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DEDICATION

I dedicate this work :
To my parents.
To my brothers
To my sisters
To my grandparents
To my entire family. To all those who taught me
To all my friends.

Daoudi Omar
Ghardaia 9 juillet 2025

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Abstract

In this memoir, we have investigated solutions to stochastic fractional differential equations (SFDE), encompassing existence and uniqueness analysis. We have presented fundamental concepts of fractional calculus (fractional differentiation and integration) as well as stochastic calculus (SFDE and SDE), followed by solving a stochastic fractional differential equation.

Keywords: Special functions, fractional calculus, Caputo, FDE, stochastic calculus, SDE, fractional stochastic differential equations (FSDE).

Résumé

Dans ce mémoire, nous avons étudié les solutions des équations différentielles fractionnaires stochastiques, y compris les questions d'existence et d'unicité. Nous avons présenté les concepts fondamentaux du calcul différentiel et intégral fractionnaire, ainsi que du calcul différentiel et intégral stochastique (EDF et EDS). Ensuite, nous avons résolu une équation différentielle fractionnaire stochastique.

Mots-clés : Fonctions spéciales, calcul fractionnaire, caputo, EDF, calcul stochastique, EDS, équations différentielles stochastiques d'ordre fractionnaire (EDFS).

ملخص

في هذه المذكرة قمنا بدراسة حلول المعادلات التفاضلية الكسرية العشوائية بما فيه الوجود والوحدانية حيث قدمنا أهم المفاهيم الأساسية حول التفاضل والتكامل الكسري وكذلك التفاضل والتكامل العشوائي(EDF و EDS) ، ثم حل معادلة تفاضلية كسرية عشوائية.

الكلمات المفتاحية: دوال خاصة، حساب كسري، كابوتو، EDF، حساب عشوائي، EDS، معادلات تفاضلية عشوائية من الرتبة الكسرية EDFS .

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INTRODUCTION

With the advancement of mathematical sciences and their applications, mathematicians and researchers see that the field of calculus is not limited to integer orders alone, but even extends to non-integer orders, known as fractional calculus.

Fractional calculus is a generalization of traditional calculus to include arbitrary non-integer orders.This topic dates back to the era when Leibniz and Newton invented differential calculus. One owes to Leibniz in a letter to L'Hôpital, dated September 30, 1695 [32], the exact birthday of the fractional calculus and the idea of the fractional derivative, who sought to understand the meaning of Leibniz's currently popular notation, $\frac{d^n y}{dx^n}$, for the nth derivative when n is a real number. L'Hôpital wondered what would happen if n took on a fractional value, such as $\frac{1}{2}$ or even $\frac{1}{12}$.

The concept of differentiation and integration to noninteger order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known Leibniz (1695) mentions it in a letter to L'Hôpital in 1695. The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832), Riemann (1847), and Holmgren (1864), Euler (1730), Lagrange (1772), and others made contributions even earlier.

In the letters to J. Wallis and J. Bernoulli (in 1697), Leibniz mentioned the possible approach to fractional-order differentiation in that sense that for non-integer values of n the definition could be the following :

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

In 1730, Euler mentioned interpolating between integral orders of a derivative and suggested to use the following relationship :

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n},$$

where $\Gamma(\cdot)$ is the (Eulers) Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,$$

$\alpha > 0$. Also for negative or non-integer (rational) values of n : Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained :

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \sqrt{\frac{4x}{\pi}} = \frac{2}{\sqrt{\pi}} x^{1/2}.$$

While traditional calculus is based on integer-order differentiation and integration, the concept of fractional calculus has enormous potential to transform how we perceive, model, and control the 'nature' around us. Numerous theoretical and experimental studies demonstrate that certain electrochemical [16], thermal [5], and viscoelastic [37] systems are governed by non-integer-order differential equations. Consequently, classical models based on integer-order derivation prove inadequate. For this reason, models based on non-integer-order differential equations have been developed .[14]

It's theory of stochastic integrals and stochastic differential equations can be traced back to the 1940s, beginning with an early important paper (worldcat, 1942) written by Kiyosi It. Differential calculus establishes the theory of ordinary differential equations, which describes a class of models of systemsthat change with time. Random perturbations were introduced in these equations through a non-differentiable Brownian motion .

Stochastic differential equations have been used in science, geometry, biology, and nearly all applied sciences. There are many articles about the existence and uniqueness of solutions of stochastic differential equations inthe existing literature (see, e.g., [4, 24, 12, 13]).
More recently, Chang et al. Using a semi group theory and a fixed-point technique, [11] investigated mean-square almost automorphic mild solutionsto non-autonomous stochastic differential equations in Hilbert spaces. El-Borai et al. [23] considered the uniqueness and continuity of solutions for a fractional stochastic integral equation.
In [22], the authors studiedan abstract fractional-order stochastic differential equations with delay driven by Brownian motion and stablished existence and uniqueness of the solution. Mourad K Zhou study the existence and uniqueness of mild solutions offractional order stochastic differential equation in hilbert space.
With the stochastic component, these equations gain another layer, capturing the uncertainty and stochasticity foundin countless natural systems. Itis then natural to apply stochastic fractional differential equations to model such phenomena whose dy namic behavior is driven jointly by deterministic and random drivers. The first chapter of this thesis constitutes a brief introduction and history of the concept of stochastic fractional differential equation ofthe Caputo type. Thisis done through a progression of step by step concepts.

In this section, we presented fundamental definitions and concepts that enable us to understand and solve stochastic fractional differential equations. We discussed specific functions such as the Gamma function, the Beta function, and certain properties related to Laplace transforms. Additionally, we explored the (Banach) fixed-point theorem and addressed fractional calculus, which includes definitions of fractional derivatives such as Caputo, Riemann-Liouvilleand fractional differential equations (FDEs) with examples
Chapter 02 :We established fundamental definitions for stochastic calculus, which included general reminders about probabilities, expectations, conditional expectations, filtrations, stochastic processes, and more. We also covered Brownian motion and stochastic integrals, along with some examples of stochastic equations.
Chapter 03
We began with basic definitions of stochastic calculus, which includes general reminders on probability,expectation, conditional expectation, etc. We also covered Brownian motion and stochastic integrals, along with some examples of stochastic equations.
Chapter 4 In this chapter, we studied solutions (existence, uniqueness, and stability) under specific conditions for fractional-order stochastic differential equations of the Caputo type using various methods. Additionally, we examined the continuity of the solution along the domain $[0, \infty)$.
We also studied the solutions of neutral fractional stochastic functional differential equations, focusing on proving the existence of a unique solution to these equations under specific conditions, thereby ensuring both existence and uniqueness of the solution.

CHAPITRE 1

FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

1.1 Préliminaire(Useful function)

In this section, we highlight the significant roles of the Gamma, Beta, and Mittag-Leffler functions in the theory of fractional calculus as well as in their respective applications[19].

1.1.1 Gamma Function

One of the basic functions of fractional calculus is the Gamma function denoted Γ.

Definition 1.1.1
The Gamma function Γ is defined by the following integral :

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0$$

For positive integer values n, the Gamma function becomes $\Gamma(\alpha) = (\alpha - 1)!$ and thus can be seen as an extension of the factorial function to real values.

Example 1.1.1
We have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.Indeed :

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt.$$

Let $u = \sqrt{t}$, then $t = u^2$ and $dt = 2udu$, thus :

$$\begin{aligned} \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt &= 2 \int_0^{+\infty} \frac{e^{-u^2}}{u} udu \\ &= 2 \int_0^{+\infty} e^{-u^2} du. \end{aligned}$$

Knowing that $\int_0^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$ (Gaussian integral), it follows that :

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proposition 1.1.1
An important property of the Gamma function is the following recurrence relation :

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0$$

Proof :
we have $\Gamma(\alpha+1) = \int_0^{+\infty} t^\alpha e^{-t} dt$.
An integration by parts aplied to the definition of gamma function :

$$\begin{aligned}
u = t^\alpha &\longrightarrow u' = \alpha t^{\alpha-1} \\
v' = e^{-t} &\longrightarrow v = -e^{-t} \\
\Gamma(\alpha+1) &= \int_0^\infty t^\alpha e^{-t} dt \\
&= \left[-t^\alpha e^{-t}\right]_0^\infty + \alpha \int_0^\infty t^{\alpha-1} e^{-t} dt \\
&= \alpha\Gamma(\alpha).
\end{aligned}$$

Proposition 1.1.2
$\forall \alpha \in \mathbb{N}, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}$

Proof :
According to Proposition 1.1.1 for all $n \in \mathbb{N}$:

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) \\
&\vdots \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{2^n}(2n-1)(2n-3)\cdots(1)\sqrt{\pi} \\
&= \frac{(2n)(2n-1)(2n-2)\cdots 2 \times 1}{2^n(2n)(2n-2)\cdots 2}\sqrt{\pi} \\
&= \frac{(2n)!}{2^{2n} n!}\sqrt{\pi} \\
&= \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}.
\end{aligned}$$

This remains true when $n = 0$, therefore :

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}, \quad \forall n \in \mathbb{N}.$$

Proposition 1.1.3

$$\Gamma\left(n+1+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}.$$

Proof :
$\Gamma\left(n+1+\frac{1}{2}\right)=\Gamma\left(n+\frac{3}{2}\right)=\frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}$,

$$\begin{aligned}\Gamma\left(n+1+\frac{1}{2}\right) &= \Gamma\left(n+\frac{1}{2}+1\right)\\ &= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\\ &= \left(n+\frac{1}{2}\right)\frac{\sqrt{\pi}(2n)!}{4^n n!}\\ &= \frac{(2n+1)\sqrt{\pi}(2n)!}{2\times 4^n n!}\\ &= \frac{(2n+1)(2n+2)\sqrt{\pi}(2n)!}{2(2n+2)\times 4^n n!}\\ &= \frac{\sqrt{\pi}(2n+2)!}{2\times 2(n+1)\times 4^n n!}\\ &= \frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}.\end{aligned}$$

