamma(z+1)=z\Gamma(z),\quad z>0.$$

1.2.2 Beta Function

Definition 1.4. [86] The beta function is usually defined by

$$\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, \quad (Re(p)>0, Re(q)>0).$$

In the following we will enumerate the basic properties of the Beta function :

Properties 1. [86]

1. The following formula which expresses the Beta function in terms of the Gamma function :
$$\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q>0.$$
2. For every $p>0$ and $q>0$, we have :
$$\mathrm{B}(p,q)=\mathrm{B}(q,p).$$
3. For every $p>0$ and $q>1$, the Beta function B satisfies the property :
$$\mathrm{B}(p,q)=\frac{q-1}{p+q-1}\mathrm{B}(p,q-1).$$
4. For any natural numbers m,n we obtain :
$$\mathrm{B}(m,n)=\frac{(n-1)!(m-1)!}{(n+m-1)!}.$$

1.2.3 Mittag-Leffler Function

The third function is a direct generalization of the exponential series, and it was defined by the mathematician Mittag-Leffler in 1903 [77].

Definition 1.5. [40, 86] The one-parameter Mittag-Leffler function $\mathbb{E}_\alpha(\cdot)$, is defined as :

$$\mathbb{E}_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad (z\in\mathbb{R},\ Re(\alpha)>0).$$

For $\alpha=1$, this function coincides with the series expansion of e^z, i.e.,

$$\mathbb{E}_1(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$$

While linear ordinary differential equations present in general the exponential function as a solution, the Mittag-Leffler function occurs naturally in the solution of fractional-order differential equations [53]. For this reason, in recent times, the Mittag-Leffler function has become an important function in the theory of the fractional calculus and its applications.

It is also common to represent the Mittag-Leffler function in two arguments. This generalization of Mittag-Leffler function was studied by Wiman in 1905 [107].

Definition 1.6. [40, 86] The Two-parameter Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$, is defined as :

$$\mathbb{E}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad (z,\beta\in\mathbb{C},\ Re(\alpha)>0).$$

For particular values of α and β it results :

$$\mathbb{E}_{0,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1)}=\sum_{k=0}^{\infty}z^k, \tag{1.5}$$

$$\mathbb{E}_{1}(z)=\mathbb{E}_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z, \tag{1.6}$$

$$\mathbb{E}_{1,0}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k)}=\sum_{k=0}^{\infty}\frac{z^k}{(k-1)!}=z\sum_{k=0}^{\infty}\frac{z^{k-1}}{(k-1)!}=ze^z, \tag{1.7}$$

$$\mathbb{E}^{(n)}_{\alpha,\beta}(z)=\frac{d^n}{dz^n}\mathbb{E}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{z^k}{\Gamma(\alpha(k+n)+\beta)}, \tag{1.8}$$

$$\frac{d}{dz}\mathbb{E}_{\alpha,1}(az^{\alpha})=\sum_{k=1}^{\infty}\frac{a^k z^{\alpha k-1}}{\Gamma(\alpha k)}=az^{\alpha-1}\sum_{k=0}^{\infty}\frac{(az^{\alpha})^k}{\Gamma(\alpha k+\alpha)}=az^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(az^{\alpha}), \tag{1.9}$$

$$\frac{d}{dz}\left(z^{\beta-1}\mathbb{E}_{\alpha,\beta}(az^{\alpha})\right)=z^{\beta-2}\mathbb{E}_{\alpha,\beta-1}(az^{\alpha}). \tag{1.10}$$

Lemma 1.7. *[86] Let $0<\alpha<2$, β be an arbitrary complex number and μ be an arbitrary real number such that $\frac{\alpha\pi}{2}<\mu<\min\{\pi,\alpha\pi\}$.*

Then, for an arbitrary integer $p\geq 1$, we have the following expansions :

$$\mathbb{E}_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp(z^{1/\alpha})-\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+O(|z|^{-1-p}), \tag{1.11}$$

when $|\arg(z)|\leq\mu$ and $|z|\to\infty$;

$$\mathbb{E}_{\alpha,\beta}(z)=-\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+O(|z|^{-1-p}), \tag{1.12}$$

when $\mu\leq|\arg(z)|\leq\pi$ and $|z|\to\infty$.

In particular If $\alpha=\beta$ we have

$$\mathbb{E}_{\alpha,\alpha}(z)=\frac{1}{\alpha}z^{(1-\alpha)/\alpha}\exp(z^{1/\alpha})-\sum_{k=2}^{p}\frac{z^{-k}}{\Gamma(\alpha-\alpha k)}+O(|z|^{-1-p}), \tag{1.13}$$

when $|\arg(z)|\leq\mu$ and $|z|\to\infty$;

$$\mathbb{E}_{\alpha,\alpha}(z)=-\sum_{k=2}^{p}\frac{z^{-k}}{\Gamma(\alpha-\alpha k)}+O(|z|^{-1-p}), \tag{1.14}$$

when $\mu\leq|\arg(z)|\leq\pi$ and $|z|\to\infty$.

1.3 Elements From Fractional Calculus Theory

In this section, we recall some definitions of Hadamard type fractional integral, derivatives, and caputo-Hadamard fractional derivatives that include all we use throughout this thesis. We conclude it by some necessary lemma, theorems and properties [19, 27, 28, 46, 53].

1.3.1 Fractional Integrals

Definition 1.8. [53]Let $0 \leq a \leq b \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The Hadamard fractional integrals of order $\alpha \in \mathbb{C}\,(Re(\alpha) > 0)$ are defined by

$$ {}^{H}I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t},\ (a < x < b), \tag{1.15}$$

Proposition 1.9. *[19]If $Re(\alpha) > 0$, $Re(\beta) > 0$ and $a < b < \infty$, then we have*

$$\left({}^{H}I^{\alpha}_{a^+} \left(\ln \frac{t}{a}\right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{x}{a}\right)^{\beta+\alpha-1}. \tag{1.16}$$

1.3.2 Fractional Derivatives

The Hadamard fractional derivatives

Definition 1.10. [53] The Hadamard fractional derivatives of order $\alpha \in \mathbb{C}\,(Re(\alpha) \geq 0)$ on (a,b) is defined by

$$\begin{aligned} ({}^{H}D^{\alpha}_{a^+} y)(x) &= \delta^n ({}^{H}I^{n-\alpha}_{a^+} y)(x) \\ &= (x\frac{x}{dx})^n \frac{1}{\Gamma(n-\alpha)} \int_a^x (\ln \frac{x}{t})^{n-\alpha+1} y(t) \frac{dt}{t} \quad (a < x < b) \end{aligned} \tag{1.17}$$

where $n = [Re(\alpha)] + 1$.

Proposition 1.11. *[53] If $Re(\alpha) > 0$, $Re(\beta) > 0$ and $0 < a < b < \infty$, then*

$$\left({}^{H}D^{\alpha}_{a^+} \left(\ln \frac{t}{a}\right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}. \tag{1.18}$$

In particular, if $\beta = 1$ and $Re(\alpha) \geq 0$, then the Hadamard fractional derivatives of a constant, in general, are not equal to zero :

$$\left({}^{H}D^{\alpha}_{a^+} \right)(1) = \frac{1}{\Gamma(1-\alpha)} \left(\ln \frac{x}{a}\right)^{-\alpha},$$

when $0 < Re(\alpha) < 1$. On the other hand, for $j = [Re(\alpha)] + 1$,

$$\left({}^{H}D^{\alpha}_{a^+} \left(\ln \frac{t}{a}\right)^{\alpha-j} \right)(x) = 0.$$

Lemma 1.12. *[19] Let $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$. If $f(x) \in AC_\delta^n[a,b]$ $(0 < a < b < \infty)$, then the Hadamaed fractional derivative ${}^H D_{a^+}^\alpha$ exist almost everywhere on $[a,b]$ and can be represented in the forms*

$$({}^H D_{a^+}^\alpha f)(x) = \sum_{k=0}^{n-1} \frac{(\delta^k f)(a)}{\Gamma(1+k-\alpha)} (\ln \frac{x}{a})^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (\ln \frac{x}{t})^{n-\alpha-1} (\delta^n f)(t) dt.$$

In particular, when $0 < Re(\alpha) < 1$, then, for $f(x) \in AC[a,b]$,

$$({}^H D_{a^+}^\alpha f)(t) = \frac{f(a)}{\Gamma(1-\alpha)} (\ln \frac{x}{a})^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_a^x (\ln \frac{x}{t})^{-\alpha} f'(t) \frac{dt}{t}$$

Proposition 1.13. *[27]Let $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha) > Re(\beta) > 0$. If $0 < a < b < \infty$, then, for $f \in L^p(a,b)\,(1 \le p \le \infty)$;*

$${}^H D_{a^+}^\beta \, {}^H I_{a^+}^\alpha f = {}^H I_{a^+}^{\alpha-\beta} f.$$

In particular, if $\beta = m \in \mathbb{N}$, then

$${}^H D_{a^+}^m \, {}^H I_{a^+}^\alpha f = {}^H I_{a^+}^{\alpha-m} f.$$

Theorem 1.14. *[53] Let $Re(\alpha) > 0$, $n = -[-Re(\alpha)]$ and $0 < a < b < \infty$,also let $({}^H I_{a^+}^{n-\alpha} f)(x)$ be the Hadamard type fractional integral of the form (1.15). If $f(x) \in L(a,b)$ and $({}^H I_{a^+}^{n-\alpha} f)(x) \in AC_\delta^n[a,b]$, then*

$$\left({}^H I_{a^+}^\alpha \, {}^H D_{a^+}^\alpha f\right)(x) = f(x) - \sum_{k=1}^{n} \frac{(\delta^{n-k}(I_{a^+}^{n-\alpha} f))(a)}{\Gamma(\alpha-k+1)} \left(\ln \frac{x}{a}\right)^{\alpha-k}.$$

In particular, if $\alpha = n \in \mathbb{N}$ and $f(x) \in AC_\delta^n[a,b]$, then

$$\left({}^H I_{a^+}^n \, {}^H D_{a^+}^\alpha f\right)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\delta^k f)(a)}{k!} \left(\ln \frac{x}{a}\right)^k.$$

The Caputo-Hadamard fractional derivatives

Definition 1.15. [39, 46] Let $Re(\alpha) \ge 0$ and $n = [Re(\alpha)] + 1$. If $f(x) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$. We define the Caputo-Hadamard fractional derivative as follow :

$${}^{cH} D_{a^+}^\alpha = {}^H D_{a^+}^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} (\ln \frac{t}{a})^k \right](x).$$

In particular, if $0 < Re(\alpha) < 1$, we have

$${}^{cH} D_{a^+}^\alpha = {}^H D_{a^+}^\alpha \left[f(t) - f(a)\right](x).$$

Theorem 1.16. *[39, 46] Let $Re(\alpha) \ge 0$, $n = [Re(\alpha) + 1]$. If $f(x) \in AC_\delta^n[a,b]$,where $0 < a < b < \infty$. Then ${}^{cH} D_{a^+}^\alpha f(x)$ exist everywhere on $[a,b]$ and*

1. *if* $\alpha \notin \mathbb{N}_0$,

$$ {}^{cH}D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\delta^n f(t)\frac{dt}{t} = {}^{H}I^{n-\alpha}_{a^+}\delta^n f(x),. \quad (1.19) $$

2. *if* $\alpha = n \in \mathbb{N}_0$,

$$ {}^{cH}D^{\alpha}_{a^+}f(x) = \delta^n f(x). \quad (1.20) $$

In particular,

$$ {}^{cH}D^{0}_{a^+}f(x) = f(x). \quad (1.21) $$

Lemma 1.17. *[46] Let* $Re(\alpha) \geq 0$, $n = [Re(\alpha)+1]$ *and* $f \in C[a,b]$.
If $Re(\alpha) \neq 0$ *or* $\alpha \in \mathbb{N}$, *then*

$$ {}^{cH}D^{\alpha}_{a^+}({}^{H}I^{\alpha}_{a^+}f)(x) = f(x). $$

Lemma 1.18. *[39, 46] Let* $f \in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $\alpha \in C$, *then*

$$ {}^{H}I^{\alpha}_{a^+}({}^{cH}D^{\alpha}_{a^+}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{\delta^k f(a)}{k!}\log(\frac{t}{a})^k. $$

Proposition 1.19. *[39, 46] Let* $Re(\alpha) \geq 0$, $n = [Re(\alpha)+1]$ *and* $Re(\beta) > 0$. *Then*

$$ {}^{cH}D^{\alpha}_{a}(\ln\frac{x}{a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{x}{a})^{\beta-\alpha-1} \quad Re(\beta) > n, \quad (1.22) $$

Theorem 1.20. *[39, 46][(Semigroup property for Caputo-Hadamard derivatives)] Let* $f(x) \in C^{m+n}_\delta[a,b]$, $0 < a < b < \infty$. *Moreover, let* $\alpha \geq 0, \beta \geq 0$ *such that* $n-1 < \alpha < n$, $m-1 < \beta < m$. *Then*

$$ {}^{cH}D^{\alpha}_{a}\,{}^{cH}D^{\beta}_{a}f(x) = {}^{cH}D^{\alpha+\beta}_{a^+}f(x). $$

Theorem 1.21. *[39, 46] Let* $f(x) \in C^n_\delta[a,b]$, $0 < a < b < \infty$, *and* $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$ *such that* $Re(\alpha) \geq 0, Re(\beta) \geq 0$. *Then*

$$ {}^{cH}D^{\alpha}_{a}\,{}^{H}I^{\beta}_{a}f(x) = {}^{H}I^{\alpha-\beta}_{a}f(x). $$

Lemma 1.22. *[5] Suppose that* f *is continuous. Then the initial value problem (IVP)*

$$ \begin{cases} {}^{cH}D^{\alpha}_{a}x(t) = f(t,x,y), & t > a,\ a > 0,\ 0 < \alpha < 1 \\ x(a) = x_a, \end{cases} $$

is equivalent to the following Volterra integral equation

$$ x(t) = x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s),y(s))}{s}ds. $$

1.4 Laplace transforms of Fractional Derivatives

Although Hadamard fractional calculus was early introduced, its Laplace transform seems not be available except the recent study [61]. It is known that the usual Laplace transform starts from the origin $t = 0$ so it is expediently applied to the (integer-order) calculus and the Rieman-Liouville calculus. Since the Hadamard calculus begins at $t = a > 0$ due to its (weakly) singular kernel involved logarithmic function $\ln t$, the classical Laplace transform can no be used. For this kind of calculus, we especially customize a kind of integral transform. In this part, we first introduce an modified Laplace transform.