Proposition 1.1.4
Taking into account that the Γ *function can be written as*

$$\Gamma(n)=\frac{\Gamma(n+1)}{n},$$

it results that the Γ *function can be defined also for negative values of* n*, in the interval* $-1<n<0$*.*
-The following particular values for the Γ *function can be useful for calculation purposes :*

$$\Gamma(1)=1,$$

$$\Gamma(0)=+\infty,$$

$$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi},$$

$$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi},$$

$$\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\sqrt{\pi}.$$

Proposition 1.1.5
By the principle of analytic continuation, the function can be extended over $\mathbb{C}\setminus\mathbb{Z}$*. For* $\alpha>0$ *and* $\alpha\notin\mathbb{N}$*, we have :*

$$(-1)^j\binom{\alpha}{j}=\frac{\Gamma(j-\alpha)}{(j+1)\Gamma(-\alpha)}.$$

1.1.2 Beta Function

Definition 1.1.2
For $\alpha > 0, \gamma > 0$, the Euler Beta function is defined by

$$\beta(\alpha,\gamma) = \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt.$$

Proposition 1.1.6 *[30]*
the Beta function is linked to the Gamma function by the following relationship :

$$\beta(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \quad \alpha > 0, \gamma > 0$$

Proof :

$$\begin{aligned}\Gamma(\alpha)\Gamma(\gamma) &= \int_0^\infty \int_0^\infty t_1^{\alpha-1}e^{-t_1}t_2^{\gamma-1}e^{-t_2}dt_1dt_2 \\ &= \int_0^\infty t_1^{\alpha-1}\left(\int_0^\infty e^{-(t_1+t_2)}t_2^{\gamma-1}dt_2\right)dt_1,\end{aligned}$$

By performing the change of variable $t_2' = t_1 + t_2$, we find

$$\begin{aligned}\Gamma(\alpha)\Gamma(\gamma) &= \int_0^\infty t_1^{\alpha-1}\int_{t_1}^\infty (t_2'-t_1)^{\gamma-1}e^{-t_2'}dt_2'dt_1 \\ &= \int_{t_1}^\infty e^{-t_2'}\int_0^\infty (t_2'-t_1)^{\gamma-1}t_1^{\alpha-1}dt_1dt_2',\end{aligned}$$

if we set $t_1' = \frac{t_1}{t_2}$, we obtain

$$\begin{aligned}\Gamma(\alpha)\Gamma(\gamma) &= \int_0^\infty e^{-t_2'}dt_2'\int_0^1 (t_2'-t_1't_2')^{\gamma-1}(t_1't_2')^{\alpha-1}t_2'dt_1' \\ &= \int_0^\infty e^{-t_2'}dt_2'\int_0^1 (t_2'(1-t_1'))^{\gamma-1}(t_1't_2')^{\alpha-1}t_2'dt_1' \\ &= \int_0^\infty e^{-t_2'}dt_2'\left((t_2')^{\gamma-1}(t_2')^{\alpha-1}t_2'\int_0^1 (1-t_1')^{\gamma-1}(t_1')^{\alpha-1}dt_1'\right) \\ &= \int_0^\infty e^{-t_2'}dt_2'\left((t_2')^{\alpha+\gamma-1}\beta(\alpha,\gamma)\right) \\ &= \int_0^\infty e^{-t_2'}(t_2')^{\alpha+\gamma-1}dt_2'\beta(\alpha,\gamma) \\ &= \Gamma(\alpha+\gamma)\beta(\alpha,\gamma).\end{aligned}$$

this gives the desired result

Remark 1.1.1
The Beta function is symmetric, that is : $\beta(\alpha,\gamma) = \beta(\gamma,\alpha),\ \forall\ Re(\alpha),\ Re(\gamma) > 0.$

Proof :
From the definition of the Beta function, γ have :

$$\beta(\alpha,\gamma) = \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt.$$

Let : $u = 1-t \implies t = 1-u$ and $dt = -du$. We then obtain :

$$\begin{aligned}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt &= -\int_1^0 (1-u)^{\alpha-1}u^{\gamma-1}du \\ &= \int_0^1 (1-u)^{\alpha-1}u^{\gamma-1}du \\ &= \beta(\gamma,\alpha).\end{aligned}$$

1.1.3 Mittag-Leffler Function

The exponential function e^{α} holds a fundamental position in the theory of integer-order differential equations.G.M. Mittag-Leffler introduced a generali$\alpha$of this exponential function using a single parameter, which is represented by the following function :

Definition 1.1.3 *[8]*
For $\alpha \in \mathbb{C}$ and α a strictly positive real number, the Mittag-Leffler function $E_{\gamma}(\alpha)$ is defined by the following series expansion :

$$E_{\gamma}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(\gamma k + 1)},$$

For any $\gamma > 0$ and $\beta > 0$, the generaliα Mittag-Leffler function $E_{\gamma,\beta}(\alpha)$ can be defined with two parameters γ and β as follows :

$$E_{\gamma,\beta}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(\gamma k + \beta)} \quad \alpha > 0, \beta > 0$$

Example 1.1.2

$$E_{1,1}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(k+1)} = \sum_{k=0}^{+\infty} \frac{\alpha^k}{k!} = e^{\alpha}.$$

$$E_{1,2}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(k+2)} = \sum_{k=0}^{+\infty} \frac{\alpha^k}{(k+1)!} = \frac{1}{\alpha} \sum_{k=0}^{+\infty} \frac{\alpha^{k+1}}{(k+1)!} = \frac{1}{\alpha} \left(e^{\alpha} - 1\right).$$

$$E_{1,3}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(k+3)} = \frac{1}{\alpha^2} \sum_{k=0}^{+\infty} \frac{\alpha^{k+2}}{(k+2)!} = \frac{1}{\alpha^2} \left(e^{\alpha} - \alpha - 1\right).$$

Remark 1.1.2
for all $\gamma > 0$ and $k \in \mathbb{N}$ we have :

$$E_{\gamma,1}(\alpha) = \sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(\alpha k + 1)} = E_{\gamma}(\alpha).$$

1.1.4 The Laplace Transform

In this section, we will discuss the Laplace transform along with its fundamental properties. The Laplace[29] transform is a wonderful tool for solving ordinary and partial differential equations and systems.

Definition 1.1.4
A function f has exponential order α if there exist constants $M > 0$ and α such that for some $t_0 \geq 0$, $|f(t)| \leq Me^{\alpha t}, \quad t \geq t_0$..we define the Laplace Transform of f

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt,$$

is called the Laplace transform of the function f.

Definition 1.1.5
The inversion of the Laplace transform is performed by means of an integral in the complex plane, for t positive,

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+\infty} e^{st} F(s) ds.$$

Where γ is chosen such that the integral converges, which implies that γ is greater than the real part of the singularity of $F(s)$.

Proposition 1.1.7
Linearity : *One of the most basic and useful properties of the Laplace operator L is that of* linearity, *namely, if* $f_1 \in L$ *for* $\mathcal{R}e(s) > \alpha$, $f_2 \in L$ *for* $\mathcal{R}e(s) > \beta$, *then* $f_1 + f_2 \in L$ *for* $\mathcal{R}e(s) > \max\{\alpha, \beta\}$, *and*

$$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2) \quad (1.11)$$

for arbitray constants $c_1 + c_2$
Derivation : *The Laplace transform of an integer-order derivative is :*

$$\begin{aligned} L\left\{f^{(n)}(t)\right\} &= s^n L\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \\ &= s^n L\{f(t)\} - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \end{aligned}$$

Integration :

$$L\left\{\int_a^t f(u)du\right\} = \frac{1}{s}L\{f\} + \frac{1}{s}\int_a^0 f(u)du.$$

Convolution :

$$L\{f * g\} = L\{f\} \times L\{g\}.$$

- *The Laplace transform of the function* $t^{\alpha-1}$ *is :*

$$L\left\{t^{\alpha-1}\right\}(s) = \Gamma(s)s^{-\alpha}.$$

1.2 Riemann-Liouville Fractional Integral

This section introduces the elementary definitions and some properties of the Riemann-Liouville fractional integral.
Let f be a real, continuous, and integrable function on the interval $[a, b]$. We consider the integral

$$\begin{aligned} \mathcal{I}^1 f(t) &= \int_a^t f(\tau)d\tau, \\ \mathcal{I}^2 f(t) &= \int_a^t \mathcal{I}^1 f(u)du, \\ &= \int_a^t \left(\int_a^u f(s)ds\right) du, \\ &= \int_a^t \left(\int_s^t du\right) f(s)ds, \\ &= \int_a^t (t-s)f(s)ds. \end{aligned}$$

By repeatedly applying this process n times, we obtain, according to Cauchy's formula :

$$\mathcal{I}^n f(t) = \int_a^t dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds.$$

And, using the generalization of the factorial function via the Gamma function : $\Gamma(n) = (n-1)!$. Riemann realized that the right-hand side could make sense even when n takes on non-integer values. He defined the fractional integral as follows : Let $f \in C[a, b], \alpha \in \mathbb{R}_+$. The Riemann-Liouville fractional integral of f of order α, denoted by $\mathcal{I}^{\alpha}_{a^+} f$, is defined by :

Definition 1.2.1
let $f \in C[a,b], \alpha \in \mathbb{R}_+$, we it the Riemann-Liouville fractional (left-sided) integral of order α, denoted by $\mathcal{I}_{a^+}^{\alpha} f$ the function defined by :

$$\mathcal{I}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \tag{1.1}$$

the right-sided Riemann-Liouville fractional integral of the function f of order α, denoted by $\mathcal{I}_b^{\alpha} f$ the function defined by :

$$\mathcal{I}_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s) ds, \tag{1.2}$$

Remark 1.2.1
For the remainder of this work, we will exclusively utilize the left-sided integral and employ the notation α.
throughout what follows, we will only use the left-sided integral and denote it as $\mathcal{I}_a^{\alpha}$.

Example 1.2.1
Calculate $\mathcal{I}_a^{\alpha} t^{\mu}$, for $\mu > -1$:

$$\mathcal{I}_a^{\alpha} t^{\mu} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu}\, ds \quad \textit{for} \quad \alpha > 0, \mu > -1.$$

Let's make the change of variable $s = t\tau$, $ds = t\, d\tau$, we have :

$$\begin{aligned}
\mathcal{I}_a^{\alpha} t^{\mu} &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau t)^{\alpha-1} (\tau t)^{\mu} t\, d\tau, \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-\tau)^{\alpha-1} \tau^{\mu} t^{\mu} t\, d\tau, \\
&= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\mu}\, d\tau, \\
&= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} B(\mu+1, \alpha), \\
&= \frac{1}{\Gamma(\alpha)} t^{\alpha+\mu} \frac{\Gamma(\mu+1)\Gamma(\alpha)}{\Gamma(\alpha+\mu+1)}.
\end{aligned}$$

Therefore,

$$\mathcal{I}_a^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}, \quad \alpha > 0, \mu > -1.$$

Lemma 1.2.1
For $\alpha = 0$, we have :

$$\mathcal{I}_a^0 f(x) = f(x).$$

Proof :

$$\begin{aligned}
\mathcal{I}_a^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x f(t) d\left(-\frac{(x-t)^{\alpha}}{\alpha}\right) \\
&= \frac{1}{\Gamma(\alpha+1)} \left[f(a)(x-a)^{\alpha} + \int_a^x f'(t)(x-t)^{\alpha} dt \right].
\end{aligned}$$

We obtain

$$\begin{aligned}\mathcal{I}_a^0 f(x) &= 1\left[f(a)1+\int_a^x f'(t)dt\right]\\ &= f(a)+f(x)-f(a)\\ &= f(x).\end{aligned}$$

Therefore

$$\mathcal{I}_a^0 f(x) = f(x).$$

Proposition 1.2.1
Let f be an integrable and bounded function, and let α and μ be two strictly positive real numbers. Then

$$\mathcal{I}_a^{\alpha}\left[\mathcal{I}_a^{\mu} f(t)\right] = \mathcal{I}_a^{\alpha+\mu} f(t), \tag{1.3}$$

Proof

$$\begin{aligned}\mathcal{I}_a^{\alpha}\left[\mathcal{I}_a^{\mu} f(t)\right] &= \frac{1}{\Gamma(\alpha)}\int_0^{t-\alpha} s^{\mu-1}\mathcal{I}_a^{\alpha} f(t-s)ds,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\mu)}\int_0^{t-a} s^{\mu-1}ds\int_a^{t-s}(t-s-u)^{\alpha-1}f(u)du,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\mu)}\int_a^{t} f(u)du\int_a^{t-u} t^{\mu-1}(t-u-s)^{\alpha-1}ds,\end{aligned}$$

let $\quad s = v(t-u)$
Then $\quad ds = (t-u)dv$
Hence, it follows that :

$$\begin{aligned}&= \frac{1}{\Gamma(\alpha)\Gamma(\mu)}\int_a^{t} f(u)du\int_0^{1}(v(t-u))^{\mu-1}(t-u-v(t-u))^{\alpha-1}(t-u)dv,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\mu)}\int_a^{t}(t-u)^{\alpha+\mu-1}f(u)du\int_0^{1} v^{\mu-1}(1-v)^{\alpha-1}dv,\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\mu)}\int_a^{t}(t-u)^{\alpha+\mu-1}f(u)du\beta\left(\mu,\alpha\right),\\ &= \frac{1}{\Gamma\left(\alpha+\mu\right)}\int_a^{t}(t-u)^{\alpha+\mu-1}f(u)du,\\ &= \mathcal{I}_a^{\alpha+\mu} f(t).\end{aligned}$$

Proposition 1.2.2
For $\alpha > 0$, the Riemann-Liouville integral is linear, i.e. :

$$\mathcal{I}_a^{\alpha}(\lambda f(t)+\mu g(t)) = \lambda\mathcal{I}_a^{\alpha} f(t)+\mu\mathcal{I}_a^{\alpha} g(t). \tag{1.4}$$

Proof

$$\begin{aligned}\mathcal{I}_a^{\alpha}(\lambda f+\mu g)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^{x}(x-\tau)^{\alpha-1}(\lambda f+\mu g)(\tau)d\tau\\ &= \frac{1}{\Gamma(\alpha)}\left[\int_a^{x}(x-\tau)^{\alpha-1}\lambda f(\tau)d\tau+\int_a^{x}(x-\tau)^{\alpha-1}\mu g(\tau)d\tau\right]\\ &= \frac{\lambda}{\Gamma(\alpha)}\int_a^{x}(x-\tau)^{\alpha-1}f(\tau)d\tau+\frac{\mu}{\Gamma(\alpha)}\int_a^{x}(x-\tau)^{\alpha-1}g(\tau)d\tau\\ &= \lambda\mathcal{I}_a^{\alpha} f(x)+\mu\mathcal{I}_a^{\alpha} g(x).\end{aligned}$$

Therefore, $\mathcal{I}_a^{\alpha}$ is a linear operator.

Proposition 1.2.3
the Laplace transform of the Riemann-liouville fractional integral for $a = 0$ of a function f,that has the Laplace transform $F(s)$ in the half-plane $\mathrm{Re}(s) > 0$, is given by :

$$L\left(\mathcal{I}^{\alpha} f\right)(s) = s^{-\alpha}F(s). \tag{1.5}$$

1.3 fractional derivative in the sense of Riemann-Liouville

Definition 1.3.1 *[18]*
let f be an integrable function over $[a,b]$ then the fractional derivative of order α (with $n-1<\alpha<n, n\in\mathbb{N}^*$) in the sense of Riemann-Liouville ${}^{RL}\mathcal{D}_a^\alpha f$ is defined by :

$$ {}^{RL}\mathcal{D}_a^\alpha f(t) = \frac{d^n}{dt^n}\left\{\mathcal{I}_a^{n-\alpha} f(t)\right\} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds. \tag{1.6}$$

Example 1.3.1
The Riemann-Liouville fractional derivative of $f(t)=(t-a)^p$. Let α be a non-integer with $0\le n-1<\alpha<n$ and $p>-1$, then we have :

$$\begin{aligned} {}^{RL}\mathcal{D}_a^\alpha f(t) &= {}^{RL}\mathcal{D}_a^\alpha (t-a)^p,\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{(\tau-a)^p}{(t-\tau)^{\alpha-n+1}}d\tau, \end{aligned}$$

By changing the variable $\tau = a+s(t-a)$ we have :

$$\begin{aligned} {}^{RL}\mathcal{D}_a^\alpha (t-a)^p &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}(t-a)^{n+p-\alpha}\int_0^1 (1-s)^{\alpha-n+1}s^p ds,\\ &= \frac{\Gamma(n+p-\alpha+1)\beta(n-\alpha,p+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)}(t-a)^{p-\alpha},\\ &= \frac{\Gamma(n+p-\alpha+1)\beta(n-\alpha,p+1)\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)\Gamma(n+p-\alpha+1)}(t-a)^{p-\alpha},\\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t-a)^{p-\alpha}. \end{aligned}$$

Special case
if $p=0$

$$ {}^{RL}\mathcal{D}_a^\alpha (t-a)^0 = {}^{RL}\mathcal{D}_a^\alpha 1 = \frac{1}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$$

Remark 1.3.1
The non-integer order derivative of a constant function in the Riemann-Liouville sense is neither zero nor constant, However, we have :

$$ {}^{RL}\mathcal{D}_a^\alpha C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \tag{1.7}$$

On note $\frac{d^n}{dt^n}$ by $\mathcal{D}^n$.

Proposition 1.3.1 *[30]*
if $\alpha=n\in\mathbb{N}$ we have :

$$ {}^{RL}\mathcal{D}_a^0 f(t) = f(t), {}^{RL}\mathcal{D}_a^1 f(t) = f^{(1)}(t), {}^{RL}\mathcal{D}_a^2 f(t) = f^{(2)}(t), \ldots, {}^{RL}\mathcal{D}_a^n f(t) = f^{(n)}(t). \tag{1.8}$$

Composition with the fractional integral

Proposition 1.3.2 *[30]*
let $\alpha>0$ et $n=[\alpha]+1$ then for every integer $m\in\mathbb{N}^*$ we have :

$$ {}^{RL}\mathcal{D}_a^\alpha f(t) = {}^{RL}\mathcal{D}_a^m\left(\mathcal{I}^{m-\alpha} f(t)\right), \quad \textit{for } m>\alpha \tag{1.9}$$

Proof
as $m \geq n$, we have :

$$\begin{aligned} {}^{RL}\mathcal{D}_a^m \left(\mathcal{I}_a^{m-\alpha} f(t)\right) &= {}^{RL}\mathcal{D}_a^n {}^{RL}\mathcal{D}_a^{m-n} \mathcal{I}_a^{m-n} \mathcal{I}_a^{n-\alpha} f(t), \\ &= {}^{RL}\mathcal{D}_a^n \mathcal{I}_a^{n-\alpha} f(t), \\ &= {}^{RL}\mathcal{D}_a^\alpha f(t). \end{aligned}$$