Definition 1.23. [60, 61] For a given function f defined on $[a,\infty)(a > 0)$, the modified Laplace transform of $f(t)$ is defined as

$$\tilde{f}(s) = \mathbb{L}_a\{f(t)\} = \int_a^\infty e^{-s\ln\frac{t}{a}} f(t)\frac{dt}{t},\ s \in \mathbb{C}.$$

The following theorem guarantees the existence of the modified Laplace transform of a given function f satisfying suitable conditions.

Theorem 1.24. *[60] For a given function f defined on $[a,\infty)(a > 0)$, if*

1. *$f(t)$ is continuous or piecewise continuous on every finite subinterval of $[a,\infty)$,*
2. *there exist a positive constant $M > 0$ and $\sigma > 0$ such that for a given large $T > a$*

$$|f(t)| \le Mt^\sigma, \quad \text{when} \quad t > T,$$

then the modified Laplace transform of $f(t)$ exists with $Re(s) > \sigma$.

Definition 1.25. [60, 61]

The inverse modified Laplace transform of $\tilde{f}(s)$ is given by

$$f(t) = \mathbb{L}_a^{-1}\{\tilde{f}(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\ln\frac{t}{a}}\tilde{f}(s)\,ds,\ \ c > 0,\ i^2 = -1.$$

The following differential property can be proved by direct calculations.

Properties 2. [61]If $\mathbb{L}_a\{f(t)\} = \tilde{f}(s)$, then

$$\mathbb{L}_a\{\delta^n f(t)\} = s^n\tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\delta^k f(a),\ \ t > a > 0, n \in \mathbb{Z}^+.$$

Now we introduce the convolution and the corresponding property.

Definition 1.26. [61] For given functions f and g defined on $[a,\infty)(a > 0)$, the integral $\int_a^t f(a\frac{t}{w})g(w)\,\frac{dw}{w}$ is called the convolution of $f(t)$ and $g(t)$, i.e.,

$$f(t) * g(t) = (f * g)(t) = \int_a^t f(a\frac{t}{w})g(w)\,\frac{dw}{w}. \tag{1.23}$$

Properties 3. [61] If $\mathbb{L}_a\{f(t)\} = \tilde{f}(s)$ and $\mathbb{L}_a\{g(t)\} = \tilde{g}(s)$, then

$$\mathbb{L}_a\{f(t) * g(t)\} = \mathbb{L}_a\{f(t)\}\mathbb{L}_a\{g(t)\} = \tilde{f}(s)\tilde{g}(s).$$

Or equivalently,

$$\mathbb{L}_a^{-1}\{\tilde{f}(s)\tilde{g}(s)\} = \mathbb{L}_a^{-1}\{\tilde{f}(s)\} * \mathbb{L}_a^{-1}\{\tilde{g}(s)\} = f(t) * g(t).$$

Next, we present modified Laplace transforms of Hadamard integral and derivative, which will be useful in the coming section.

Lemma 1.27. *[61] Let $n-1 < \alpha < n \in \mathbb{Z}^+$, Then the following equalities hold :*

$$\mathbb{L}_a\{D_a^{-\alpha} f(t)\} = s^{-\alpha}\mathbb{L}\{f(t)\},$$

$$\mathbb{L}_a\{D_a^{\alpha} f(t)\} = s^{\alpha}\mathbb{L}_a\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}[\delta^k D_{a,t}^{-(n-\alpha)} f(t)]|_{t=a},$$

$$\mathbb{L}_a\{{}^cD_a^{\alpha} f(t)\} = s^{\alpha}\mathbb{L}_a\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\delta^k f(a).$$

1.4.1 Modified Laplace transform of Mittag-Leffler function

we give modified Laplace transform of Mittag-Leffler function. By means of the following formula [86]

$$\int_0^{\infty} e^{-st} t^{\alpha j+\beta-1} E_{\alpha,\beta}^{j}(\pm\lambda t^{\alpha})dt = \frac{j!s^{\alpha-\beta}}{(s^{\alpha}\pm\lambda)^{j+1}},\ \ Re(s) > |\lambda|^{\frac{1}{\alpha}},$$

applying the change of variable $t = \ln\frac{w}{a}$ gives

$$\int_a^{\infty} e^{-s\ln\frac{w}{a}} (\ln\frac{w}{a})^{\alpha j+\beta-1} E_{\alpha,\beta}^{j}\left(\pm\lambda(\ln\frac{w}{a})^{\alpha}\right)\frac{dw}{w} = \frac{j!s^{\alpha-\beta}}{(s^{\alpha}\pm\lambda)^{j+1}},\ \ Re(s) > |\lambda|^{\frac{1}{\alpha}}.$$

1.5 Graph theory

We gather together some basic concepts and theorems on graph theory (see[64, 96]). Let $G = (V, E)$ be a non-empty directed graph, i.e., $V = \{1, 2, \ldots, n\}$ is a set of vertices and E is a set whose elements are arcs (i, j) leading from initial vertex i to the terminal vertex j. A subgraph $H = (V, F)$ of G where F is includes in E is said to be spanning if H contains all vertices of G. A directed graph G is weighted if each arc (i, j) is assigned a positive weight a_{ij}. A directed path is a subgraph $P = (I, X)$ of the form $I = i_1, i_2 \ldots i_m,\ X = \{(i_k, i_{k+1}) : k = 1, 2, \ldots, m-1\}$ where the i_k are all distinct. If P is closed, namely $i_m = i_1$, we say that P is a directed cycle. A connected graph T is a tree if it has no cycles. A tree T is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A graph G is said to be strongly connected if from one vertex to other vertex there is a directed path.

Given a weighted digraph G with n vertices, we define the weight matrix $A = (a_{ij})_{n\times n}$ whose elements a_{ij} are the weight of arc (j,i), and we write (G,A). We define the weight $w(G)$ of G as the product of the weights of all its arcs. A weighted digraph (G,A) is said to be balanced if $w(C) = w(-C)$ for each directed cycle C. Here $-C$ denotes the reverse of C and is constructed by reversing the direction of all arcs in C. The Laplacian matrix of (G,A) is defined as

$$L = \begin{bmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \sum_{k\neq n} a_{nk} \end{bmatrix}$$

Let c_i denote the cofactor of the i-th diagonal element of L. The following result is standard in graph theory.

Proposition 1.28. *[64] Assume $n \geq 2$. Then*

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} \omega(\mathcal{T}) \quad i = 1,2,\ldots,n, \tag{1.24}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of (G,A) that are rooted at vertex i, and $\omega(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if (G,A) is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.

Theorem 1.29. *[64] Assume $n \geq 2$. Let c_i as defined in Proposition 1.28. Then the following identity holds :*

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i,x_j) = \sum_{Q\in\mathbb{Q}} w(Q) \sum_{(s,r)\in E_{c_Q}} F_{rs}(x_r,x_s). \tag{1.25}$$

Here $F_{ij}(x_i,x_j)$, $1 \leq i,j \leq n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of (G,A), $w(Q)$ is the weight of Q, and C_Q denotes the directed cycle of Q.

1.6 Topics of functional analysis

In this section, we present main preliminaries of functional analysis that will be used in this thesis.

1.6.1 fixed point theorems

The theory of fixed point is one of the most powerful tools of modern mathematics. The theorems which are concerning with the existence of solutions for differential equations.

Definition 1.30. A point $x \in X$ is called a fixed point of a function $f : X \to X$, if

$$f(x) = x,\ x \in X.$$

Banach[20] proved that a contraction mapping in the field of a complete normed space possesses a unique fixed point .This theorem is probably the most well-known fixed-point theorem.
This theorem is outstanding among fixed point theorems , because it is not only guarantees existence of a fixed point, but also its uniqueness, an approximation method actually to find the fixed point, a priori and a posteriori estimates for the rate of convergence.

Definition 1.31 (**Banach Fixed Point Theorem**). Let X be a Banach space and f be a contraction mapping with Lipschitz constant k. Then f has an unique fixed point.

1.6.2 Generalized metric space

In this part, we define generalized metric space (or vector metric spaces).
If $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \dots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$. For $x \in \mathbb{R}^n$, $(x)_i = x_i$, $i = 1, \dots, n$.

Definition 1.32. [42] Let X be a nonempty set. By a generalized metric on X (or vector-valued metric) we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties :
(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$.
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$.
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.
Note that for any $i \in \{1, \dots, n\}$ $(d(u, v))_i = d_i(u, v)$ is a metric space in X.

We call the pair (X, d) a generalized metric space. For $r = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered at x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered at x_0 with radius $r = (r_1, \dots, r_n) > 0$, $r_i > 0$, $i = 1, \dots, n$.

1.6.3 Generalized Banach space

Definition 1.33. [42] Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ By a vector-valued norm on E we mean a map $||.|| : E \to \mathbb{R}^n_+$ with the following properties :
(i) $||x|| \geq 0$,for all $x \in E$; if $||x|| = 0$ then $x = 0$
(ii) $||\lambda x|| = |\lambda| ||x||$ for all $x \in E$ and $\lambda \in \mathbb{R}$,
(iii) $||x + y|| \leq ||x|| + ||y||$, $x, y \in E$.
The pair $(E, ||.||)$ is called a generalized normed space. If the generalized metric generated by $||.||$($i.e$ $d(x, y) = |x - y|$) is complete then the space $(E, ||.||)$ is called a generalized Banach space, where

$$||x - y|| = \begin{pmatrix} ||x - y||_1 \\ \cdots \\ ||x - y||_n \end{pmatrix}.$$

Notice that $||.||$ is a generalized Banach space on E if and only if $||.||_i, i = 1, \ldots, n$ are norms on E.

Remark 1.34. [42] In generalized metric space in the sense of Perov, the notions of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

1.6.4 Matrix convergence

Definition 1.35. [42] A square matrix A of real numbers is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

Lemma 1.36. *(see [38]) Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent :*

- *A is a matrix convergent to zero ;*
- *The eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$; where I denote the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R}_+)$,*
- *The matrix $I - A$ is non-singular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;*
- *The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnegative elements ;*
- *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.*

1.6.5 Fixed Point Theorems in Vector Metric and Banach Spaces

Definition 1.37. [42, 81] Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a matrix A convergent to zero such that

$$d(N(x), N(y)) \leq Ad(x,y), \quad \forall x, y \in X.$$

Theorem 1.38. *(Perov's fixed point theorem) [81, 83]. Let (X,d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix A. Then N has a unique fixed point x^* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x^*) \leq A^k (I - A)^{-1} d(x_0, N(x_0)) \quad \forall k \in \mathbb{N}.$$

1.7 Stability

We introduce in this section some definitions and results concerning the stability theory.

Definition 1.39. [67] The constant x_e is an equilibrium of fractional differential system

$$ {}^{cH}D_a^{\alpha} x(t) = f(t, x(t)),\ t \in [a, \infty), \tag{1.26}$$

if and only if $f(t, x_e) = 0$, for all $t > a$.

For convenience, we state all definitions and theorems for stability when the equilibrium point is the origin of $\mathbb{R}^n$, i.e. $x_e = 0$.

Definition 1.40. [6, 7]The zero solution of the system (1.26) is said to be

1. **stable** if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$ there exist $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \delta$ implies $||x(t; t_0, x_0)|| < \epsilon$ for $t \geq t_0$;
2. **uniformly stable** if for every $\epsilon > 0$ and there exist $\delta = \delta(\epsilon) > 0$ such that for $t_0 \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^n$ with $||x_0|| < \delta$ the inequality $||x(t; t_0, x_0)|| < \epsilon$ holds for $t \geq t_0$;
3. **attractive** if there exists $\beta > 0$ such that for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ such that for any $x_0 \in \mathbb{R}^n$ with $||x_0|| < \beta$ the inequality $||x(t; t_0, x_0)|| < \epsilon$ holds for $t \geq t_0 + T$;
4. **uniformly attractive** if for $\beta > 0$: for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ such that for any $t \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^n$ with $||x_0|| < \beta$ the inequality $||x(t; t_0, x_0)|| < \epsilon$ holds for $t \geq t_0 + T$;
5. **asymptotically stable** if the zero solution is stable and attractive.
6. **uniformly asymptotically stable** if the zero solution is uniformly stable and uniformly attractive.

Definition 1.41. [67] The zero solution of the system (1.26) is said to be Mittag-Leffler stable if

$$||x(t)|| \leq \left[m(x(a)) E_\alpha \left(-\lambda \left(\ln \frac{t}{a} \right)^\alpha \right) \right]^\gamma, \ t > a,$$

where a is the initial time, $\alpha \in (1,2)$, $\lambda \geq 0$, $\gamma > 0$, $m(0) = 0$, $m(x) \geq 0$ and m(x) is locally Lipschitz on $x \in B \subset \mathbb{R}^n$.

Definition 1.42. [67] The zero solution of the system (1.26) is said to be Generalized Mittag-Leffler stable if

$$||x(t)|| \leq \left[m(x(a)) \left(\ln \frac{t}{a} \right)^{-\rho} E_{\alpha,1-\rho} \left(-\lambda \left(\ln \frac{t}{a} \right)^\alpha \right) \right]^\gamma, \ t > a,$$

where a is the initial time, $\alpha \in (1,2)$, $-\alpha < \rho < 1 - \alpha$, $\lambda \geq 0$, $\gamma > 0$, $m(0) = 0$, $m(x) \geq 0$ and m(x) is locally Lipschitz on $x \in B \subset \mathbb{R}^n$.

Remark 1.43. [67] Mittag-Leffler Stability and Generalized Mittag-Leffler Stability imply asymptotic stability.

Lemma 1.44. *(Gronwall Inequality)[88] Suppose that $g(t)$ and $\varphi(t)$ are continuous in $[t_0, t_1]$, $g(t) \geq 0$; $\lambda \geq 0$ and $r \geq 0$ are two constants. If*

$$\varphi(t) = \lambda + \int_{t_0}^{t} [g(\tau)\varphi(\tau) + r] d\tau,$$

then

$$\varphi(t) \leq (\lambda + r(t_1 - t_0)) \exp\left(\int_{t_0}^{t} g(\tau) d\tau \right), \quad t_0 \leq t \leq t_1.$$

Definition 1.45. [64] A function φ is said to belong to class $\mathcal{K}$ if $\varphi \in C[\mathbb{R}^+, \mathbb{R}^+]$, $\varphi(0) = 0$ and φ is strictly increasing.

Chapitre 2

Existence results and Ulam-Hyers stability to impulsive coupled system fractional differential equations [1]

2.1 Introduction

In this chapter we establish existence, uniqueness and Ulam stability of a impulsive coupled system of fractional differential equations of the form :

$$\begin{cases} \left({}^{cH}D_a^{\alpha}x\right)(t) & = f_1(t,x(t),y(t) \quad t\in[a,T], \quad t\neq t_k,\, k=1,...,m, \\ \left({}^{cH}D_a^{\beta}y\right)(t) & = f_2(t,x(t),y(t) \quad t\in[a,T], \quad t\neq t_k,\, k=1,...,m, \\ \Delta x(t_k) & = I_k(x(t_k^-),y(t_k^-)), \qquad\qquad k=1,...,m, \\ \Delta y(t_k) & = \overline{I}_k(x(t_k^-),y(t_k^-)), \qquad\qquad k=1,...,m, \\ x(a)=x_a, & \\ y(a)=y_a, & \end{cases} \tag{2.1}$$

where $0<\alpha,\beta<1$, $a>0$. Here $a=t_0\leq t_1\leq\cdots\leq t_m\leq t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim\limits_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim\limits_{h\to 0}x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively. $x_a,y_a\in\mathbb{R}$, $f_1,f_2:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and I_k , $\overline{I}_k\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are a given functions.

The results are based the Perov fixed point theorem. We also establish the Ulam-Hyers stability results of the proposed coupled system . An example is included to show the applicability of our results. We have given and proved the results in this chapter taking into account the numerous books and articles focused on of existence and uniqueness of solutions by Using the Perov fixed point theorem [24, 26, 37, 38, 49, 81, 83, 85, 103]. For the stability of Ulam, we have taken into consideration the articles (see[16, 17, 105, 109, 110, 111]) and references therein.

1. **H. Belbali**, M. Benbachir, Existence results and Ulam-Hyers stability to impulsive coupled system fractional differential equations, Turkish Journal of Mathematics(2021), 45 : 1368-1385.

2.2 Existence Results

let $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$. In order to define a solution for problem (2.1), consider the following space of picewise continuous functions for a given $T > a > 0$,

$$PC(J,\mathbb{R}) = \big\{ y\colon\ [a,T] \to \mathbb{R},\ y_k \in C(J_k,\mathbb{R}) \text{ for } k = 0, \ldots m+1, \\ \text{and there exist } \ y(t_k^-) \text{ and } y(t_k^+) \text{ with } \ y(t_k) = y(t_k^-),\ \ k = 1, \cdots, m \big\}.$$

This set is a Banach space with the norm $\|y\|_{PC} = \sup\limits_{t\in[a,T]} |y(t)|$.

Set $J' = J \backslash \{t_1, \ldots, t_m\}$.

Before proceeding to the main results, we give the following lemma.

Lemma 2.1. *Let $0 < \alpha < 1$ and let $f \in C[J \times \mathbb{R} \times \mathbb{R}]$, a function x is a solution of the fractional integral equation*

$$x(t) = \begin{cases} x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds & if\, t \in [a, t_1], \\ x_a + \frac{1}{\Gamma(\alpha)} \sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s)}{s} ds + \sum\limits_{i=1}^{k} I_i(x(t_i^-), y(t_i^-)) & if\, t \in (t_k, t_{k+1}], \quad k = 1, \ldots, m, \end{cases} \tag{2.2}$$

if and only if x is a solution of the impulsive fractional IVP

$$\left({}^{cH}D_a^\alpha x\right)(t) = f(t, x, y) \qquad for\ each\, t \in J, \tag{2.3}$$

$$\Delta x(t_k) = I_k(x(t_k^-), y(t_k^-)), k = 1, \ldots, m, \tag{2.4}$$

$$x(a) = x_a. \tag{2.5}$$

Proof. Assume x satisfies (2.3)-(2.5). Using conditions (2.4), (2.5) and Lemma 1.22, we obtain :
If $t \in [a, t_1]$, then

$$x(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} x(t) &= x(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &= \Delta x(t_1) + x(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &= I_1(x(t_1^-), y(t_1^-)) + x_a + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then

$$\begin{aligned} x(t) &= x(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &= \Delta x(t_2) + x(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &= x_a + I_1(x(t_1^-), y(t_1^-)) + I_2(x(t_2^-), y(t_2^-)) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t_1} \left(\ln \frac{t_1}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds. \end{aligned}$$

Repeating the same process for $t \in [t_k, t_{k+1}]$ and $k = 3, ..., m$, then we get

$$\begin{aligned} x(t) = x_a &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s)}{s} ds \\ &+ \sum_{i=1}^{k} I_i(x(t_i^-), y(t_i^-)). \end{aligned}$$

Conversely, assume that x satisfies the impulsive fractional integral equation (2.2). If $t \in [a, t_1]$ then $x(a) = x_a$ and using the fact that ${}^{cH}D_a^\alpha$ is the left inverse of ${}^{H}I_a^\alpha$ and using the fact that ${}^{cH}D_a^\alpha C = 0$, where C is a constant, we obtain

$$^{cH}D_a^\alpha x(t) = f(t,x,y) \text{ for all } t \in [a,t_1] \cup [t_k, t_{k+1}], k = 1, \dots, m.$$

Also, we can easily show that $\Delta x|_{t=t_k} = I_k(x(t_k^-, y(t_k^-))$ for $k = 1, \dots, m$.

□

Now, we first define the solution to our problem.

Lemma 2.2. *A function $(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ is said to be a solution of (2.1) if and only if*

$$\begin{cases} x(t) = & x_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s,x(s),y(s))}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s,x(s),y(s))}{s} ds + \sum_{i=1}^{k} I_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], \quad k = 1, ..., m, \\ y(t) = & y_a + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s,x(s),y(s))}{s} ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s,x(s),y(s))}{s} ds + \sum_{i=1}^{k} \overline{I}_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], \quad k = 1, ..., m. \end{cases} \tag{2.6}$$

The following assumptions are needed in the sequel.
(H_1) There exist constants $k_i > 0$, $i = 1, \cdots, 4$, such that

$$|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \leq k_1|x - \overline{x}| + k_2|y - \overline{y}|, \quad \text{for all} \quad x, \overline{x}, y, \overline{y} \in \mathbb{R},$$

$$|f_1(t,0,0)| = M_1,$$

and

$$|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \leq k_3|x-\overline{x}| + k_4|y-\overline{y}|, \quad \text{for all} \quad x,\overline{x},y,\overline{y} \in \mathbb{R},$$

$$|f_2(t,0,0)| = M_2.$$

(H_2) There exist constants a_{1i}, a_{2i}, b_{1i}, $b_{2i} \geq 0$, $i = 1, \cdots, m$, such that

$$|I_i(x,y) - I_i(\overline{x},\overline{y})| \leq a_{1i}|x-\overline{x}| + a_{2i}|y-\overline{y}|, \quad \text{for all} x,\overline{x},y,\overline{y} \in \mathbb{R},$$

and

$$|\overline{I}_i(x,y) - \overline{I}_i(\overline{x},\overline{y})| \leq b_{1i}|x-\overline{x}| + b_{2i}|y-\overline{y}|, \quad \text{for all} x,\overline{x},y,\overline{y} \in \mathbb{R}.$$

We will use the Perov fixed point theorem to prove the existence of a solution of the problem (2.1).

Theorem 2.3. *Assume that (H_1) - (H_2) are satisfied and the matrix*

$$A = \begin{pmatrix} A_\alpha k_1 + \sum\limits_{i=1}^{k} a_{1i} & A_\alpha k_2 + \sum\limits_{i=1}^{k} a_{2i} \\ A_\beta k_3 + \sum\limits_{i=1}^{k} b_{1i} & A_\beta k_4 + \sum\limits_{i=1}^{k} b_{2i} \end{pmatrix}, k = 1,...,m \tag{2.7}$$

converges to zero, where $A_\alpha = \frac{2}{\Gamma(\alpha+1)}\left(\ln\frac{T}{a}\right)^\alpha$, $A_\beta = \frac{2}{\Gamma(\beta+1)}\left(\ln\frac{T}{a}\right)^\beta$.
Then the problem (2.1) has a unique solution.

Proof.

Consider operator $T : PC \times PC \to PC \times PC$ defined by

$$T(x,y) = (T_1(x,y), T_2(x,y)),$$

where

$$\begin{aligned} T_1(x,y)(t) =& x_a + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_1(s,x(s),y(s))}{s}ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_1(s,x(s),y(s))}{s}ds + \sum_{i=1}^{k} I_i(x(t_i),y(t_i)), t \in (t_k,t_{k+1}], k = 1,...,m, \end{aligned}$$

and

$$\begin{aligned} T_2(x,y)(t) =& y_a + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\beta-1}\frac{f_2(s,x(s),y(s))}{s}ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}\frac{f_2(s,x(s),y(s))}{s}ds + \sum_{i=1}^{k} \overline{I}_i(x(t_i),y(t_i)), t \in (t_k,t_{k+1}], k = 1,...,m. \end{aligned}$$

Now, we first show that T is well defined. Given $(x,y) \in PC \times PC$, $t \in [a,T]$, we have

$$\begin{aligned}\|T_1(x,y)\|_{PC} \leq & |x_a| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{(k_1\|x\|_{PC} + k_2\|y\|_{PC})}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{\|f_1(s,0,0)\|_\infty}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{(k_1\|x\|_{PC} + k_2\|y\|_{PC})}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\|f_1(s,0,0)\|_\infty}{s} ds \\ & + \sum_{i=1}^{k} |I_i(x(t_i), y(t_i)| \\ & \leq |x_a| + \frac{2}{\Gamma(\alpha+1)} (\ln \frac{T}{a})^\alpha ((k_1\|x\|_{PC} + k_2\|y\|_{PC}) + M_1) + \sum_{i=1}^{k} [a_{1i}\|x\|_{PC} + a_{2i}\|y\|_{PC}].\end{aligned}$$

And, we can also proof as below that :

$$\|T_2(x,y)\|_{PC} \leq |y_a| + \frac{2}{\Gamma(\beta+1)} (\ln \frac{T}{a})^\alpha ((k_3\|x\|_{PC} + k_4\|y\|_{PC}) + M_2) + \sum_{i=1}^{k} [b_{1i}\|x\|_{PC} + b_{2i}\|y\|_{PC}].$$

Thus

$$\begin{aligned}\begin{pmatrix} \|T_1(x,y)\|_{PC} \\ \|T_2(x,y)\|_{PC} \end{pmatrix} & = \begin{pmatrix} |x_a| + A_\alpha M_1 \\ |y_a| + A_\beta M_2 \end{pmatrix} \\ & + \begin{pmatrix} A_\alpha k_1 + \sum_{i=1}^{k} a_{1i} & A_\alpha k_2 + \sum_{i=1}^{k} a_{2i} \\ A_\beta k_3 + \sum_{i=1}^{k} b_{1i} & A_\beta k_4 + \sum_{i=1}^{k} b_{2i} \end{pmatrix} \begin{pmatrix} \|x\|_{PC} \\ \|y\|_{PC} \end{pmatrix}, k = 1, ..., m.\end{aligned}$$

This implies that T is well defined.

Clearly, fixed points of T are solutions of problem (2.1).

We show that T is a contraction. Let $(x,y),(x,y) \in PC \times PC$. Then $(H_1$) and $(H_2$) imply

$$\|T_1(x,y)-T_1(\overline{x},\overline{y})\|_{PC} \leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{|f_1(s,x(s),y(s))-f_1(s,\overline{x(s)},\overline{y(s)})|_{PC}}{s}ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{|f_1(s,x(s),y(s))-f_1(s,\overline{x(s)},\overline{y(s)})|_{PC}}{s}ds$$
$$+\sum_{i=1}^{k}|I_i(x(t_i),y(t_i)-\overline{I_i}(\overline{x(t_i)},\overline{y(t_i)})|_{PC}$$
$$\leq \frac{1}{\Gamma(\alpha)}(k_1\|x-\overline{x}\|_{PC}+k_2\|y-\overline{y}\|_{PC})\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{ds}{s}$$
$$+(k_1\|x-\overline{x}\|_{PC}+k_2\|y-\overline{y}\|_{PC})\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}$$
$$+\sum_{i=1}^{k}(a_{1i}\|x-\overline{x}\|_{PC}+a_{2i}\|y-\overline{y}\|_{PC})$$
$$\leq\left(A_\alpha k_1+\sum_{i=1}^{k}a_{1i}\right)\|x-\overline{x}\|_{PC}+\left(A_\alpha k_2+\sum_{i=1}^{k}a_{2i}\right)\|y-\overline{y}\|_{PC}.$$

Similarly, we have

$$\|T_2(x,y)-T_2(\overline{x},\overline{y})\|_{PC}\leq\left(A_\beta k_3+\sum_{i=1}^{k}b_{1i}\right)\|x-\overline{x}\|_{PC}+\left(A_\beta k_4+\sum_{i=1}^{k}b_{2i}\right)\|y-\overline{y}\|_{PC}.$$

It follows that

$$\|T(x,y)-T(\overline{x},\overline{y})\|_{PC}\leq\ A\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix},\quad \text{for all}\quad (x,y),(\overline{x},\overline{y})\in PC\times PC.$$

Hence, by Theorem (1.38), the problem (2.1) has a unique solution.

□

2.3 Ulam-Hyers Stability

In this section, we introduce Ulam's type stability concepts for problem (2.1). We consider the following inequality

$$\begin{cases} |\left({}^{cH}D_a^{\alpha}u\right)(t)-f_1(t,u,v)|\leq\epsilon_\alpha & t\in J',\\ |\Delta u(t_k)-I_k(u(t_k),v(t_k))|\leq\epsilon_\alpha, & k=1,...,m,\\ |\left({}^{cH}D_a^{\beta}v\right)(t)-f_2(t,u,v)|\leq\epsilon_\beta & t\in J',\\ |\Delta v(t_k)-\overline{I}_k(u(t_k),v(t_k))|\leq\epsilon_\beta & k=1,...,m,\end{cases}\tag{2.8}$$

We adopt the following definitions from [90].