Lemma 1.3.1
let $\alpha > 0$ and $f \in L^1[a,b]$, then the equality :

$$ {}^{RL}\mathcal{D}_a^\alpha \mathcal{I}_a^\alpha f(t) = f(t). \tag{1.10}$$

is true almost every $t \in [a,b]$

Proof
Using the definition, we have :

$$\begin{aligned} {}^{RL}\mathcal{D}_a^\alpha \mathcal{I}_a^\alpha f(t) &= {}^{RL}\mathcal{D}^n \mathcal{I}_a^{n-\alpha} \mathcal{I}_a^\alpha f(t), \\ &= {}^{RL}\mathcal{D}_a^n \mathcal{I}_a^n f(t), \\ &= f(t). \end{aligned}$$

Theorem 1.3.1
let $\alpha, \beta > 0$ and $n-1 \leq \alpha < n, m-1 \leq \alpha < m$ such that $(n, m \in \mathbb{N})$ then :

1. *if $\alpha > \beta > 0$, then for $f \in L^1[a,b]$ the equality :*
$$ {}^{RL}\mathcal{D}_a^\beta \left(\mathcal{I}_a^\alpha\right) f(t) = \mathcal{I}_a^{\alpha-\beta} f(t) \tag{1.11}$$
is valid almost everywhere on $[a,b]$.
2. *if there exist a function $\varphi \in L^1[a,b]$ tel such that $f = \mathcal{I}_a^\alpha \varphi$ then :*
$$ \mathcal{I}_a^\alpha {}^{RL}\mathcal{D}_a^\alpha f(t) = f(t), \tag{1.12}$$
for almost every $x \in [a,b]$.
3. *if $\beta \geq \alpha > 0$ and the fractional derivative $\mathcal{D}_a^{s-\alpha} f$ exist, then :*
$$ {}^{RL}\mathcal{D}_a^\beta \left(\mathcal{I}_a^\alpha\right) f(t) = {}^{RL}\mathcal{D}_a^{\beta-\alpha} f(t), \tag{1.13}$$

Proof
Using definition 1.3.1 and proposition 1.2.1 we obtain :

1. for $\alpha > \beta > 0$, then for all $n \geq m$, we have :
$$\begin{aligned} {}^{RL}\mathcal{D}_a^\beta \left(\mathcal{I}_a^\alpha\right) f(t) &= {}^{RL}\mathcal{D}_a^n \mathcal{I}_a^{n-\beta} \left(\mathcal{I}_a^\alpha\right) f(t), \\ &= {}^{RL}\mathcal{D}_a^n \left(\mathcal{I}_a^{n+\alpha-\beta}\right) f(t), \\ &= {}^{RL}\mathcal{D}_a^n \left(\mathcal{I}_a^n \left(\mathcal{I}_a^{\alpha-\beta}\right) f(t)\right), \\ &= \mathcal{I}_a^{\alpha-\beta} f(t). \end{aligned}$$
almost for every $t \in [a,b]$
2. by relation (1.20), we obtain :
$$\begin{aligned} \mathcal{I}_a^\alpha {}^{RL}\mathcal{D}_a^\alpha f(t) &= \mathcal{I}_a^\alpha \left({}^{RL}\mathcal{D}_a^\alpha \mathcal{I}_a^\alpha \varphi(t)\right), \\ &= \mathcal{I}_a^\alpha \varphi(t), \\ &= f(t). \end{aligned}$$
3. we have :
$$\begin{aligned} {}^{RL}\mathcal{D}_a^\beta \left(\mathcal{I}_a^\alpha\right) f(t) &= {}^{RL}\mathcal{D}_a^m \mathcal{I}_a^{m-\beta} \mathcal{I}_a^\alpha f(t), \\ &= {}^{RL}\mathcal{D}_a^m \mathcal{I}_a^{m-(\beta-\alpha)} f(t), \\ &= {}^{RL}\mathcal{D}_a^{\beta-\alpha} f(t). \end{aligned}$$
Exist for $i-1 \leq \beta - \alpha < i$ et $i \leq m$

Composition with integer order derivatives

Theorem 1.3.2
let $\alpha, \beta > 0$ *and* $n-1 \leq \alpha < n, m-1 \leq \alpha < m$ *such that* $(n, m \in \mathbb{N})$ *then : for* $\alpha > 0,\ k \in \mathbb{N}^*$*. if the fractional derivatives* $\mathcal{D}_a^{\alpha} f$ *and* $\mathcal{D}_a^{k+a} f$ *exist, then :*

$$^{RL}\mathcal{D}_a^{k}\left(^{RL}\mathcal{D}_a^{\alpha} f(t)\right) = {}^{RL}\mathcal{D}_a^{k+\alpha} f(t) \tag{1.14}$$

Proof
we have :

$$\begin{aligned} ^{RL}\mathcal{D}_a^{k}\left(^{RL}\mathcal{D}_a^{\alpha} f(t)\right) &= {}^{RL}\mathcal{D}_a^{k}\mathcal{D}_a^{n}\mathcal{I}_a^{n-\alpha} f(t), \\ &= {}^{RL}\mathcal{D}_a^{k+n}\mathcal{I}_a^{k+n-\alpha+k-k} f(t), \\ &= {}^{RL}\mathcal{D}_a^{k+n}\mathcal{I}_a^{k+n-(\alpha+k)} f(t), \\ &= {}^{RL}\mathcal{D}_a^{k+\alpha} f(t). \end{aligned}$$

Hence the result.

Proposition 1.3.3
for $\alpha > 0, n \in \mathbb{N}^*$*. if the fractional derivative* $\mathcal{D}_a^{n+\alpha} f$ *and* $1 \leq k \leq n-1$ *exist, then*

$$^{RL}\mathcal{D}_a^{\alpha}\left(^{RL}\mathcal{D}^{n} f(t)\right) = \left(^{RL}\mathcal{D}_a^{n+\alpha} f(t)\right) - \sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)}. \tag{1.15}$$

Remark 1.3.2
Fractional differentiation and conventional differentiation commute only if : $f^{(k)}(a) = 0$ *for all* $k = 0, 1, 2, \ldots, n-1$*.*

composition with fractional derivatives

Proposition 1.3.4
for all $n-1 \leq \alpha < n$ *and* $m-1 \leq \beta < m$ *we have :*

$$^{RL}\mathcal{D}_a^{\alpha}\left(^{RL}\mathcal{D}_a^{\beta} f(t)\right) = {}^{RL}\mathcal{D}_a^{\alpha+\beta} f(t) - \sum_{k=1}^{m}\left[^{RL}\mathcal{D}_a^{\beta-k} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(-\alpha-k+1)}. \tag{1.16}$$

Proposition 1.3.5
for all $n-1 \leq \alpha < n$ *and* $m-1 \leq \beta < m$ *we have :*

$$^{RL}\mathcal{D}_a^{\beta}\left(^{RL}\mathcal{D}_a^{\alpha} f(t)\right) = {}^{RL}\mathcal{D}_a^{\alpha+\beta} f(t) - \sum_{k=1}^{n}\left[^{RL}\mathcal{D}_a^{\alpha-k} f(t)\right]_{t=a} \frac{(t-a)^{-\beta-k}}{\Gamma(-\beta-k+1)}. \tag{1.17}$$

assume that if $\alpha = \beta$ *and* $\left[^{RL}\mathcal{D}_a^{\beta-k} f(t)\right]_{t=a}$ *for all* $k = 1, 2, \ldots, m$ *and* $\left[^{RL}\mathcal{D}_a^{\alpha-k} f(t)\right]_{t=a}$ *for all* $k = 1, 2, \ldots, n$

1.4 fractional Derivation in the Grunwald-Letnikov sense

Fractional calculus is itself a sub-branch ofanalysis, which is a generalisation of differentiation and integration to non-integer order.and fractional order integration and differentiation operations. Several definitions of fractional derivatives which exist, unfortunately are notall equal. In this chapter, the most following are listed commonly used definitions, such as Riemann-Liouville, Liouville, Caputo, and Grunwald-Letnikov.
This definition is based on the calculation of derivatives using finite differences.[19]
let $f : \mathbb{R} \to \mathbb{R}^n$. for $h > 0$, denote the τ_h left translation operator :

$$\tau_h f(t) = f(t-h), \tag{1.18}$$

thus, we have

$$f'(t) = \lim_{h\to 0} \frac{1}{h}(f(t) - f(t-h)) = \lim_{h\to 0} \frac{1}{h}\left(id - \tau_h\right) f(t).$$

By denoting $\tau_h^2 = \tau_h \circ \tau_h$, we have : $\tau_h^2 f(t) = f(t-2h)$.
Regarding the second derivative

$$\begin{aligned} f''(t) &= \lim_{h\to 0}\left(\frac{1}{h}\left(id - \tau_h\right)\right)^2 f(t) \\ &= \lim_{h\to 0} \frac{1}{h^2}\left(id - 2\tau_h - \tau_h^2\right) f(t) \\ &= \lim_{h\to 0} \frac{1}{h^2}(f(t) - 2f(t-h) + f(t-2h)). \end{aligned}$$

More generally,the derivative n^{ime} of f is given by

$$\begin{aligned} f^{(n)}(t) &= \lim_{h\to 0} \frac{1}{h^n}(id - \tau_h)^n f(t) \\ &= \lim_{h\to 0} \frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k} id^{n-k}(-\tau_h)^k f(t) \\ &= \lim_{h\to 0} \frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k} f(t-kh) \end{aligned} \tag{1.19}$$

where

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!},$$

It is possible to extender to $k > n$, by setting $\left(\begin{array}{c} n \\ k \end{array}\right) = 0$. the formula 1.19 then becomes :

$$f^{(n)}(t) = \lim_{h\to 0} \frac{1}{h^n}\sum_{k=0}^{\infty}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) f(t-kh).$$

the generalization of this formula using the Gamma function, for α non integer (with $0 \le n-1 < \alpha < n$) by setting for $\alpha \in \mathbb{R}^+/\mathbb{N}$ et $k \in \mathbb{N}$, Note that $\left(\begin{array}{c} \alpha \\ k \end{array}\right) = 0$ even if $k > \alpha$

$${}^{G}\mathcal{D}_a^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k \left(\begin{array}{c} \alpha \\ k \end{array}\right) f(t-kh),$$

According to proposition 1.1.5, we have

$$(-1)^k \left(\begin{array}{c} \alpha \\ k \end{array}\right) = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)},$$

this gives us :

$$ {}^{G}\mathcal{D}_a^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} f(t-kh), $$

and

$$ \begin{aligned} {}^{G}\mathcal{D}_a^{-\alpha} f(t) &= \lim_{h\to 0} \frac{1}{h^{-\alpha}} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)} f(t-kh), \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \end{aligned} $$

if f is of class C^n then using integration by parts, we obtain :

$$ {}^{G}\mathcal{D}_a^{-\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k+\alpha}}{\Gamma(k+\alpha+1)} + \frac{1}{\Gamma(n+\alpha)} \int_a^t (t-\tau)^{n+\alpha-1} f^{(n)}(\tau) d\tau. $$

and also :

$$ {}^{G}\mathcal{D}_a^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. $$

Example 1.4.1
The derivative $g(t) = (t-b)^q$ in the sense of Grunwald-Letnikov.Let q be non-integer and $0 \le m-1 < \beta < m$ with $q > m-1$.Then we have : $g^{(k)}(b) = 0, \quad$ for $k = 0, 1, \ldots, m-1$, and $g^{(m)}(\xi) = \frac{\Gamma(q+1)}{\Gamma(q-m+1)}(\xi-b)^{q-m}$. Thus

$$ {}^{G}\mathcal{D}_b^{\beta}(t-b)^q = \frac{\Gamma(q+1)}{\Gamma(m-\beta)\Gamma(q-m+1)} \int_b^t (t-\xi)^{m-\beta-1}(\xi-b)^{q-m} d\xi. $$

Taking $\xi = b + s(t-b)$, we have :

$$ \begin{aligned} {}^{G}\mathcal{D}_b^{\beta}(t-b)^q &= \frac{\Gamma(q+1)}{\Gamma(m-\beta)\Gamma(q-m+1)}(t-b)^{q-\beta} \int_0^1 (1-s)^{m-\beta-1} s^{q-m} ds \\ &= \frac{\Gamma(q+1)\beta(m-\beta, q-m+1)}{\Gamma(m-\beta)\Gamma(q-m+1)}(t-b)^{q-\beta} \\ &= \frac{\Gamma(q+1)\Gamma(m-\beta)\Gamma(q-m+1)}{\Gamma(m-\beta)\Gamma(q-m+1)\Gamma(q-\beta+1)}(t-b)^{q-\beta} \\ &= \frac{\Gamma(q+1)}{\Gamma(q-\beta+1)}(t-b)^{q-\beta}. \end{aligned} $$

Remark 1.4.1
the derivative of a constant function in the sense of Grunwald-letnikov is neither zero nor constant.
if $f(t) = C$ and α is non integer we have : $f^{(k)}(t) = 0$ for $k = 1, 2, \ldots, n$

$$ \begin{aligned} {}^{G}\mathcal{D}_a^{\alpha} f(t) &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \underbrace{\sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}}_{0} \\ &+ \underbrace{\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau}_{0} \\ &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}. \end{aligned} $$

Composition with derivatives of integer order

Proposition 1.4.1
for m a positive integer and α non integer with :

$$\frac{d^m}{dt^m}\left({}^G\mathcal{D}_a^\alpha f(t)\right) = {}^G\mathcal{D}_a^{m+\alpha} f(t), \tag{1.20}$$

And

$${}^G\mathcal{D}_a^\alpha\left(\frac{d^m}{dt^m}(f(t))\right) = {}^G\mathcal{D}_a^{m+\alpha} f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)}. \tag{1.21}$$

Proof
For m a positive integer and α non integer with $(n-1<\alpha<n)$ we have :

$$\begin{aligned}\frac{d^m}{dt^m}\left({}^G\mathcal{D}_a^\alpha f(t)\right) =& \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-(\alpha+m)}}{\Gamma(k-(\alpha+m)+1)}\\ &+\frac{1}{\Gamma(n+m-(p+m))}\int_a^t (t-\tau)^{n+m-(p+m)-1} f^{(n+m)}(\tau)\,d\tau\end{aligned}$$

then :

$$\frac{d^m}{dt^m}\left({}^G\mathcal{D}_n^\alpha f(t)\right) = {}^G\mathcal{D}_a^{m+\alpha} f(t)$$

but :

$$\begin{aligned}{}^G\mathcal{D}_a^\alpha\left(\frac{d^m}{dt^m}f(t)\right) =& \sum_{k=0}^{m-1}\frac{f^{(m+k)}(a)(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-p))}\int_a^t (t-\tau)^{n-\alpha-1} f^{n+m}(\tau)d\tau,\\ =& \sum_{k=0}^{m+m-1}\frac{f^{(k)}(a)(t-a)^{k-(\alpha+m)}}{\Gamma(k-(\alpha+m)+1)}\\ &+\frac{1}{\Gamma(n+m-(p+m))}\int_a^t (t-\tau)^{m+m-(p+m)-1} f^{n+m}(\tau)d\tau\\ &-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-(\alpha+m)}}{\Gamma(k-(\alpha+m)+1)}\\ =& {}^G\mathcal{D}_a^{m+\alpha} f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)}.\end{aligned}$$

Remark 1.4.2
It is deduced that fractional differentiation and conventional differentiation commute only if : $f^{(k)}(a)=0$ for all $k=0,1,2,\ldots,m-1$.

composition with fractional derivatives

Proposition 1.4.2

1. *if $\alpha'<0$ and $\alpha\in\mathbb{R}$ then :*

$${}^G\mathcal{D}_a^\alpha\left({}^G\mathcal{D}_a^{\alpha'}(f(t))\right) = {}^G\mathcal{D}_a^{\alpha+\alpha'} f(t),$$

2. *if $0\le m-1<\alpha'<m$ and $\alpha<0$ then :*

$${}^G\mathcal{D}_a^\alpha\left({}^G\mathcal{D}_a^{\alpha'}(f(t))\right) = {}^G\mathcal{D}_a^{\alpha+\alpha\prime} f(t),$$

only if $f^{(k)}(a)=0$ for all $k=0,1,2,\ldots,m-2$

3. if $0 \leq m-1 < \alpha' < m$ and $0 \leq n-1 < \alpha < n$ then :

$$\begin{aligned} {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) &= {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) \\ &= {}^{G}\mathcal{D}_a^{\alpha+\alpha'} f(t), \end{aligned}$$

only if $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, r-2$ with $r = \max(m, n)$

Proof

1. if $\alpha < 0$ and $\alpha' < 0$ then :

$$\begin{aligned} {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) &= \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(\tau))\right) d\tau, \\ &= \frac{1}{\Gamma(-\alpha)\Gamma(-\alpha')} \int_a^t (t-\tau)^{-\alpha-1} d\tau \int_a^t (\tau-s)^{-\alpha'-1} f(s) ds, \\ &= \frac{1}{\Gamma(-\alpha)\Gamma(-\alpha')} \int_a^t f(s) ds \int_a^t (\tau-s)^{-\alpha'-1}(t-\tau)^{-\alpha-1} d\tau, \\ &= \frac{1}{\Gamma(-(\alpha+\alpha'))} \int_a^t (t-s)^{-\alpha-\alpha'-1} f(s) ds, \\ &= {}^{G}\mathcal{D}_\alpha^{\alpha+\alpha'} f(t). \end{aligned}$$

if $\alpha' < 0$ and $0 \leq n-1 < \alpha < n$ we have $\alpha = n + (\alpha - n)$with $(\alpha - n) < 0$ then :

$$\begin{aligned} {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) &= \frac{d^n}{dt^n}\left\{{}^{C}\mathcal{D}_a^{\alpha-n}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right)\right\}, \\ &= \frac{d^n}{dt^n}\left({}^{G}\mathcal{D}_a^{\alpha'+\alpha-n}(f(t))\right), \\ &= {}^{G}\mathcal{D}_a^{\alpha+\alpha'} f(t). \end{aligned}$$

2. for $0 \leq m-1 < \alpha' < m$ and $\alpha < 0$ we have :

$$ {}^{G}\mathcal{D}_a^{\alpha'} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha'}}{\Gamma(k-\alpha'+1)} + \frac{1}{\Gamma(m-\alpha')} \int_a^t (t-\tau)^{m-\alpha'-1} f(m)(t) d\tau, $$

and $(t-a)^{k-\alpha'}$ they have non integrable singularities then ${}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha}(f(t))\right)$ only exists $f_{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, m-2$ this case we have :

$$ {}^{G}\mathcal{D}_a^{\alpha'} f(t) = \frac{f^{(m-1)}(a)(t-a)^{m-1-\alpha'}}{\Gamma(m-\alpha')} + {}^{G}\mathcal{D}_a^{\alpha'-m} f^{m}(t), $$

then :

$$\begin{aligned} {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) &= \frac{f^{(m-1)}(a)(t-a)^{m-1-\alpha'-\alpha}}{\Gamma(m-\alpha'-\alpha)} + {}^{G}\mathcal{D}_a^{\alpha+\alpha'-m} f^{m}(t), \\ &= \frac{f^{(m-1)}(a)(t-a)^{m-1-(\alpha'+\alpha)}}{\Gamma(m-\alpha'-\alpha)}, \\ &+ \frac{1}{\Gamma(m-(\alpha'+\alpha))} \frac{1}{\Gamma(m-\alpha')} \int_a^t (t-\tau)^{m-(\alpha+\alpha')-1} f(m)(t) d\tau, \\ &= {}^{G}\mathcal{D}_a^{\alpha+\alpha'} f(t). \end{aligned}$$