Definition 2.4. Problem (2.1) is Ulam-Hyers stable if there exists a real number $\lambda_{\alpha,\beta} = (\lambda_\alpha, \lambda_\beta) > 0$ such that for each $\epsilon = (\epsilon_\alpha, \epsilon_\beta) > 0$ and for each solution $(u, v) \in PC(J, \mathbb{R})$ of inequality (2.8) there exists a solution $(x, y) \in PC(J, \mathbb{R})$ of problem (2.1) with

$$|(u, v) - (x, y)| \leq \epsilon.\lambda_{\alpha,\beta}.$$

Definition 2.5. Problem (2.1) is generalized Ulam-Hyers stable if there exists $\phi_{\alpha,\beta} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\phi_{\alpha,\beta}(0) = 0$ such that for each solution $(u, v) \in PC(J, \mathbb{R})$ of inequality (2.8) there exists a solution $(x, y) \in PC(J, \mathbb{R})$ of problem (2.1) with

$$|(u, v) - (x, y)| \leq \phi_{\alpha,\beta}(\epsilon).$$

Remark 2.6. A function $(u, v) \in PC(J, \mathbb{R})$ is a solution of inequality (2.8) if and only if there is $(g_1, g_2) \in PC(J, \mathbb{R})$ and a sequence g_{1k}, g_{2k}, $k = 1, 2, \ldots, m$ (which depend on (u, v)) such that

(i) $|g_1(t)| \leq \epsilon_\alpha$, $|g_2(t)| \leq \epsilon_\beta$, $|g_{1k}(t)| \leq \epsilon_\alpha$, $|g_{2k}(t)| \leq \epsilon_\beta$, $k = 1, 2, \ldots, m$,

(ii)

$$\begin{cases} ({}^{cH}D_a^\alpha u(t) = f_1(t, u, v) + g_1(t) & t \in J', \\ \left({}^{cH}D_a^\beta v\right)(t) = f_2(t, u, v)| + g_2(t) & t \in J', \\ \Delta u(t_k) = I_k(u(t_k), v(t_k)) + g_{1k}, & k = 1, ..., m, \\ \Delta v(t_k) = \overline{I}_k(u(t_k), v(t_k)) + g_{2k} & k = 1, ..., m. \end{cases}$$

Lemma 2.7. *Suppose (u, v) is the solution of the inequality (2.8), then we have the system of inequalities given as*

$$\begin{cases} \Big| u(t) - u_a - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s))}{s} ds \\ - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s))}{s} ds - \sum_{i=1}^{k} I_i(u(t_i), v(t_i)) \Big| \leq \lambda_\alpha \varepsilon_\alpha, \\ \Big| v(t) - v_a - \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s))}{s} ds \\ - \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s))}{s} ds - \sum_{i=1}^{k} \overline{I}_i(u(t_i), v(t_i)) \Big| \leq \lambda_\beta \varepsilon_\beta. \end{cases}$$

Proof. By using lemma 2.6, we have

$$\begin{cases} ({}^{cH}D_a^\alpha u)(t) = f_1(t, u, v) + g_1(t) & t \in J', \\ \left({}^{cH}D_a^\beta v\right)(t) = f_2(t, u, v) + g_2(t) & t \in J', \\ \Delta u(t_k) = I_k(u(t_k), v(t_k)) + g_{1k}, & k = 1, ..., m, \\ \Delta v(t_k) = \overline{I}_k(u(t_k), v(t_k)) + g_{2K} & k = 1, ..., m. \end{cases} \tag{2.9}$$

Then, the solution of (2.9) is given by

$$\begin{cases} u(t) & = u_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s)) + g_1(s)}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s)) + g_1(s)}{s} ds + \sum_{i=1}^{k} I_i(u(t_i), v(t_i)) + g_{1i}, \\ v(t) & = v_a + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s)) + g_2(s)}{s} ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s)) + g_2(s)}{s} ds + \sum_{i=1}^{k} \overline{I}_i(u(t_i), v(t_i)) + g_{2i}. \end{cases} \tag{2.10}$$

From first equation of the system (2.10), we have

$$\begin{aligned}&\Bigg|u(t)-u_a-\tfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\alpha-1}\tfrac{f_1(s,u(s),v(s))}{s}ds-\tfrac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{f_1(s,u(s),v(s))}{s}ds\\&-\sum_{i=1}^{k}I_i(u(t_i),v(t_i))\Bigg|\le\tfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\alpha-1}\tfrac{|g_1(s)|}{s}ds+\tfrac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{|g_1(s)|}{s}ds+\sum_{i=1}^{k}|g_{1i}|\\&\le\tfrac{2\varepsilon_\alpha}{\Gamma(\alpha+1)}\left(\ln\tfrac{T}{a}\right)^{\alpha}+k\varepsilon_\alpha\\&\le\left(\tfrac{2}{\Gamma(\alpha+1)}\left(\ln\tfrac{T}{a}\right)^{\alpha}+k\right)\varepsilon_\alpha=\lambda_\alpha\varepsilon_\alpha.\end{aligned}$$

Repeating the same procedure for second equation of the system (2.10), we have

$$\begin{aligned}&\Bigg|v(t)-v_a-\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds\\&-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds-\sum_{i=1}^{k}\overline{I}_i(u(t_i),v(t_i))\Bigg|\le\lambda_\beta\varepsilon_\beta.\end{aligned}$$

where $\frac{2}{\Gamma(\beta+1)}\left(\ln\frac{T}{a}\right)^{\beta}+k=\lambda_\beta$.

□

Let us set
$\Lambda_1:=A_\alpha k_1+\sum_{i=1}^{k}a_{1i},\ \Lambda_2:=A_\alpha k_2+\sum_{i=1}^{k}a_{2i},$
$\Lambda_1^*:=A_\beta k_3+\sum_{i=1}^{k}b_{1i},\ \Lambda_2^*:=A_\beta k_4+\sum_{i=1}^{k}b_{2i}.$

Theorem 2.8. *If the assumptions (H1) -(H2) hold, and suppose that*

$$\Lambda_1<1,\Lambda_2^*<1\,and\,\Lambda:=1-\frac{\Lambda_2\Lambda_1^*}{(1-\Lambda_1)(1-\Lambda_2^*)}\neq0.$$

Then problem (2.1) is Ulam-Hyers and generalized Ulam-Hyers stable.

Proof. Let $(u,v)\in PC(J,\mathbb{R})$ be any solution of the inequality (2.8) and let $(x,y)\in PC(J,\mathbb{R})$ be the unique solution of the following :

$$\begin{cases}\left({}^{cH}D_a^{\alpha}x\right)(t) &=f_1(t,x,y)\quad t\in[a,T], & t\neq t_k,\,k=1,...,m,\\ \left({}^{cH}D_a^{\beta}y\right)(t) &=f_2(t,x,y)\quad t\in[a,T], & t\neq t_k,\,k=1,...,m,\\ \Delta x(t_k) &=x(t_k^+)-x(t_k^-)=I_k(x(t_k),y(t_k)), & k=1,...,m,\\ \Delta y(t_k) &=y(t_k^+)-y(t_k^-)=\overline{I}_k(x(t_k),y(t_k)), & k=1,...,m,\\ x(a)=x_a, \\ y(a)=y_a,\end{cases}\tag{2.11}$$

then, in view of Lemma 2.1, the solution of (2.11) is provided by

$$\begin{cases} x(t) = & u_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s,x(s),y(s))}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s,x(s),y(s))}{s} ds + \sum_{i=1}^{k} I_i(x(t_i), y(t_i)), \\ y(t) = & v_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_2(s,x(s),y(s))}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_2(s,x(s),y(s))}{s} ds + \sum_{i=1}^{k} \overline{I}_i(x(t_i), y(t_i)). \end{cases}$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned} \left\|u - x\right\|_{PC} \leq & \Big| u(t) - u_a - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s))}{s} ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s,u(s),v(s))}{s} ds \\ & - \sum_{i=1}^{k} I_i(u(t_i), v(t_i)) \Big| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{\left|f_1(s,u(s),v(s)) - f_1(s,x(s),y(s))\right|}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\left|f_1(s,u(s),v(s)) - f_1(s,x(s),y(s))\right|}{s} ds + \sum_{i=1}^{k} \left|I_i(u(t_i), v(t_i)) - I_i(x(t_i), y(t_i))\right| \\ \leq & \lambda_\alpha \varepsilon_\alpha + A_\alpha \left(k_1 \|u - x\|_{PC} + k_2 \|v - y\|_{PC}\right) + \sum_{i=1}^{k} \left(a_{1i} \|u - x\|_{PC} + a_{2i} \|v - y\|_{PC}\right) \\ \leq & \lambda_\alpha \varepsilon_\alpha + \left[A_\alpha k_1 + \sum_{i=1}^{k} a_{1i}\right] \|u - x\|_{PC} + \left[A_\alpha k_2 + \sum_{i=1}^{k} a_{2i}\right] \|v - y\|_{PC} \\ \leq & \lambda_\alpha \varepsilon_\alpha + \Lambda_1 \|u - x\|_{PC} + \Lambda_2 \|v - y\|_{PC}. \end{aligned}$$

Thus, we get

$$\left\|u - x\right\|_{PC} \leq \frac{\lambda_\alpha \varepsilon_\alpha}{1 - \Lambda_1} + \frac{\Lambda_2}{1 - \Lambda_1} \|v - y\|_{PC}. \tag{2.12}$$

In addition, for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned} \left\|v - y\right\|_{PC} \leq & \Big| v(t) - v_a - \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s))}{s} ds - \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s,u(s),v(s))}{s} ds \\ & - \sum_{i=1}^{k} \overline{I}_i(u(t_i), v(t_i)) \Big| + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{\left|f_2(s,u(s),v(s)) - f_2(s,x(s),y(s))\right|}{s} ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{\left|f_2(s,u(s),v(s)) - f_2(s,x(s),y(s))\right|}{s} ds + \sum_{i=1}^{k} \left|\overline{I}_i(u(t_i), v(t_i)) - \overline{I}_i(x(t_i), y(t_i))\right| \\ \leq & \lambda_\beta \varepsilon_\beta + A_\beta \left(k_3 \|u - x\|_{PC} + k_4 \|v - y\|_{PC}\right) + \sum_{i=1}^{k} \left(b_{1i} \|u - x\|_{PC} + b_{2i} \|v - y\|_{PC}\right) \\ \leq & \lambda_\beta \varepsilon_\beta + \left[A_\beta k_3 + \sum_{i=1}^{k} b_{1i}\right] \|u - x\|_{PC} + \left[A_\beta k_4 + \sum_{i=1}^{k} b_{2i}\right] \|v - y\|_{PC} \\ \leq & \lambda_\beta \varepsilon_\beta + \Lambda_1^* \|u - x\|_{PC} + \Lambda_2^* \|v - y\|_{PC}. \end{aligned}$$

Thus, we get

$$\left\|v - y\right\|_{PC} \leq \frac{\lambda_\beta \varepsilon_\beta}{1 - \Lambda_2^*} + \frac{\Lambda_1^*}{1 - \Lambda_2^*} \|v - y\|_{PC}. \tag{2.13}$$

The equivalent matrix of Equations (2.12) and (2.13) is given as :

$$\begin{bmatrix} 1 & -\frac{\Lambda_2}{1-\Lambda_1} \\ -\frac{\Lambda_1^*}{1-\Lambda_2^*} & 1 \end{bmatrix} \begin{bmatrix} \|u - x\|_{PC} \\ \|v - y\|_{PC} \end{bmatrix} \leq \begin{bmatrix} \frac{\lambda_\alpha \varepsilon_\alpha}{1-\Lambda_1} \\ \frac{\lambda_\beta \varepsilon_\beta}{1-\Lambda_2^*} \end{bmatrix}.$$

Solving the above inequality, we get

$$\begin{bmatrix} \|u-x\|_{PC} \\ \\ \|v-y\|_{PC} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Lambda} & \frac{\Lambda_1^*}{\Lambda(1-\Lambda_2^*)} \\ \\ \frac{\Lambda_2}{\Lambda(1-\Lambda_1)} & \frac{1}{\Lambda} \end{bmatrix} \begin{bmatrix} \frac{\lambda_\alpha \varepsilon_\alpha}{1-\Lambda_1} \\ \\ \frac{\lambda_\beta \varepsilon_\beta}{1-\Lambda_2^*} \end{bmatrix}.$$

Further simplification of above system gives

$$\|u-x\|_{PC} \leq \frac{\lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)^2},$$
$$\|v-y\|_{PC} \leq \frac{\lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)^2},$$

from which we have

$$\|u-x\|_{PC} + \|v-y\|_{PC} \leq \frac{\lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)^2} + \frac{\lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)^2}. \quad (2.14)$$

Let $\varepsilon = \max\{\varepsilon_\alpha, \varepsilon_\beta\}$, then from (2.14) we have

$$\|(u,v)-(x,y)\|_{PC} \leq \lambda_{\alpha,\beta}\varepsilon,$$

where

$$\lambda_{\alpha,\beta} = \left[\frac{\lambda_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta}{\Lambda(1-\Lambda_2^*)^2} + \frac{\lambda_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha}{\Lambda(1-\Lambda_1)^2}\right].$$

Hence, problem (2.1) is Ulam-Hyers stable.
Over and above, if we write

$$\|(u,v)-(x,y)\|_{PC} \leq \psi_{\alpha,\beta}(\varepsilon), \text{ where } \psi_{\alpha,\beta}(\varepsilon) = \varepsilon.\lambda_{\alpha,\beta}, \text{ and } \psi_{\alpha,\beta}(0) = 0.$$

Then problem (2.1) is generalized Ulam-Hyers stable.

□

2.4 Example

Example 2.9. Consider the following differential equation system

$$\begin{cases} ({}^{cH}D^{\frac{1}{2}}x)(t) = \frac{\sin(x+y)}{20(\ln t+1)}, & t \in [1,e],\ t \neq \frac{5}{3}, \\ \left({}^{cH}D^{1/2}y\right)(t) = \frac{\arctan t}{3+|x+y|}, & t \in [1,e],\ t \neq \frac{5}{3}, \\ \Delta x(\frac{5}{3}) = \exp^{-\frac{5}{3}}\left(\sin x(\frac{5}{3}) + y(\frac{5}{3})\right), \\ \Delta y(\frac{5}{3}) = \frac{|x(\frac{5}{3})+y(\frac{5}{3})|}{10}, \\ x(1) = \frac{1}{2}, \\ y(1) = \frac{3}{2}. \end{cases} \quad (2.15)$$

Here, we have

$$f_1(t,x,y) = \frac{\sin(x+y)}{20(\ln t+1)}, \qquad f_2(t,x,y) = \frac{\arctan t}{3+|x+y|},$$

and we simply check that

$$\forall x,y,\overline{x},\overline{x}\in\mathbb{R};\quad \Big|f_1(t,x,y)-f_1(t,\overline{x},\overline{y})\Big|\leq\frac{1}{20}\Big|x-\overline{x}\Big|+\frac{1}{20}\Big|y-\overline{y}\Big|,\quad \forall t\in[1,e],$$

$$\forall x,y,\overline{x},\overline{x}\in\mathbb{R};\quad \Big|f_2(t,x,y)-f_2(t,\overline{x},\overline{y})\Big|\leq\frac{\pi}{18}\Big|x-\overline{x}|+\frac{\pi}{18}\Big|y-\overline{y}\Big|,\quad \forall t\in[1,e],$$

$$\left|I\left(x(\frac{5}{3}),y(\frac{5}{3})\right)-I\left(\overline{x}(\frac{5}{3}),\overline{y}(\frac{5}{3})\right)\right|\leq e^{\frac{-5}{3}}\Big|x-\overline{x}\Big|+e^{\frac{-5}{3}}\Big|y-\overline{y}\Big|,$$

$$\left|\overline{I}\left(x(\frac{5}{3}),y(\frac{5}{3})\right)-\overline{I}\left(\overline{x}(\frac{5}{3}),\overline{y}(\frac{5}{3})\right)\right|\leq\frac{1}{10}\Big|x-\overline{x}\Big|+\frac{1}{10}\Big|y-\overline{y}\Big|.$$

Therefore the matrix

$$A=\begin{pmatrix}0.3. & 0.3\\ 0.49 & 0.49\end{pmatrix}$$

converges to zero since its eigenvalues are $\lambda_1=0.79<1$, $\lambda_2=0<1$. From Theorem (2.3), the problem (2.15) has an unique solution.