3. for $0 \leq m-1 < \alpha' < m$ and $0 \leq n-1 < \alpha < n$ we have :

$$ {}^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) = \frac{d^n}{dt^n}\left\{{}^{G}\mathcal{D}_a^{\alpha-n}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right)\right\}. $$

if $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \dots, m-2$ then :

$$^{G}\mathcal{D}_a^{\alpha-n}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) = {}^{G}\mathcal{D}_a^{\alpha+\alpha'-m} f(t),$$

therefore :

$$\begin{aligned} ^{G}\mathcal{D}_a^{\alpha}\left({}^{G}\mathcal{D}_a^{\alpha'}(f(t))\right) &= \frac{d^n}{dt^n}{}^{G}\mathcal{D}_a^{\alpha+\alpha'-n} f(t), \\ &= {}^{G}\mathcal{D}_a^{\alpha+\alpha'} f(t). \end{aligned}$$

The Laplace transform of fractional derivative in the sense of Grunwald-Letnikov

let f be a function that has the Laplace transform $F(s)$. for $0 \leq \alpha < 1$ we have :

$$^{G}\mathcal{D}_0^{\alpha} f(t) = \frac{f(0)t^{-\alpha}}{\Gamma(n-\alpha)} + \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{-\alpha} f'(\tau)d\tau, \tag{1.22}$$

then :

$$\begin{aligned} L\left[{}^{G}\mathcal{D}_0^{\alpha} f(t)\right](s) &= \frac{f(0)}{s^{1-\alpha}} + \frac{1}{s^{1-\alpha}}[sF(s) - f(0)] \\ &= s^{\alpha}F(s), \end{aligned} \tag{1.23}$$

for $\alpha \geq 1$ there does not exist a Laplace transform in the classical sense, but in the sense of distributions, we also have :

$$L\left[{}^{G}\mathcal{D}_0^{\infty} f(t)\right](s) = s^{\alpha}F(s). \tag{1.24}$$

1.5 Fractional Derivative in the Caputo sense

The de nition of the Riemann-Liouville type fractional derivation played an impor tant role in the development of the theory of fractional derivatives and integrals be cause of their applications in pure mathematics (solution of integer order di erential equations, de nition of new classes function, summation of series, etc.). However, modern technology requires some revision of the well-known pure mathematical ap proach. Much work has appeared, especially on the theory of viscoelasticity and solid mechanics, where fractional derivatives are used for a good description of material properties. Mathematical modeling based on rheological models naturally leads to di erential equations of fractional order, and to the need to formulate the initial conditions of such equations.The applied problems require de nitions of fractional derivatives authorizing the use of physically interpretable initial conditions, which contain $f(a)$;$f(b)$,etc... Despite the fact that initial value problems with such initial conditions can be solved mathematically, the solution of this problem was proposed by M.Caputo(in the sixties) in his de nition which he adapted with Mainardi in the structure of the theory of viscoelastics :Therefore we introduces a fractional deriva tive which is more restrictive than that of Riemann-Liouvile derivative.[18]

Definition 1.5.1
for any α , a strictly positive real number,the caputo fractional derivative ${}^{C}\mathcal{D}_a^{\alpha} f$ of order α on $[a, b]$, is defined as :

$$\begin{aligned} ^{C}\mathcal{D}_a^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}}ds, \\ &= \mathcal{I}_a^{n-\alpha} f^{(n)}(t). \end{aligned} \tag{1.25}$$

Example 1.5.1
the caputo derivative of $f(t) = (t-a)^p$. let α be non integer $0 \leq n-1 < \alpha < n$ and $p > -1$ then we have :

$$\begin{aligned} {}^{C}\mathcal{D}_a^{\alpha} f(t) &= {}^{C}\mathcal{D}_a^{\alpha}(t-a)^p, \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} \int_a^t (\tau-a)^{p-n}(t-\tau)^{n-\alpha-1} d\tau, \end{aligned}$$

Taking $\tau = a + s(t-a)$ we get :

$$\begin{aligned} {}^{C}\mathcal{D}_a^{\alpha}(t-a)^p &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha} \int_0^1 (1-s)^{n-\alpha-1} s^{p-n} ds, \\ &= \frac{\Gamma(p+1)\beta(n-\alpha, p-n+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha}, \\ &- \frac{\Gamma(p+1)\Gamma(n-\alpha)\Gamma(p-n+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)\Gamma(p-\alpha+1)}(t-a)^{p-\alpha} m \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t-a)^{p-\alpha}. \end{aligned}$$

composition with the fractional integral

Theorem 1.5.1 *[41]*
let $\alpha > 0$, and f is a continuous function on $[a, +\infty)$ in $\mathbb{R}$ we have :

$$\mathcal{I}_a^{\alpha}\left({}^{C}\mathcal{D}_a^{\alpha} f(t)\right) = f(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!}. \tag{1.26}$$

Theorem 1.5.2 *[26]*
let $\alpha > 0$ and f be a continuous function on $[a, +\infty)$ in $\mathbb{R}$ we have :

$${}^{C}\mathcal{D}_a^{\alpha}\left(\mathcal{I}_a^{\alpha} f\right)(t) = f(t). \tag{1.27}$$

Remark 1.5.1
the caputo derivative operator can be considered as a left-inverse of the fractional integration operator, but it does not constitute a right-inverse.

Remark 1.5.2
the conclusion of theorem 1.5.1 indicates that differentiating a function f in the Caputo is equivalent to a fractional derivative of the remainder in the Taylor expansion of f.

Theorem 1.5.3
Si $\alpha = n \in \mathbb{N}$ we have :

$${}^{C}\mathcal{D}_a^{\alpha} f(t) = f^{(n)}(t). \tag{1.28}$$

that is to say :

$${}^{C}\mathcal{D}_a^{0} f(t) = f(t), {}^{C}\mathcal{D}_a^{1} f(t) = f^{(1)}, {}^{C}\mathcal{D}_a^{2} f(t) = f^{(2)}, \ldots, {}^{C}\mathcal{D}_a^{n} f(t) = f^{(n)}(t). \tag{1.29}$$

1.6 Relationship between the Riemann-Liouville Fractional Derivative and the Caputo Fractional Derivative

The following theorem establishes the connection between the caputo fractional and Riemann-Liouville fractional derivatives.

Theorem 1.6.1
let $\alpha \geq 0$ (with $m-1 \leq \alpha < n$ and $m \in \mathbb{N}^*$) if f has $m-1$ derivatives at and if ${}^C\mathcal{D}_a^\alpha f$ and $\mathcal{D}_a^\alpha f$ exist, then : for almost every $t \in [a, +\infty)$:

$$ {}^C\mathcal{D}_a^\alpha f(t) = {}^{RL}\mathcal{D}_a^\alpha f(t) - \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a). \tag{1.30}$$

Proof
we have :

$$\begin{aligned}
\mathcal{I}_a^\alpha f(t) &= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-s)^\alpha f'(s) ds, \\
&= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + \mathcal{I}_a^{\alpha+1} f'(t), \\
\mathcal{I}_a^\alpha f(t) &= \sum_{j=1}^{m-1} \frac{(t-a)^{\alpha+j}}{\Gamma(\alpha+1+j)} f(j)(a) + \mathcal{I}_a^{\alpha+n} f^n(t),
\end{aligned}$$

Setting $n = m$ and $\alpha = m - \alpha$ we find :

$$\mathcal{I}_a^{m-\alpha} f(t) = \mathcal{I}_a^{2m-\alpha} f^{(m)}(t) = \sum_{j=1}^{m-1} \frac{(t-a)^{m-\alpha+j}}{\Gamma(m-\alpha+1+j)} f^{(j)}(a),$$

then

$$\begin{aligned}
\frac{d^m}{dt^m}\left[\mathcal{I}_a^{m-\alpha} f(t)\right] &= \frac{d^m}{dt^m}\left[\mathcal{I}_a^{2m-\alpha} f(t)^{(m)} + \sum_{j=1}^{m-1} \frac{(t-a)^{m-\alpha+j}}{\Gamma(m-\alpha+1+j)} f^{(j)}(a)\right], \\
&= \frac{d^m}{dt^m}\left[\mathcal{I}_a^{2m-\alpha} f^{(m)}(t)\right] + \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a), \\
&= \mathcal{I}_a^{m-\alpha} f^{(m)}(t) + \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a),
\end{aligned}$$

Therefore

$$\mathcal{D}_a^\alpha f(t) = {}^{RL}\mathcal{D}_a^\alpha f(t) - \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a).$$

Corrolaire 1.6.1
for $\alpha > 0$, we deduce that if $f^{(k)}(a) = 0$for $k = 0, 1, 2, \ldots, n-1, (n = [\alpha] + 1)$ then we will have

$$\mathcal{D}_a^\alpha f(t) = {}^C\mathcal{D}_a^\alpha f(t). \tag{1.31}$$

1.7 General properties of fractional derivatives

Linearity

Proposition 1.7.1 *[10]*
let f, g be two continuous functions on $[a, b]$, Fractional differentiation is a linear operation,i.e., for any : $\forall\gamma, \lambda \in \mathbb{R}, \alpha > 0,$ we have

$$\mathcal{D}_a^\alpha[\lambda f(t) + \gamma g(t)] = \lambda\mathcal{D}_a^\alpha f(t) + \gamma\mathcal{D}_a^\alpha g(t). \tag{1.32}$$

Where $\mathcal{D}^\alpha$ denotes any sense of fractional derivative.

Example 1.7.1
- the linearity of fractional derivative in the sense of Grunwald-Letnikov :
let $\alpha, \beta \in \mathbb{C}$ we have :

$$\begin{aligned}
{}^G\mathcal{D}_a^\alpha[\lambda f(t) + \gamma g(t)] =& \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)[\lambda f(t-kh) + \gamma g(t-kh)],\\
=&\lambda\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)f(t-kh),\\
&+\gamma\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)g(t-kh),\\
=&\lambda{}^G\mathcal{D}_a^\alpha f(t) + \gamma{}^G\mathcal{D}_a^\alpha g(t),
\end{aligned}$$

- the linearity of fractional derivative in the sense of Riemann-Liouville :
Let $\alpha, \beta \in \mathbb{C}$ we have :

$$\begin{aligned}
{}^{RL}\mathcal{D}_a^\alpha[\lambda f(t) + \gamma g(t)] =&\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{[\lambda f(s) + \gamma g(s)]}{(t-s)^{\alpha-n+1}},\\
=&\frac{\lambda}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{f(s)}{(t-s)^{\alpha-n+1}},\\
&+\frac{\gamma}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{g(s)}{(t-s)^{\alpha-n+1}}m\\
=&\lambda{}^{RL}\mathcal{D}_a^\alpha f(t) + \gamma{}^{RL}\mathcal{D}_a^\alpha g(t).
\end{aligned}$$

Leibniz Rule

for an integer n we have

$$\frac{d^n}{dt^n}(f(t)g(t)) = \sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)f^k(t)g^{n-1}(t). \tag{1.33}$$

the generalization of this formula gives us

$$\mathcal{D}^\alpha(f(t)g(t)) = \sum_{k=0}^{n}\left(\begin{array}{c}\alpha\\k\end{array}\right)f^k(t)\mathcal{D}^{\alpha-k}g(t) + R_n^\alpha(t), \tag{1.34}$$

where $n \geq \alpha + 1$ and

$$R_{\mathrm{n}}^\alpha(t) = \frac{1}{n!\Gamma(-\alpha)}\int_a^t(t-\tau)^{-\alpha-1}g(\tau)d\tau\int_\tau^t(\tau-\xi)^n f^{(n+1)}(\xi)d\xi, \tag{1.35}$$

with $\lim_{n\to\infty} R_n^\alpha(t) = 0$
if f and g are continuous on $[a,t]$ include all their derivatives ,the formula becomes :

$$\mathcal{D}^\alpha(f(t)g(t)) = \sum_{k=0}^{n} \left(\begin{array}{c} \alpha \\ k \end{array} \right) f^k(t)\mathcal{D}^{\alpha-k}g(t) + R_n^\alpha(t). \tag{1.36}$$

$\mathcal{D}^\alpha$ is the fractional derivative in the sense of Grunwald-Letnikov and in the sense of Riemann-Liouville.

Definition 1.7.1 *[25]*
let $\alpha > 0$, $\alpha \notin \mathbb{N}, n = [\alpha] + 1$ and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, then :

$$^{RL}\mathcal{D}^\alpha y(t) = f(t, y(t)), \tag{1.37}$$

is called a Riemann-Liouville fractional differential equation.
Similarly,

$$^{C}\mathcal{D}^\alpha y(t) = f(t, y(t)), \tag{1.38}$$

is called a caputo fractional differential equation.

1.8 Riemann-Liouville fractional differential equation

Starting with the homogeneous Riemann-Liouville type equation.

Lemma 1.8.1 *[27]*
let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L(0,1)$, then the Riemann-Liouville fractional differential equation is :

$$\mathcal{D}_{0^+}^\alpha u(t) = 0, \quad 0 < t < 1, \tag{1.39}$$

admits a unique solution

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}.$$

where $C_m \in \mathbb{R}$, with $m = 1, 2, ..., n$.

Lemma 1.8.2 *[27]*
Suppose that

$$u \in C(0,1) \cap L(0,1), \quad \text{and} \quad \mathcal{D}_{0^+}^\alpha u \in C(0,1) \cap L(0,1).$$

Then :

$$\mathcal{I}_{0^+}^\alpha \mathcal{D}_{0^+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}, \tag{1.40}$$

where $C_m \in \mathbb{R}$, with $m = 1, 2, ..., n$.

Proof
Let $\alpha > 0$. For all $u \in C(0,1) \cap L(0,1)$, we have :

$$\begin{aligned} \mathcal{I}_{0^+}^\alpha \mathcal{D}_{0^+}^\alpha u(t) &= u(t) - \sum_{k=1}^{n} \frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{n-k})(0)}{\Gamma(\alpha - k + 1)} t^{\alpha-k}, \\ &= u(t) - \left[\frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-1)})(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-2)})(0)}{\Gamma(\alpha-1)} t^{\alpha-2} + ... + \frac{(\mathcal{I}_{0^+}^{n-\alpha} u)(0)}{\Gamma(\alpha-n+1)} t^{\alpha-n} \right], \end{aligned}$$

We define $C_m = -\dfrac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-m)})(0)}{\Gamma(\alpha - m + 1)} \in \mathbb{R}$, for each $m = 1, 2, ..., n$, we find the equality (1.40).

Lemma 1.8.3
Let $1 < \alpha \leq 2,$ *and* $y \in C([0,1])$.
Then the unique solution to the boundary value problem

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.41}$$

is given by :

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

such as :

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if} \quad 0 \leq s \leq t \leq 1 \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & \text{if} \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{1.42}$$

Proof
Applying $\mathcal{I}^{\alpha}_{0+}$, to equation 1.41, we obtain :

$$\mathcal{I}^{\alpha}_{0+}[\mathcal{D}^{\alpha}_{0+}u(t) + y(t)] = 0 \leftrightarrow \mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t) + \mathcal{I}^{\alpha}_{0+}y(t) = 0.$$

According to Lemma 1.8.2, for $1 < \alpha \leq 2$ $(n = [\alpha] + 1 = 2)$, we have :

$$\mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}, \ C_1, C_2 \in \mathbb{R}$$

Thus,

$$u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \mathcal{I}^{\alpha}_{0+}y(t) = 0.$$

which implies

$$u(t) = -\mathcal{I}^{\alpha}_{0+}y(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2},$$

Therefore, the general solution of equation 1.41 is given by :

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - C_1 t^{\alpha-1} - C_2 t^{\alpha-2}. \tag{1.43}$$

The boundary conditions imply that :

$$\begin{cases} u(0) = 0 \Rightarrow 0 = -0 - 0 - \lim_{t\to 0} C_2 t^{\alpha-2} & \Rightarrow C_2 = 0, \\ u(1) = 0 \Rightarrow 0 = -\dfrac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds - C_1 & \Rightarrow C_1 = -\dfrac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds. \end{cases}$$

The integro-differential equation 1.43 is equivalent to :

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds, \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_t^1 (1-s)^{\alpha-1} y(s)ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_t^1 (1-s)^{\alpha-1} y(s)ds, \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1} - (t-s)^{(\alpha-1)}}{\Gamma(\alpha)} y(s)ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} y(s)ds, \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

The proof is complete.

1.9 Caputo fractional differential equation

Starting with the homogeneous Caputo-type equation.

Lemma 1.9.1 *[30]*
Let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L(0,1)$, then the Caputo-type fractional differential equation is :

$$^{C}\mathcal{D}^{\alpha}_{0+}u(t) = 0, \tag{1.44}$$

admits a unique solution

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}.$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2, \ldots, n-1$.

Proof
let $\alpha > 0$, we have :

$$^{C}\mathcal{D}^{\alpha}_{0+}t^m = 0, \quad \text{for} \quad m = 0, 1, 2, \ldots, n-1.$$

So the fractional differential equation 1.44 admits a particular solution, such as

$$u(t) = C_m t^m, \quad \text{for} \quad m = 0, 1, 2, \ldots, n-1. \tag{1.45}$$

where $C_m \in \mathbb{R}$. The general solution of 1.44, given as a sum of particular solutions 1.45, i.e.,

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}.$$

Lemma 1.9.2 *[30]*
Assume that $u \in C^n([0,1])$. then :

$$\mathcal{I}^{\alpha}_{0+} + {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + ... + C_{n-1}t^{n-1}. \tag{1.46}$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2, \ldots, n-1$.

Proof
Let $\alpha > 0$. for all $u \in C^n([0,1])$ we have

$$\begin{aligned}\mathcal{I}^{\alpha}_{0+}{}^{C}\mathcal{D}^{\alpha}_{0+}u(t) &= u(t) - \sum_{k=0}^{n-1}\frac{u^k(0)}{k!}t^k \\ &= u(t) - \left[u(0) + u'(0)t + \frac{u''(0)}{2}t^2 + ... + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}\right]\end{aligned}$$

We pose $C_m = -\dfrac{u^{(m)}(0)}{m!} \in \mathbb{R}$, for each $m = 0, 1, 2, ..., n-1$,We easily find the equality 1.46

Lemma 1.9.3
let $1 < \alpha \leq 2$, and $y \in C([0,1])$.
Then the unique solution to the boundary value problem is :

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) = y(t), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \ u(1) + u'(1) = 0 \end{cases}, \tag{1.47}$$

is given by :

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

such as :

$$G(t,s) = \begin{cases} \frac{(1-t)(1-s)^{\alpha-1} + (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if} \quad 0 \leq s \leq t \leq 1 \\ \frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if} \quad 0 \leq t \leq s \leq 1 \end{cases} \tag{1.48}$$

Proof
Applying $\mathcal{I}^{\alpha}_{0^+}$, to equation 1.47 we obtain :

$$\mathcal{I}^{\alpha}_{0+}\left[{}^{C}\mathcal{D}^{\alpha}_{0+}u(t)-y(t)\right]=0 \Leftrightarrow \mathcal{I}^{\alpha}_{0+}\,{}^{C}\mathcal{D}^{\alpha}_{0+}u(t)-\mathcal{I}^{\alpha}_{0+}y(t)=0.$$