On the other hand, we have $\Lambda_2=\Lambda_1=0.3$, $\Lambda_1^*=\Lambda_2^*=0.49$. Therefore

$$\Lambda=1-\frac{0.3*0.49}{(1-0.3)(1-0.49)}=0.58\neq 0$$

So, the coupled system (2.15) is Ulam-Hyers stable, generalized Ulam-Hyers stable.

Chapitre 3

Stability for coupled systems on networks with Caputo-Hadamard fractional derivative [1]

3.1 Introduction

Coupled systems of fractional differential equations on networks (CSFDENs) have been investigated extensively due to their wide applications in different fields such as engineering, physics, epidemiology, signal and image processing, artificial intelligence, pattern classification, etc [95, 93, 116, 113]. A network can be described as a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics are given by a system of differential equations called the vertex system. The directed arcs indicate interactions between vertex systems. In 2010, Li et al. [64] introduced a new method based on graph theory and Lyapunov technique to study the stability and synchronization of neural networks. Since then, this technique has attracted considerable interest [66, 31].

Suo et al[96] studied the stability of the following system :

$$\begin{cases} x_i' = f_i(t,x_i) + \sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), & t \neq t_k, \\ \Delta x_i = I_k(x_i), & t = t_k,\ k = 1,2,\dots \\ x_i(t_0^+) = x_{i0}, & \end{cases}$$

where $i = 1,2,\dots,n$, $0 < t_1 < t_2 < \cdots < t_k < \dots$, and $t_k \to \infty$ as $k \to \infty$; f_i is continuous on $(t_{k-1},t_k] \times \mathbb{R}^{mi}$, g_{ij} is continuous on $(t_{k-1},t_k] \times \mathbb{R}^{mi} \times \mathbb{R}^{mj}$, and $I_k \in C[\mathbb{R}^{mi},\mathbb{R}^{mi}]$.

Zhang et al[112] studied the global stability of the following impulsive coupled system on a digraph G

$$\begin{cases} D^{\mu} x_p = -\omega_p x_p + \sum\limits_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum\limits_{q=1}^{n} a_{pq}(x_p(t) - x_q(t)), & t \geq 0,\ t \neq t_k, \\ \Delta x_p(t_k) = I_k(x_p(t_k)), & \\ x(t_k^-) = x_{t_k}, & k = 1,2,\dots \end{cases}$$

1. **H. Belbali**, M. Benbachir, Stability for coupled systems on networks with Caputo-Hadamard fractional derivative. Journal of Mathematical Modeling 2020 ; 9 (1) : 107-118.

where D^{μ} is the Caputo fractional derivative of order $0<\mu<1$, $p,q=1,2,\ldots,n$, $f_q(x)$ is a function satisfying the Lipschitz condition.

Li et al[63] considered the stability of the coupled systems fractional differential equations on networks

$$\begin{cases} {}^{c}D^{q}x_i = f_i(t,x_i)+\sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), \quad t\geq t_0,\ i=1,2,\ldots,n, \\ x_i(t_0)=x_{i0}, \end{cases}$$

where ${}^{c}D^{q}$ is the Caputo's fractional derivative of order q, $0<q<1$, $x_i\in\mathbb{R}^{mi}$ and $f_i:\mathbb{R}_+\times\mathbb{R}^{mi}\to\mathbb{R}^{mi}$ and $g_{ij}:\mathbb{R}_+\times\mathbb{R}^{mi}\times\mathbb{R}^{mj}\to\mathbb{R}^{mi}$.

Motivated by the works of the papers mentioned above, in this chapter, we establish the stability and uniform asymptotic stability of the trivial solution for coupled systems of fractional differential equations on networks with Caputo-Hadamard fractional derivative, of the form

$$\begin{cases} {}^{cH}D^{\alpha}x_i = f_i(t,x_i)+\sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), \quad t> t_0, \\ x_i(t_0)=x_{i0}, \end{cases}$$

where $1<\alpha\leq 2$, $i=1,2,\ldots,n$, $f_i:\mathbb{R}_+\times\mathbb{R}^{m_i}\to\mathbb{R}^{m_i}$, $g_{ij}:\mathbb{R}_+\times\mathbb{R}^{m_i}\times\mathbb{R}^{m_j}\to\mathbb{R}^{m_i}$. We assume that the functions f_i and g_{ij} satisfy the Lipschitz conditions.
The results are based on graph theory and the classical Lyapunov technique, we prove stability and uniform asymptotic stability under suitable sufficient conditions. We also provide an example to illustrate the obtained results.

3.2 Stability analysis for coupled systems of fractional differential equations on networks

Consider a network represented by digraph G with n vertices ($n\geq 2$). Assume that the i-th vertex dynamic is described by a system of fractional differential equations as follows :

$$\begin{cases} {}^{cH}D^{\alpha}x_i = f_i(t,x_i), \quad t>t_0,\ i\in I, \\ x_i(t_0)=x_{i0}, \end{cases} \tag{3.1}$$

where α, $1<\alpha\leq 2$, $x_i\in\mathbb{R}^{mi}$ and $f_i:\mathbb{R}_+\times\mathbb{R}^{mi}\to\mathbb{R}^{mi}$. Let $g_i:\mathbb{R}_+\times\mathbb{R}^{mi}\times\mathbb{R}^{mj}\to\mathbb{R}^{mi}$ represents the influence of the vertex j on vertex i and $g_{ij}=0$, if there exists no arc from j to i in G. Then we obtain the following coupled system on graph G :

$$\begin{cases} {}^{cH}D^{\alpha}x_i = f_i(t,x_i)+\sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), \quad t> t_0, \\ x_i(t_0)=x_{i0}, \end{cases} \tag{3.2}$$

$i=1,2,\ldots,n$. Here functions f_i and g_{ij} satisfy the global Lipschitz conditions so that initial-value problem (3.2) has an unique solution. Equation (3.2) has a trivial solution $(x_1,x_2,\ldots,x_n)=0$ for $t\geq t_0$.

Let $D_i \subset \mathbb{R}^{mi}$ be an open set. We define $V_i : \mathbb{R} \times D_i \to \mathbb{R}$ the Lyapunov function for each vertex system (3.1). We are particularly interested in constructing Lyapunov functions $V : \mathbb{R} \times D \to \mathbb{R}$ for coupled system (3.2) of form

$$V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i).$$

Where $D = D_1 \times D_2 \times \cdots \times D_n \subset \mathbb{R}^n, \ m = m_1 + m_2 + \cdots + m_n$
The following result gives a general and systematic approach for such construction.

Theorem 3.1. *Suppose that the following assumptions are satisfied.*

1. *There exist functions $V_i(t,x_i), F_{ij}(x_i,x_j)$, and a matrix $A = (a_{ij})_{n\times n}$ in which $a_{ij} > 0$ such that*

$$^{cH}D^{\alpha} V_i(t,x_i) \leq \sum_{i=1}^{n} a_{ij} F_{ij}(x_i,x_j) \ \ t > t_0, \ x_i \in \mathbb{R}^{mi}, \ 1 < i < n. \tag{3.3}$$

2. *Along each directed cycle C of the weighted digraph (G,A)*

$$\sum_{(s,r)\in E(c)} a_{ij} F_{rs}(x_r,x_s) \leq 0 \ ; \ t \geq t_0 \ , x_r \in \mathbb{R}^{mr}, x_s \in \mathbb{R}^{ms}. \tag{3.4}$$

3. *Constants c_i are given in (1.24).*

Then function $V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i)$ is a Lyapunov function for (3.2) .

Proof. For $V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i)$, we have

$$\begin{aligned} ^{cH}D^{\alpha} V(t,x) &= {}^{cH}D^{\alpha} \sum_{i=1}^{n} c_i . V_i(t,x_i), \\ &\leq \sum_{i=1}^{n} c_i \, {}^{cH}D^{\alpha} V_i(t,x_i). \end{aligned}$$

According to condition (1), we have

$$^{cH}D^{\alpha} V(t,x) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} c_i a_{ij} F_{ij}(t,x_i,x_j).$$

Applying Theorem 1.29

$$^{cH}D^{\alpha} V(t,x) = \sum_{Q\in\mathbb{Q}} w(Q) \sum_{(j,i)\in E_{c_Q}} F_{ij}(t,x_i,x_j).$$

According to condition (2) and $w(Q) > 0$, we have

$$^{cH}D^{\alpha} V(t,x) \leq 0.$$

The proof is therefore complete. □

Note that if (G,A) is balanced, then

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i,x_j) \leq \frac{1}{2} \sum_{Q\in\Omega} w(Q) \sum_{(s,r)\in E_{c_Q}} \left[F_{rs}(x_r,x_s) + F_{sr}(x_s,x_r)\right].$$

Proposition 3.2. *Suppose that (G,A) is balanced. Then the conclusion of Theorem 3.1 holds if condition (2) is replaced by the following one :*

$$[F_{rs}(x_r,x_s)+F_{sr}(x_s,x_r)] \leq 0 \quad t \geq t_0\,, x_r \in \mathbb{R}^{mr}, x_s \in \mathbb{R}^{ms}. \tag{3.5}$$

Theorem 3.3. *Assume the following conditions are satisfied.*

- *There exist functions $V_i \in C^1[\mathbb{R}^+ \times D_i, \mathbb{R}^+]$, $F_{ij}(t,x_i,x_j)$ a matrix $A=(a_{ij})_{n\times n}$ in which $a_{ij} \leq 0$ and $b_i > 0$ such that for $i=1,2,\dots,n$*

$$^{cH}D^{\alpha}V_i(t,x_i) \leq -b_iV_i(t,x_i)+\sum_{j=1}^{n} a_{ij}F_{ij}(x_i,x_j) \quad t > t_0. \tag{3.6}$$

- *Either (2) holds, or if (G,A) is balanced and (3.5) holds.*
- *There exists a $\delta_0^{(i)} > 0$ and a function $d_i \in \mathcal{K}$ such that*

$$V_i(t,x_i) \leq d_i(\| x_i \|), \quad \text{provided} \quad \| x_i \| < \delta_0^{(i)}. \tag{3.7}$$

- *Constants c_i are given in (1.24).*

Then, the function $V(t,x)=\sum\limits_{i=1}^{n} c_iV_i(t,x_i)$ is a Lyapunov function for (3.2) and the trivial solution of (3.2) is uniformly asymptotically stable.

Proof. For $V(t,x)=\sum\limits_{i=1}^{n} c_iV_i(t,x_i)$, according to condition (2) and (3.6), we have

$$\begin{aligned} ^{cH}D^{\alpha}V(t,x) &= {}^{cH}D^{\alpha}\sum_{i=1}^{n} c_iV_i(t,x_i), \\ &\leq \sum_{i=1}^{n} c_i\,{}^{cH}D^{\alpha}V_i(t,x_i), \\ &\leq \sum_{i=1}^{n} c_i\left[-b_iV_i(t,x_i)+\sum_{j=1}^{n} a_{ij}F_{ij}(x_i,x_j)\right], \\ &\leq -\sum_{i=1}^{n} c_ib_iV_i(t,x_i), \\ &\leq -bV(t,x), \end{aligned}$$

where $b=\min\{b_1,b_2,\dots,b_n\}$.

So that the trivial solution is asymptotically stable. On the other hand, as $d_i \in \mathcal{K}$, we get (3.7) independent of t. Therefore the number δ can be chosen independent of t_0.

Define

$$\delta_0=\min\{\delta_0^{(1)},\delta_0^{(2)},\dots,\delta_0^{(n)}\},$$

$$b(\| x \|)=n\min\{c_1b_1(\| x_1 \|),c_2b_2(\| x_2 \|),\dots,c_nb_n(\| x_n \|)\},$$

and

$$d(\| x \|)=n\min\{c_1d_1(\| x_1 \|),c_2d_2(\| x_2 \|),\dots,c_nd_n(\| x_n \|)\}.$$

For every $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) < \delta_0$ such that $d(\| x_0 \|) < b(\varepsilon)$ provided that $\| x_0 \| < \delta$.

if $\parallel x_0 \parallel < \delta$, then according to (3.7), we have

$$V(t,x) \leq \sum_{i=1}^{n} c_i V_i(t_0, x_{i0}) \leq \sum_{i=1}^{n} c_i d_i(\parallel x_i \parallel) \leq \sum_{i=1}^{n} \frac{1}{n} d(\parallel x_0 \parallel) \leq b(\varepsilon).$$

Since $V_i(t,x_i)$ is a positive definite function, we deduce that there exists $b_i(.) \in \mathcal{K}$ such that

$$V_i(t,x_i) \geq b_i(\parallel x_i \parallel).$$

Then

$$V(t,x) = \sum_{i=1}^{n} c_i V_i(t,x_i) \geq \sum_{i=1}^{n} c_i b_i(\parallel x_i \parallel) \geq \sum_{i=1}^{n} \frac{1}{n} b(\parallel x \parallel) = b(\parallel x \parallel).$$

So, we have

$$b(\parallel x \parallel) \leq V(t,x) \leq b(\varepsilon).$$

Then $\parallel x \parallel \leq \varepsilon$. This implies that the trivial solution of (3.2) is uniformly stable. We conclude that the trivial solution of (3.2) is uniformly asymptotically stable.

□

3.3 Example

Example 3.4. We consider the following coupled system of fractional differential equation on digraph G :

$$\begin{cases} {}^{cH}D^{\alpha} x_i = -\omega_i x_i + f_i(x_i) + \sum\limits_{j=1}^{n} \beta_{ij}(x_i - |x_j|), \\ x_i(t_0) = x_{i0}, \end{cases} \tag{3.8}$$

$i,j = 1,\ldots,n,\ 0 < \alpha < 1,\ \omega_i > 0$, where x_i is n-dimensional column vectors, f_i is continuous and there exists a Lipschitz constant $L_i > 0$ such that $|f_i(x_i) - f_i(y_i) \leq L_i|x_i - y_i|$ for all $x_i \neq y_i$ In addition, $f_i(0) = 0,\ \ \beta_{ij} \leq 0,\ \ \beta_{ij} = -\beta_{ji}$ and $\beta_{ij} \neq 0$ if $i \neq j$.

Suppose that the following conditions hold :

1. (G,A) is strongly connected and balanced.
2. $\gamma_i = \omega_i - L_i > 0,\ \ i = 1,2,\ldots,n.$

Then the trivial solution of system(3.8) is uniformly asymptotically stable.

Proof. Let us consider $V_i(t,x_i(t)) = |x_i(t)|$, then we get

$$\mu|x_i(t)| \geq |x_i(t)| \text{ for all } \ \mu \geq 1,$$

and, we put $\mu|x_i(t)| \in K$ such that

$$V_i(t,x_i(t)) \leq \mu|x_i(t)|,\ \mu \geq 1.$$

Therefore (3.7) holds .
If $x_i(t) = 0$, then ${}^{cH}D^{\alpha}|x_i| = 0$.
If $x_i(t) > 0$, then

$$\begin{aligned} {}^{cH}D^{\alpha}|x_i(t)| =& \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n x(s) \frac{ds}{s}, \\ =& {}^{cH}D^{\alpha} x_i(t). \end{aligned}$$

If $x_i(t) < 0$, then

$$\begin{aligned} {}^{cH}D^{\alpha}|x_i(t)| &= -\frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\left(s\frac{d}{ds}\right)^n x(s)\frac{ds}{s}, \\ &= -{}^{cH}D^{\alpha}x_i(t). \end{aligned}$$

Therefore ${}^{cH}D^{\alpha}|x_i(t)| = sgn(x_i(t))\,{}^{cH}D^{\alpha}x_i(t)$. According to (3.8), we have

$$\begin{aligned} {}^{cH}D^{\alpha}|x_i(t)| =& sgn(x_i(t))^{cH}D^{\alpha}x_i(t), \\ =& sgn(x_i(t))\left(-\omega_i x_i + f_i(x_i) + \sum_{j=1}^{n}\beta_{ij}(x_i - |x_j|)\right), \\ =& -\omega_i|x_i| + f_i(|x_i|) + \sum_{j=1}^{n}\beta_{ij}(|x_i| - |x_j|), \\ \leq& -\omega_i|x_i| + L_i|x_i| + \sum_{j=1}^{n}\beta_{ij}(|x_i| - |x_j|), \\ \leq& (-\omega_i + L_i)\,|x_i| + \sum_{j=1}^{n}\beta_{ij}(|x_i| - |x_j|), \\ \leq& -\gamma_i V_i(t, x_i) + \sum_{j=1}^{n} a_{ij}F_{ij}(x_i, x_j), \end{aligned}$$

where $F_{ij}(x_i, x_j) = sgn(\beta_{ij})(|x_i| - |x_j|)$.

It is easy to show that

$$\begin{aligned} F_{ij}(x_i, x_j) =& sgn(\beta_{ij})(|x_i| - |x_j|), \\ =& -sgn(\beta_{ji})(|x_j| - |x_i|), \\ =& -F_{ji}(x_j, x_i). \end{aligned}$$

Thus along each directed cycle C of the weighted digraph (G, A)

$$\sum_{(i,j)\in E(c_Q)} [F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)] = 0.$$

According to Theorem 3.3, we can conclude that (3.8) is uniformly asymptotically stable.

□

Chapitre 4

Stability results for linear fractional differential system with Caputo-Hadamard derivative [1]

4.1 Introduction

This chapter is devoted to proving the stability of linear fractional differential system with Caputo-Hadamard derivative. The results are obtained by the Laplace transform, the asymptotic expansion of the Mittag-Leffler function and the Gronwall inequality. Further, examples are provided to illustrate our results.
In 1996, Matignon [74], studied for the first time the stability of autonomous linear fractional differential systems with the Caputo derivative. This criterion was developed by several authors. Deng et al[35], studied the stability of some fractional systems with multiple time delays. Recently, Qian et al[87], have investigated the stability of fractional differential systems with Riemann-Liouville derivative. In [114], authors derived the same results to [74] for the different case of order of the fractional derivative.

Thus, motivated by the results mentioned, in this chapter we discuss the stability of linear fractional order systems with Caputo-Hadamard derivativ .

In section 4.2 we consider the stability of the following linear autonomous caputo-Hadamard fractional differential system :

$$\begin{cases} {}^{cH}D^{\alpha}_{a,t}x(t) = Ax(t), & t > a > 0,\ 0 < \alpha < 1, \\ x(a) = x_0, \end{cases}$$

where $x(t) \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$ and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

In section 4.3 we consider the stability of perturbed fractional differential system

$$\begin{cases} {}^{cH}D^{\alpha}_{a,t}x(t) = Ax(t) + B(t)x(t), & t > a > 0,\ 0 < \alpha < 1, \\ x(a) = x_0, \end{cases}$$

1. **H. Belbali**, M. Benbachir, Stability results for linear fractional order systems with Caputo-Hadamard derivative. (submitted).

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $B(t) : [a,\infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

The main tools in our analysis are the properties of Mittag-Leffler functions, the Laplace transform and the Gronwall inequality.

4.2 Stability of Autonomous Linear Fractional Differential Systems

In this section, we consider the stability of the following linear autonomous caputo-Hadamard fractional differential system :

$$\begin{cases} {}^{cH}D^{\alpha}x(t) = Ax(t), \quad t > a > 0,\ 0 < \alpha < 1, \\ x(a) = x_0, \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$ and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

Then, by analyzing the solutions of the above initial value problem (4.1), one can find the following result when A is diagonalizable and has all non-zero eigenvalues.

Theorem 4.1. *The fractional differential system (4.1) is asymptotically stable, If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \tag{4.2}$$

and $(I - As^{-\alpha})$ is an invertible matrix.

Proof. Using the modified Laplace transform on the both sides of system (4.1) yields

$$s^{\alpha}X(s) - s^{\alpha-1}\delta^0 x(a) = AX(s).$$

Thus

$$X(s) = (Is^{\alpha} - A)^{-1}s^{\alpha-1}x_0.$$

Therefore it follows from the inverse modified Laplace transform that we can get the solution of system (4.1),

$$x(t) = x_0\, E_{\alpha}\left(A\left(\ln\frac{t}{a}\right)^{\alpha}\right). \tag{4.3}$$

Since A is diagonalizable, then there exists an invertible matrix P such that $D = P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then

$$E_{\alpha}\left(A\left(\ln\frac{t}{a}\right)^{\alpha}\right) = PE_{\alpha}\left(D\left(\ln\frac{t}{a}\right)^{\alpha}\right)P^{-1} = P\psi_{\alpha}(\frac{t}{a})P^{-1},$$

where

$$\psi_{\alpha}(\frac{t}{a}) = diag\left[E_{\alpha}\left(\lambda_1\left(\ln\frac{t}{a}\right)^{\alpha}\right), E_{\alpha}\left(\lambda_2\left(\ln\frac{t}{a}\right)^{\alpha}\right), \ldots, E_{\alpha}\left(\lambda_n\left(\ln\frac{t}{a}\right)^{\alpha}\right)\right].$$

According to (4.2) and (1.12), $E_{\alpha}\left(\lambda_i\left(\ln\frac{t}{a}\right)^{\alpha}\right)$ is given as

$$E_{\alpha}\left(\lambda_i\left(\ln\frac{t}{a}\right)^{\alpha}\right) = -\sum_{k=1}^{p}\frac{\left(\lambda_i\left(\ln\frac{t}{a}\right)^{\alpha}\right)^{-k}}{\Gamma(1-\alpha k)} + O\left(\left|\left(\lambda_i\left(\ln\frac{t}{a}\right)^{\alpha}\right)\right|^{-1-p}\right) \to 0 \,\text{as}\, t \to +\infty,\ 1 \le i \le n.$$

Hence

$$\lim_{t\to+\infty}\left\|E_\alpha\left(D\left(\ln\frac{t}{a}\right)^\alpha\right)\right\|=\lim_{t\to+\infty}\left\|diag\left[E_\alpha\left(\lambda_1\left(\ln\frac{t}{a}\right)^\alpha\right),E_\alpha\left(\lambda_2\left(\ln\frac{t}{a}\right)^\alpha\right),\ldots,E_\alpha\left(\lambda_n\left(\ln\frac{t}{a}\right)^\alpha\right)\right]\right\|=0,$$

and

$$\lim_{t\to+\infty}\left\|x(t)\right\|=\lim_{t\to+\infty}\left\|x_0E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\|=\lim_{t\to+\infty}\left\|P\left[x_0E_\alpha\left(D\left(\ln\frac{t}{a}\right)^\alpha\right)\right]P^{-1}\right\|=0,$$

for any non-zero initial value x_0. The proof is complete.

□

Theorem 4.2. *If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))|\geq\frac{\alpha\pi}{2}$$

and the critical eigenvalues satisfying $|\arg(\lambda(A))|=\frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities, then system (4.1) is stable but not asymptotically stable.

Proof. Without loss of generality, suppose there exists a critical eigenvalue, say λ_i, satisfying $|\arg(\lambda_i)|=\frac{\alpha\pi}{2}$ with algebraic and geometric multiplicity both equal to one. Then, from (4.3) the solution of system (4.1) is given by

$$\begin{aligned}x(t)=&x_0E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\\&=x_0PE_\alpha\left(D\left(\ln\frac{t}{a}\right)^\alpha\right)P^{-1}\\&=x_0PE_\alpha\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)diag(1,\ldots,1)P^{-1}.\end{aligned}$$

Next, from (1.11) we have

$$E_\alpha\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)=\frac{1}{\alpha}\exp\left\{\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)^{1/\alpha}\right\}-\sum_{k=1}^{p}\frac{\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)^{-k}}{\Gamma(1-\alpha k)}+O\left(\left|\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)\right|^{-1-p}\right).$$

Let $\lambda_i=r\left(\cos(\frac{\alpha\pi}{2})+i\sin(\frac{\alpha\pi}{2})\right)$, where r is the modulus of λ_i, and $i^2=-1$. Then,

$$\begin{aligned}E_\alpha\left(\lambda_i\left(\ln\frac{t}{a}\right)^\alpha\right)&=\frac{1}{\alpha}\exp\left\{\left[r\left(\cos(\frac{\alpha\pi}{2})+i\sin(\frac{\alpha\pi}{2})\right)\right]^{1/\alpha}\left(\ln\frac{t}{a}\right)\right\}-\sum_{k=1}^{p}\frac{\left[r\left(\ln\frac{t}{a}\right)^\alpha\left(\cos(\frac{\alpha\pi}{2})+i\sin(\frac{\alpha\pi}{2})\right)\right]^{-k}}{\Gamma(1-\alpha k)}\\&+O\left(\left|r\left(\ln\frac{t}{a}\right)^\alpha\left(\cos(\frac{\alpha\pi}{2})+i\sin(\frac{\alpha\pi}{2})\right)\right|^{-1-p}\right)\\&=\frac{1}{\alpha}\exp\left\{r^{1/\alpha}\left(\ln\frac{t}{a}\right)\left(\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})\right)\right\}-\sum_{k=1}^{p}\frac{r^{-k}\left(\ln\frac{t}{a}\right)^{-\alpha k}\left(\cos(\frac{-\alpha k\pi}{2})+i\sin(\frac{-\alpha k\pi}{2})\right)}{\Gamma(1-\alpha k)}\\&+O\left(\left(\ln\frac{t}{a}\right)^{-\alpha p-\alpha}\right)\\&=\frac{1}{\alpha}\exp\left\{ir^{1/\alpha}\left(\ln\frac{t}{a}\right)\right\}-\sum_{k=1}^{p}\frac{r^{-k}\left(\ln\frac{t}{a}\right)^{-\alpha k}\left(\cos(\frac{\alpha k\pi}{2})-i\sin(\frac{\alpha k\pi}{2})\right)}{\Gamma(1-\alpha k)}+O\left(\left(\ln\frac{t}{a}\right)^{-\alpha p-\alpha}\right).\end{aligned}$$

The absolute value of the first term of the right-hand side of the above equality equals $\frac{1}{\alpha}$, whereas the rest of the terms tend to zero as $t \to +\infty$. All these imply that the zero solution of system (4.1) is stable but not asymptotically stable.

□

Corollaire 4.2.1. If A has an eigenvalue λ_0 such that $|\arg(\lambda_0)| < \frac{\alpha\pi}{2}$, then system (4.1) is unstable.

Proof. According to (1.11), we have

$$E_\alpha\left(\lambda_0\left(\ln\frac{t}{a}\right)^\alpha\right) = \frac{1}{\alpha}\exp\left\{\lambda_0^{1/\alpha}\left(\ln\frac{t}{a}\right)\right\} - \sum_{k=1}^{p}\frac{\left(\lambda_0\left(\ln\frac{t}{a}\right)^\alpha\right)^{-k}}{\Gamma(1-\alpha k)} + O\left(\left|\lambda_0\left(\ln\frac{t}{a}\right)^\alpha\right|^{-1-p}\right)$$

$$\to +\infty \text{ as } t \to +\infty,$$

and $\lim_{t\to+\infty}||x(t)|| = \lim_{t\to+\infty}||x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)|| = +\infty$.
Hence system (4.1) is unstable.

□

4.3 Stability analysis of perturbed fractional differential system

We consider the perturbed system of (4.1) given by :

$$\begin{cases} {}^{cH}D^\alpha x(t) = Ax(t) + B(t)x(t), \quad t > a > 0,\ 0 < \alpha < 1, \\ x(a) = x_0, \end{cases} \tag{4.4}$$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $B(t) : [a,\infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$.

Theorem 4.3. *Suppose $||B(t)||$ is bounded ($||B(t)|| \leq C$ for some $C > 0$) and all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}. \tag{4.5}$$

Then system (4.4) is asymptotically stable.

Proof. Using the Laplace transform and the inverse Laplace transform, the solution of equations (4.4) can be written as

$$x(t) = x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right) + \int_a^t \left(\ln a\frac{t}{w}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\ln a\frac{t}{w}\right)^\alpha\right) B(w)x(w)\frac{dw}{w},$$

from which it follows that

$$||x(t)|| \leq \left|\left|x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right|\right| + \int_a^t \left|\left|\left(\ln a\frac{t}{w}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\ln a\frac{t}{w}\right)^\alpha\right)\right|\right| . \left|\left|B(w)\right|\right| . \left|\left|x(w)\right|\right|\frac{dw}{w}.$$

Applying Lemma 1.44, we have

$$\begin{aligned}
||x(t)|| &\le \left\|x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\| \exp\left\{\int_a^t \left\|\left(\ln a\frac{t}{w}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\ln a\frac{t}{w}\right)^\alpha\right)\right\|.\left\|B(w)\right\|\frac{dw}{w}\right\}\\
&\le \left\|x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\| \exp\left\{\int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|.\left\|B\left(a\frac{t}{\theta}\right)\right\|\frac{d\theta}{\theta}\right\}\\
&\le \left\|x_0 E_\alpha\left(A(\ln)^\alpha\right)\right\| \exp\left\{C\int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|\frac{d\theta}{\theta}\right\}.
\end{aligned}$$

Since A is similar to a diagonal matrix. Then

$$\int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|\frac{d\theta}{\theta} = \int_a^t \left\|P\Omega_\alpha(\theta)P^{-1}\right\|\frac{dw}{w},$$

where,

$$\Omega_\alpha(\theta) = diag\left\{(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_1(\ln\theta)^\alpha\right), (\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_2(\ln\theta)^\alpha\right), \dots, (\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n(\ln\theta)^\alpha\right)\right\}.$$

We shall now show that there exists a positive constant M such that

$$\int_a^t \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta} \le M, \ \ 1 \le i \le n.$$

Indeed, using (1.11) we find for $t > t_0 > a$,

$$\begin{aligned}
&\int_a^t \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta}\\
&= \int_a^{t_0} \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta} + \int_{t_0}^t \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta}\\
&= \int_a^{t_0} \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta} + \int_{t_0}^t \left|(\ln\theta)^{\alpha-1}\left(-\sum_{k=2}^p \frac{\left(\lambda_i(\ln\theta)^\alpha\right)^{-k}}{\Gamma(\alpha-\alpha k)} + O\left(\left|\lambda_i(\ln\theta)^\alpha\right|^{-1-p}\right)\right)\right|\frac{d\theta}{\theta}\\
&= \int_a^{t_0} \left|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_i(\ln\theta)^\alpha\right)\right|\frac{d\theta}{\theta} + \int_{t_0}^t \left|-\sum_{k=2}^p \frac{\lambda_i^{-k}(\ln\theta)^{-\alpha k+\alpha-1}}{\Gamma(\alpha-\alpha k)} + O\left(|\lambda_i|^{-1-p}(\ln\theta)^{-\alpha p-\alpha}\right)\right|\frac{d\theta}{\theta}\\
&\le \int_a^{t_0} (\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(|\lambda_i|(\ln\theta)^\alpha\right)\frac{d\theta}{\theta} + \int_{t_0}^t \left\{\sum_{k=2}^p \frac{|\lambda_i|^{-k}(\ln\theta)^{-\alpha k+\alpha-1}}{|\Gamma(\alpha-\alpha k)|} + O\left(|\lambda_i|^{-1-p}(\ln\theta)^{-\alpha p-\alpha}\right)\right\}\frac{d\theta}{\theta}\\
&= \sum_{k=0}^\infty \frac{|\lambda_i|^k}{\Gamma(\alpha k+\alpha)}\int_a^{t_0}(\ln\theta)^{\alpha k+\alpha-1}\frac{d\theta}{\theta} + \sum_{k=2}^p \frac{|\lambda_i|^{-k}}{|\Gamma(\alpha-\alpha k)|}\int_{t_0}^t (\ln\theta)^{-\alpha k+\alpha-1}\frac{d\theta}{\theta} + O\left(|\lambda_i|^{-1-p}\left(\ln\frac{t}{t_0}\right)^{-\alpha p-\alpha+1}\right)\\
&= \sum_{k=0}^\infty \frac{|\lambda_i|^k\left(\ln\frac{t_0}{a}\right)^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} + \sum_{k=2}^p \frac{|\lambda_i|^{-k}(\ln t)^{-\alpha k+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} - \sum_{k=2}^p \frac{|\lambda_i|^{-k}(\ln t_0)^{-\alpha k+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} + O\left(|\lambda_i|^{-1-p}\left(\ln\frac{t}{t_0}\right)^{-\alpha p-\alpha+1}\right)\\
&\to \left(\ln\frac{t_0}{a}\right)^\alpha E_{\alpha,\alpha+1}\left(|\lambda_i|\left(\ln\frac{t_0}{a}\right)^\alpha\right) + \sum_{k=2}^p \frac{|\lambda_i|^{-k}(\ln t_0)^{-\alpha k+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} < M \text{ as } t \to +\infty.
\end{aligned}$$

It immediately follows that $\int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|\frac{d\theta}{\theta} < K$ for any $t > a$.

Thus, $\exp\left\{C\int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|\frac{d\theta}{\theta}\right\}$ is bounded.

Hence, we have

$$\lim_{t\to+\infty}\left\|x(t)\right\| = \lim_{t\to+\infty}\left\|x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\| = 0,$$

which completes the proof.

□

Theorem 4.4. *If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| \geq \frac{\alpha\pi}{2}, \tag{4.6}$$

and the critical eigenvalues have the same algebraic and geometric multiplicities and $\int_a^\infty ||B(w)|| \frac{dw}{w}$. is bounded, then system (4.4) is stable.

Proof. From the proof of theorem 4.3, we have

$$||x(t)|| \leq \left\|x_0 E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\| + \int_a^t \left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\|.\left\|B(a\frac{t}{\theta})\right\|.\left\|x(a\frac{t}{\theta})\right\|\frac{d\theta}{\theta}.$$

According to the proof of Theorem 4.2, the matrix is bounded. Therefore, there exists a positive number K_1 such that $\left\|E_\alpha\left(A\left(\ln\frac{t}{a}\right)^\alpha\right)\right\| \leq K_1$. Moreover, in the same way, there exists a positive number K_2 such that $\left\|(\ln\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\ln\theta)^\alpha\right)\right\| \leq K_2$. Then we have

$$||x(t)|| \leq \left\|x_0 K_1\right\| + \int_a^t K_2.\left\|B(w)\right\|.\left\|x(w)\right\|\frac{dw}{w}.$$

Applying Lemma 1.44, we have

$$||x(t)|| \leq \left\|x_0 K_1\right\| \exp\{\int_a^t K_2.\left\|B(w)\right\|.\left|\frac{dw}{w}\right\}.$$

Thus, we derive that $||x(t)||$ is bounded according to the condition $\int_a^\infty ||B(w)|| \frac{dw}{w} < \infty$, that is, the system (4.4) is stable. The proof is completed.

□

4.4 Example

Example 4.5. Consider the perturbed system of a linear fractional differential system with Caputo-Hadamard derivative

$$^{cH}D^\alpha_{a,t}x(t) = Ax(t) + B(t), \quad t > a > 0, \tag{4.7}$$

with initial condition $x_0 = x(1,1)^T$, where $\alpha = 1/2$ and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & \frac{\sin(t)}{2} \\ \frac{\sin(t)}{2} & 0 \end{pmatrix}.$$

The eigenvalues of A are given by $\lambda = 1 \pm i\sqrt{3}$.
The eigenvalues satisfy $|\arg(\lambda_1)| = \frac{\pi}{3}$ and $|\arg(\lambda_2)| = \frac{\pi}{3}$.
Since all the eigenvalues satisfy $|\arg(\lambda(A))| > \frac{\pi}{4}$. Then according to Theorem (4.1) The homogeneous system of (4.7)is asymptotically stable.
Moreover, $\forall t > a ||B(t)|| \leq M = \frac{1}{2}$.
Since all the conditions of the Theorem 4.4 are satisfied. Hence the given system (4.7) is asymptotically stable.

Chapitre 5

Existence Theory and Generalized Mittag-Leffler Stability for a Nonlinear Caputo-Hadamard FIVP via Lyapunov Method [1]

5.1 Introduction

In this chapter, the main properties such as the existence , uniqueness and different types of stability are studied for the fractional system involving the nonlinear Caputo-Hadamard FIVP as given by :

$$\begin{cases} {}^{cH}D_c^{\ell}\phi(t) = A\phi(t) + \psi(t,\phi(t),{}^{cH}D_c^{\beta}\phi(t)), \quad t > c > 0, \\ \Theta^k\phi(t)\mid_{t=c} = \phi_k,\ k = 0,1, \end{cases} \tag{5.1}$$

where, $1 < \ell < 2, 0 < \beta < \ell - 1,\ \phi_0, \phi_1 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\Theta = t\frac{d}{dt}$ and $\psi : [c,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

First, In section 5.2, we shall give several sufficient conditions confirming the existence of solution and its uniqueness for the nonlinear Caputo-Hadamard FIVP (5.1), using Banach contraction principle. Next, In section 5.3, we study the generalized Mittag-Leffler stability for the Caputo-Hadamard system (5.1) by Lyapunov-like function and $\mathcal{K}$-class function.
Finally, we provide an illustrative example to show the applicability of our results.

The existence and uniqueness problems for fractional differential equations case has been extensively studied by many authors ; see [10, 11, 21, 23, 50, 51, 53, 86, 94]. and the references therein. For the stability, we have taken into consideration the articles [22, 32, 34, 35, 41, 74, 80, 91, 115].

1. **H. Belbali**, M. Benbachir, S. Etemad and S. Rezapour, Existence Theory and Generalized Mittag-Leffler Stability for a Nonlinear Caputo-Hadamard FIVP via Lyapunov Method (submitted)

5.2 Existences and Uniqueness of Solution

For a given $T > c > 0$, let $\mathbb{E} = C([c,T],\mathbb{R}^n)$ be a Banach space consisting of continuous n-vector mappings given on $[c,T]$ furnished with the norm

$$||\phi|| = \sup_{t\in[c,T]} |\phi(t)|.$$

Notice that the norm of an n-vector $\phi(t) = (\phi_1(t),\phi_2(t),\dots,\phi_n(t)) \in \mathbb{R}^n$ is presented as

$$||\phi(t)|| = \left(\sum_{k=1}^{n} |\phi_k(t)|^2\right)^{1/2}.$$

Based on given problem (5.1), introduce the Banach space $\mathbb{B} = \{\phi; \phi \in \mathbb{E}, {}^{cH}D_c^{\beta}\phi \in \mathbb{E}\}$ via the norm

$$||\phi||_{\mathbb{B}} = ||\phi|| + ||{}^{cH}D_c^{\beta}\phi||.$$

Now, we first derive the equivalent solution to our system.

Lemma 5.1. *For $1 < \ell < 2$, $0 < \beta < \ell - 1$ and invertible matrix $[Is^{\ell} - A]$, the solution of the nonlinear Caputo-Hadamard FIVP (5.1) is given as*

$$\phi(t) = \mathbb{E}_{\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w,\phi(w),{}^{cH}D_c^{\beta}\phi(w))\frac{dw}{w}.$$

Proof. Let $\Psi(s)$ and $\Phi(s)$ be the modified Laplace transforms of $\psi(t)$ and $\phi(t)$. Then, by using the modified Laplace transform and its properties for the nonlinear Caputo-Hadamard FIVP (5.1), we have

$$\mathbb{L}_c\{{}^{CH}D_c^{\ell}\phi(t)\} = \mathbb{L}_c\{A\phi(t)\} + \mathbb{L}_c\{\psi(t,\phi(t),{}^{cH}D_c^{\beta}\phi(t))\},$$

and so,

$$\Phi(s) = s^{\ell-1}[Is^{\ell}-A]^{-1}\phi_0 + s^{\ell-2}[Is^{\ell}-A]^{-1}\phi_1 + [Is^{\ell}-A]^{-1}F(s,\Phi(s),{}^{cH}D_c^{\beta}\Phi(s)).$$

By utilizing the inverse modified Laplace transform on the above relation, we obtain

$$\phi(t) = \mathbb{E}_{\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0 + \left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w,\phi(w),{}^{cH}D_c^{\beta}\phi(w))\frac{dw}{w},$$

and this concludes the proof.

□

We will use the Banach's contraction principale to prove the existence of a solution of the nonlinear Caputo-Hadamard FIVP (5.1).

Theorem 5.2. *Let $\psi : [c,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous function which fulfills the Lipschitz inequality*

$$||\psi(t,\phi_1(t),y_1(t))-\psi(t,\phi_2(t),y_2(t))|| \le K(||\phi_1(t)-\phi_2(t)||+||y_1(t)-y_2(t)||),\ t\in[c,T],\ K>0.$$

Then, the nonlinear Caputo-Hadamard FIVP (5.1) has a solution uniquely on $[c,T]$ if

$$\left[\frac{1}{\ell}+\frac{(T-c)\Gamma(\ell)}{Tc\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{-\beta}\right]KM_\ell\left(\ln\frac{T}{c}\right)^{\ell}<1, \tag{5.2}$$

where $||\psi(t,0,0)||\le M_0$ and $\left|\left|\mathbb{E}_{\ell,i}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right|\right|\le M_i, i\in\{1,2,\ell\}$.

Proof. Consider the operator $N:\mathbb{B}\to\mathbb{B}$ formulated by

$$N\phi(t)=\mathbb{E}_\ell\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0+\left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\psi(w,\phi(w),{}^{cH}D_c^\beta\phi(w))\frac{dw}{w}.$$

We follow the proof in some steps :

(Step 1) : N is well–defined : Given $\phi\in\mathbb{B}, t\in[c,T]$, we have

$$||N\phi(t)||\le\left|\left|\mathbb{E}_\ell\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right|\right|||\phi_0||+\left(\ln\frac{t}{c}\right)\left|\left|\mathbb{E}_{\ell,2}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right|\right|||\phi_1||$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\left|\left|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{w}{c}\right)^{\ell}\right)\right|\right|\left|\left|\psi(w,\phi(w),{}^{cH}D_c^\beta\phi(w))\right|\right|\frac{dw}{w}$$
$$\le M_1||\phi_0||+M_2\left(\ln\frac{t}{c}\right)||\phi_1||+M_\ell\Big[\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}\left[K\left(||\phi(w)||+\left|\left|{}^{cH}D_c^\beta\phi(w)\right|\right|\right)\right]\frac{dw}{w}$$
$$+\int_c^t\left(\ln\frac{w}{c}\right)^{\ell-1}||\psi(s,0,0)||\frac{dw}{w}\Big]$$
$$\le M_1||\phi_0||+M_2\left(\ln\frac{t}{c}\right)||\phi_1||+\frac{KM_\ell}{\ell}\left(\ln\frac{t}{c}\right)^{\ell}||\phi||_{\mathbb{B}}+\frac{M_0M_\ell}{\ell}\left(\ln\frac{t}{c}\right)^{\ell}.$$

Consequently we obtain,

$$||N\phi||\le M_1||\phi_0||+M_2\left(\ln\frac{T}{c}\right)||\phi_1||+\frac{M_0M_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell}+\frac{KM_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell}||\phi||_{\mathbb{B}}. \tag{5.3}$$

Applying the first derivative of $N\phi(t)$, using (1.9) and (1.10) we have

$$N'\phi(t)=A\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_0+\mathbb{E}_{\ell,1}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\phi_1$$
$$+\frac{1}{t}\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\psi(t,\phi(t),{}^{cH}D_c^\beta\phi(t)).$$

Hence,

$$
\begin{aligned}
||N'\phi(t)|| &\leq ||A|| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| ||\phi_0|| + \left\|\mathbb{E}_{\ell,1}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| ||\phi_1|| \\
&+ \frac{1}{t}\left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\mathbb{E}_{\ell,\ell}\left(A\left(\ln\frac{t}{c}\right)^{\ell}\right)\right\| \left\|\psi(t,\phi(t), {}^{cH}D_c^{\beta}\phi(t))\right\| \\
&\leq M_\ell ||A|| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi_0\right\| + M_1\left\|\phi_1\right\| + \frac{KM_\ell}{c}\left(\ln\frac{t}{c}\right)^{\ell-1}\left\|\phi\right\|_{\mathbb{B}} + \frac{M_0M_\ell}{c}\left(\ln\frac{t}{c}\right)^{\ell-1} \\
&\leq M_\ell ||A|| \left(\ln\frac{t}{c}\right)^{\ell-1} \left\|\phi_0\right\| + M_1\left\|\phi_1\right\| + KM'_\ell\left(\ln\frac{t}{c}\right)^{\ell-1}\left\|\phi\right\|_{\mathbb{B}} + M_0M'_\ell\left(\ln\frac{t}{c}\right)^{\ell-1},
\end{aligned}
$$

where $M'_\ell = \frac{M_\ell}{c}$.
Now, one can estimate

$$
\begin{aligned}
\left\|{}^{cH}D_c^{\beta}N\phi(t)\right\| &\leq \frac{1}{\Gamma(1-\beta)}\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta}\left\|N'\phi(w)\right\|\frac{dw}{w} \\
&\leq \frac{M_1||\phi_1||}{\Gamma(1-\beta)}\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta}\frac{dw}{w} \\
&+ \frac{1}{\Gamma(1-\beta)}\left[M_\ell||A||||\phi_0|| + KM'_\ell||\phi||_{\mathbb{B}} + M_0M'_\ell\right]\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta}\left(\ln\frac{w}{c}\right)^{\ell-1}\frac{dw}{w} \\
&\leq \frac{M_1||\phi_1||}{\Gamma(2-\beta)}\left(\ln\frac{t}{c}\right)^{1-\beta} + \frac{\Gamma(\ell)}{\Gamma(\ell-\beta+1)}\left[M_\ell||A||||\phi_0|| + KM'_\ell||\phi||_{\mathbb{B}} + M_0M'_\ell\right]\left(\ln\frac{t}{c}\right)^{\ell-\beta}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\left\|{}^{cH}D_c^{\beta}N\phi\right\| \leq \frac{M_1||\phi_1||}{\Gamma(2-\beta)}\left(\ln\frac{T}{c}\right)^{1-\beta} + \frac{\Gamma(\ell)}{\Gamma(\ell-\beta+1)}\Big[M_\ell||A||||\phi_0|| \\
+ KM'_\ell||\phi||_{\mathbb{B}} + M_0M'_\ell\Big]\left(\ln\frac{T}{c}\right)^{\ell-\beta}.
\end{aligned} \tag{5.4}
$$

From (5.3) and (5.4), we find that

$$
\begin{aligned}
||N\phi||_{\mathbb{B}} &\leq \left[M_1 + \frac{\Gamma(\ell)M_\ell||A||}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right]||\phi_0|| + \left[M_2\left(\ln\frac{T}{c}\right) + \frac{M_1}{\Gamma(2-\beta)}\left(\ln\frac{T}{c}\right)^{1-\beta}\right]||\phi_1|| \\
&+ \left[\frac{KM'_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell} + \frac{\Gamma(\ell)KM_\ell}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right]||\phi||_{\mathbb{B}} \\
&+ \left[\frac{M_0M_\ell}{\ell}\left(\ln\frac{T}{c}\right)^{\ell} + \frac{\Gamma(\ell)M_0M'_\ell}{\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{\ell-\beta}\right].
\end{aligned}
$$

This implies that N is well defined.

(Step 2) : N is contraction on $\mathbb{B}$; For $\phi, y \in \mathbb{B}$ and $t \in [c,T]$, we get

$$\begin{aligned}
||N\phi(t)-Ny(t)|| &\le M_\ell \int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1} ||\psi(w,\phi(w),{}^{cH}D_c^\beta\phi(w)) - \psi(w,y(w),{}^{cH}D_c^\beta y(w))||\frac{dw}{w} \\
&\le KM_\ell \int_c^t \left(\ln\frac{w}{c}\right)^{\ell-1} \left[||\phi(w)-y(w)|| + ||{}^{cH}D_c^\beta\phi(w) - {}^{cH}D_c^\beta y(w)||\right]\frac{dw}{w} \\
&\le \frac{KM_\ell}{\ell}\left(\ln\frac{t}{c}\right)^{\ell} ||\phi-y||_{\mathbb{B}}.
\end{aligned}$$

On the other hand,
$||N'\phi(t)-N'y(t)|| \le \frac{1}{t}KM_\ell\left(\ln\frac{t}{c}\right)^{\ell-1}||\phi-y||_{\mathbb{B}}$.
So

$$\begin{aligned}
||{}^{cH}D_c^\beta N\phi(t) - {}^{cH}D_c^\beta Ny(t)|| &\le \frac{1}{\Gamma(1-\beta)}\int_c^t \left(\ln\frac{t}{w}\right)^{-\beta} ||N'\phi(t)-N'y(t)||\frac{dw}{w} \\
&\le \frac{KM_\ell}{\Gamma(1-\beta)}||\phi-y||_{\mathbb{B}} \int_c^t \left(\ln\frac{t}{w}\right)^{-\beta}\left(\ln\frac{w}{c}\right)^{\ell-1}\frac{1}{w}\frac{dw}{w} \\
&\le \frac{(t-c)KM_\ell\Gamma(\ell)}{tc\Gamma(\ell-\beta+1)}\left(\ln\frac{t}{c}\right)^{\ell-\beta}||\phi-y||_{\mathbb{B}}.
\end{aligned}$$

Then,

$$||N\phi-Ny||_{\mathbb{B}} \le \left[\frac{1}{\ell} + \frac{(T-c)\Gamma(\ell)}{Tc\Gamma(\ell-\beta+1)}\left(\ln\frac{T}{c}\right)^{-\beta}\right]KM_\ell\left(\ln\frac{T}{c}\right)^{\ell}||\phi-y||_{\mathbb{B}}.$$

The contractive property for N, thanks to (5.2), is established. As a consequence, Theorem1.31 confirms the existence of unique solution for the nonlinear Caputo-Hadamard FIVP (5.1) on $[c,T]$. This completes the proof.

□

5.3 Generalized Mittag-Leffler stability

In this section, we follow our study in relation to the stability of the nonlinear Caputo-Hadamard FIVP (5.1) by terms of a Lyapunov-like function and $\mathcal{K}$-class function.

From now on, we suppose that the Lyapunov function $\mathbb{V} : [c,\infty) \times \mathbb{R}^n \to \mathbb{R}^+$ is continuously differentiable w.r.t. the time variable t, Lipschtiz w.r.t. the unknown function ϕ, and also $\mathbb{V}(t,0) = 0$.

5.3.1 Lyapunov method

Theorem 5.3. *Let $\phi = 0$ be an equilibrium point of nonlinear Caputo-Hadamard FIVP* (5.1)*, and assume that $\mathbb{V}$ satisfies*

$$c||\phi||^b \le \mathbb{V}(t,\phi(t)), \tag{5.5}$$

$$^{cH}D_c^{\ell}\mathbb{V}(t,\phi(t)) \leq -q\mathbb{V}(t,\phi(t)), \tag{5.6}$$

so that $\phi\in\mathbb{R}^n$, $c,b,q>0$. Then, the zero solution is Mittag–Leffler stable if $\mathbb{V}(c,\phi(c))\geq 0$ and $\Theta\mathbb{V}(c,\phi(c))=0$, where $\Theta=\frac{d}{dt}$.

Proof. Using the inequality (5.6), a nonnegative function $M(t)$ exists which satisfies

$$^{cH}D_c^{\ell}\mathbb{V}(t,\phi(t))+M(t)=-q\mathbb{V}(t,\phi(t)). \tag{5.7}$$

Let $\mathbb{L}_c\{\mathbb{V}(t,\phi(t))\}=\mathbb{V}(s)$. Then, the application of the Laplace transform of (5.7) gives

$$s^{\ell}\mathbb{V}(s)-s^{\ell-1}\mathbb{V}_0-s^{\ell-2}\mathbb{V}_1+M(s)=-q\mathbb{V}(s). \tag{5.8}$$

By using the inverse modified Laplace transform to (5.8), it yields

$$\begin{aligned}\mathbb{V}(t,\phi(t)) &= \mathbb{V}_0\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)\\ &\quad+\mathbb{V}_1\left(\ln\frac{t}{c}\right)\mathbb{E}_{\ell,2}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)-M(t)*\left[\left(\ln\frac{t}{c}\right)^{\ell-1}\mathbb{E}_{\ell,\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)\right].\end{aligned}$$

Since both $\left(\ln\frac{t}{c}\right)^{\ell-1}$ and $\mathbb{E}_{\ell,\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)$ are nonnegative functions and $\mathbb{V}_1=\Theta\mathbb{V}(c,\phi(c))=0$, we deduce that

$$\mathbb{V}(t,\phi(t))\leq\mathbb{V}_0\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right).$$

In accordance with (5.5), we obtain

$$||\phi(t)||\leq\left[\frac{\mathbb{V}_0}{c}\mathbb{E}_{\ell}\left(-q\left(\ln\frac{t}{c}\right)^{\ell}\right)\right]^{\frac{1}{b}},$$

for $m=\dfrac{\mathbb{V}_0}{c}\geq 0$. In this case, the zero solution of the nonlinear Caputo-Hadamard FIVP (5.1) is Mittag-Leffler stable.

□

5.3.2 Stability via $\mathcal{K}$-Class functions

Theorem 5.4. *Let $\phi=0$ be an equilibrium point of the nonlinear Caputo-Hadamard FIVP (5.1). Suppose that there exis a $\mathcal{K}$-class function φ which satisfies*

$$\mathbb{V}(t,\phi(t))\geq\varphi^{-1}(||\phi(t)||), \tag{5.9}$$

$$^{cH}D_c^{\ell}\mathbb{V}(t,\phi(t))\leq 0, \tag{5.10}$$

$$\sup_{t\geq c}\varphi\left(\mathbb{V}(c,\phi(c))+\Theta\mathbb{V}(c,\phi(c))\ln\frac{t}{c}\right)\leq M, \tag{5.11}$$

for $M\geq 0$. Then, the zero solution is stable.

Proof. By applying (5.10), there exists some $M \geq 0$ so that

$$^{cH}D_c^{\ell}\mathbb{V}(t,\phi(t)) = -M(t).$$

By using the Laplace transform and its inverse, we obtain

$$\mathbb{V}(t,\phi(t)) = \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1 - M(t) * \left[\frac{1}{\Gamma(\ell)}\left(\ln\frac{t}{c}\right)^{\ell-1}\right], \tag{5.12}$$

where $\mathbb{V}_0 = \mathbb{V}(c,\phi(c))$, and $\mathbb{V}_1 = \Theta\mathbb{V}(c,\phi(c))$.
Substituting (5.12) into (5.9), it yields

$$\varphi^{-1}(||\phi(t)||) \leq \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1 - M(t) * \left[\frac{1}{\Gamma(\ell)}\left(\ln\frac{t}{c}\right)^{\ell-1}\right] \leq \mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1.$$

Therefore

$$||\phi(t)|| \leq \varphi\left(\mathbb{V}_0 + \left(\ln\frac{t}{c}\right)\mathbb{V}_1\right).$$

Then, by equation (5.11), we get $||\phi(t)|| \leq M$, $t > c$, which confirms that the zero solution of the nonlinear Caputo-Hadamard FIVP (5.1) is stable.

□

5.4 Example

Here, we validate our results by providing the next example.

Example 5.5. According to (5.1), consider the nonlinear Caputo-Hadamard FIVP

$$\begin{cases} ^{cH}D_{1,t}^{3/2}\phi(t) = \dfrac{1}{10}\left(-|\phi(t)| - ^{cH}D_{1,t}^{1/2}|\phi(t)|\right), & t \in [1,e], \\ \Theta^k\phi(t)\mid_{t=1} = 0, k = 0,1. \end{cases} \tag{5.13}$$

Here, we have $A = 0$ and $\psi\left(t,\phi(t),^{cH}D_{1,t}^{1/2}\phi(t)\right) = \frac{1}{10}\left(-|\phi(t)| - ^{cH}D_{1,t}^{1/2}|\phi(t)|\right)$, where $\psi : [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In order to show that (5.13) has an unique solution, we simply check that

$$\left|\left|\psi\left(t,\phi(t),^{cH}D_{1,t}^{1/2}\phi(t)\right) - \psi\left(t,y(t),^{cH}D_{1,t}^{1/2}y(t)\right)\right|\right|_{\mathbb{B}} \leq \frac{1}{10}\left|\left|\phi(t) - y(t)\right|\right|_{\mathbb{B}},$$

which is satisfying the Lipschitz condition with $K = \frac{1}{10}$. Since $|\mathbb{E}_{\ell,\ell}\left(A(\ln\frac{t}{c})^{\ell}\right)| \leq M_\ell$, for $A = 0$, we have $\mathbb{E}_{\frac{3}{2},\frac{3}{2}}(0) = \frac{2}{\sqrt{\pi}}$ and

$$\left[\frac{1}{\frac{3}{2}} + \frac{(e-1)\Gamma(\frac{3}{2})}{e\Gamma(2)}\left(\ln\frac{e}{1}\right)^{-1/2}\right]\frac{1}{10}\frac{2}{\sqrt{\pi}}\left(\ln\frac{e}{1}\right)^{3/2} = 0.10 < 1.$$

From Theorem5.2, the the nonlinear Caputo-Hadamard FIVP (5.13) has a solution uniquely.

On the other hand, consider the Lyapunov function $\mathbb{V}(t,\phi(t)) = |\phi(t)|$. In this case,

$$^{cH}D^{\ell}\mathbb{V}(t,\phi(t)) = \frac{1}{10}\left(-\mathbb{V}(t,\phi(t)) - {}^{cH}D^{\beta}\mathbb{V}(t,\phi(t))\right) \leq -\frac{1}{10}\mathbb{V}(t,\phi(t)).$$

Hence, the hypotheses of Theorem 5.3 hold with $c = 0, b = 1$ and $q = \dfrac{1}{10}$. Accordingly, the zero solution of the given nonlinear Caputo-Hadamard FIVP (5.13) is Mittag-Leffler stable.

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