According to Lemma 1.9.2, for $1<\alpha\leq 2$ ($n=[\alpha]+1=2$),we have :

$$\mathcal{I}^{\alpha}_{0+}\,{}^{C}\mathcal{D}^{\alpha}_{0+}u(t)=u(t)+C_0+C_1t,\ \ C_0,C_1,C_2\in\mathbb{R},$$

thus,

$$u(t)+C_0+C_1t-\mathcal{I}^{\alpha}_{0+}y(t)=0,$$

which implies

$$u(t)=\mathcal{I}^{\alpha}_{0+}y(t)-C_0-C_1t,$$

Therefore, the general solution of equation 1.47is given by :

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds-C_0-C_1t. \tag{1.49}$$

The boundary conditions imply that :

$$\begin{cases} u(0)+u'(0)=0 & \Rightarrow C_0+C_1=0\\ u(1)+u'(1)=0 & \Rightarrow C_0+2C_1=(\mathcal{I}^{\alpha}_{0+}y)(1)+(\mathcal{I}^{\alpha}_{0+}y)'(1)\end{cases}$$

thus

$$\begin{cases} C_0=-\,(\mathcal{I}^{\alpha}_{0+}y)(1)-(\mathcal{I}^{\alpha}_{0+}y)'(1)\\ \quad =-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1(1-s)^{\alpha-1}y(s)ds-\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^1(1-s)^{\alpha-2}y(s)ds\\ C_1=(\mathcal{I}^{\alpha}_{0+}y)(1)-(\mathcal{I}^{\alpha}_{0+}y)'(1)\\ \quad =\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1(1-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^1(1-s)^{\alpha-2}y(s)ds\end{cases}$$

The integro-differential equation 1.47 is equivalent to :

$$\begin{aligned} u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds-\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\ &-\frac{t}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds\\ =&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{(1-t)}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}y(s)ds\\ &+\frac{(1-t)}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}y(s)ds+\frac{(1-t)}{\Gamma(\alpha-1)}\int_0^t(1-s)^{\alpha-2}y(s)ds\\ &+\frac{(1-t)}{\Gamma(\alpha-1)}\int_t^1(1-s)^{\alpha-2}y(s)ds\\ =&\int_0^t\left[\frac{(t-s)^{\alpha-1}+(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]y(s)ds\\ &+\int_t^1\left[\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]y(s)ds\\ =&\int_0^1G(t,s)y(s)ds.\end{aligned}$$

The proof is complete.

1.10 Existence and uniqueness of the solution

This section constitutes a preliminary part in which fundamental concepts and results of the theory of functional analysis are recalled (Banach contraction principle, equicontinuity, Schauder's theorem, Arzela-Ascoli theorem,...). Subsequently, the question of existence and uniqueness of the solution for the boundary value problem of fractional order differential equation will be addressed.

1.10.1 Fixed point theory

In this part, we have revisited some definitions and theorems from [34]

1.10.2 Banach fixed point theorem

Let us consider the following initial value problem

$$u' = f(t, u), \qquad u(t_0) = u_0. \tag{1.1}$$

By applying the integral operator, we obtain the equivalent integral equation :

$$u(t) = \int_{t_0}^{t} f(s, u(s))ds + u_0, \tag{1.2}$$

and let $\{u_n\}$ be a sequence of functions, with

$$u_1(t) = \int_{t_0}^{t} f(s, u_0)ds + u_0, \qquad u(t_0) = u_0, \tag{1.3}$$

and, in general,

$$u_{n+1}(t) = \int_{t_0}^{t} f(s, u_n(s))ds + u_0. \tag{1.4}$$

This is called Picard's method of successive approximations. One can show that converges uniformly on some interval $|t - t_0| \leq k$ to some continuous function, say $u(t)$. Taking the limit in the equation defining $u_{n+1}(t)$, we pass the limit through the integral and have

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds,$$

so that $u(t_0) = u_0$ and, upon differentiation, we obtain $u^{'}(t) = f(t, u(t))$. Thus, $u(t)$ is a solution of the initial value problem. Banach realized that this was actually a fixed point theorem with wide application. Let us define an operator B on a complete metric space $C([t_0, t_0 + k], \mathbb{R})$ with the supremum norm $\| \cdot \|$ by $u \in C$ as

$$(Bu)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds,$$

then a fixed point of B, say $B\phi = \phi$, is a solution of the initial value problem.

Definition 1.10.1 *[10]*
Let (E, d) be a complete metric space and $B : E \longrightarrow E$. The operator B is a contraction if there is a $\lambda \in [0, 1)$ such that $u, v \in E$ imply

$$d(Bu, Bv) \leq \lambda d(u, v).$$

Theorem 1.10.1 ([10] (Contraction Mapping Principle))
Let (E, d) be a complete metric space and $B : E \longrightarrow E$ a contraction operator. Then there is a unique $u \in E$ with $Bu = u$. Furthermore, if $v \in E$ and if $\{v_n\}$ is defined inductively by $v_1 = Bv$ and $v_{n+1} = Bv_n$, then $v_n \longrightarrow u$, the unique fixed point. In particular, the equation $Bu = u$ has one and only one solution.

Theorem 1.10.2 *[10]*
Let (E, d) be a complete metric space and suppose that $B : E \longrightarrow E$ such that B^m is a contraction for some fixed positive integer m. Then B has a fixed point in E.

Theorem 1.10.3 *[10]*
Let (E, d) be a compact metric space,

$$B : E \longrightarrow E \text{ and } d(Bu, Bv) < d(u, v), \text{ for } u \neq v. \tag{1.5}$$

Then B has a unique fixed point.

Theorem 1.10.4 *[10]*
If (E, d) is a complete nonempty metric space and $B : E \longrightarrow E$ is a contraction operator with fixed point u, then for any $v \in E$ we have :

(a) $d(u, v) \leq \frac{d(Bv,v)}{(1-\lambda)}$,

(b) $d(B^n v, u) \leq \frac{\lambda^n d(Bv,v)}{(1-\lambda)}$.

Theorem 1.10.5 *(Arzelà Ascoli)[2]*
Let A be a subset of $C(J; E)$; A is relatively compact in $C(J; E)$ if and only if the following conditions are satisfied :

1. *The set A is bounded, i.e., there exists a constant $k > 0$ such that :*

$$||f|| \leq k \text{ for every } x \in J \text{ and } f \in A.$$

2. *The set A is equicontinuous, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| \leq \epsilon \text{ for every } t_1, t_2 \in J \text{ and } f \in A.$$

3. *for every $x \in J$ the set $\{f(x), f \in A\} \subset E$is relatively compact.*

1.11 Cauchy problem fractional order differential equation

The existence and uniqueness of the solution to a Cauchy problem for fractional-order differential equations (using the Caputo derivative) will be studied, where the problem is given in the following form :

$$\begin{cases} {}^C\mathcal{D}^\alpha y(t) = f(t, y(t)) & t \in [0, T],\ 0 < \alpha < 1 \\ y(0) = y_0,\ y_0 \in \mathbb{R} \end{cases} \tag{1.50}$$

tell that $f : [0, T] \in \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 1.11.1 *[31]*
Let $0 < \alpha < 1$ and let $h : [0, T] \to \mathbb{R}$ be a continuous function. A function y is a solution to the Cauchy problem

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx. \tag{1.51}$$

Proof
We apply operator $\mathcal{I}^\alpha$ to equation 1.50 and we find

$$\begin{aligned} \mathcal{I}^{\alpha C}\mathcal{D}^{\alpha C}\mathcal{D}^\alpha y = \mathcal{I}^\alpha f(t) &\Rightarrow y(t) + c_0 = \mathcal{I}^\alpha h(t) \\ &\Rightarrow y(t) = \mathcal{I}^\alpha h(t) - c_0 \end{aligned}$$

The initial condition gives

$$y(0) = (\mathcal{I}^\alpha h)(0) - c_0 = -c_0 \Rightarrow c_0 = -y_0.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{I}^\alpha h(t) - (y_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx + y_0. \end{aligned}$$

in return

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx \\ &= \mathcal{I}^\alpha h(t) + y_0. \end{aligned}$$

we apply ${}^C\mathcal{D}^\alpha$ to the integral equation 1.51.

$$\begin{aligned} {}^C\mathcal{D}^\alpha y(t) &= {}^C\mathcal{D}^\alpha (\mathcal{I}^\alpha h)(t) + {}^C\mathcal{D}^\alpha (y_0) \\ &= h(t). \end{aligned}$$

Thus, it remains to verify that $y(0) = y_0$,

$$\begin{aligned} y(0) &= \mathcal{I}^\alpha h(0) + y_0 = 0 + y_0 \\ &= y_0. \end{aligned}$$

Then y is a solution to the problem 1.51.

Theorem 1.11.1 *[10]*
Let $0 < \alpha < 1$ and $f : [0;T] \times \mathbb{R} \to \mathbb{R}$ and satisfies the following Lipschitz condition :

$$|f(t,y) - f(t,z)| \leq k|y - z|, \ \forall t \in [0,T], \ \text{and } y, z \in \mathbb{R}.$$

$$\frac{kT^\alpha}{\Gamma(\alpha+1)} < 1,$$

There exists a unique solution to the Cauchy problem 1.50.

Proof
We use the Banach fixed point theorem 1.10.1.
We transform problem 1.50 into a fixed point problem (Lemma 1.11.1), considering the operator

$$\begin{aligned} F : C([0,T],\mathbb{R}) &\to C([0,T],\mathbb{R}) \\ y &\to F(y)(t) = y_0 + \frac{1}{\Gamma} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) dx. \end{aligned}$$

where $C([0,T],\mathbb{R})$ is the Banach space of continuous functions y defined on $[0,T]$ in $\mathbb{R}$, equipped with the norm

$$||y|| = \sup_{t \in [0,T]} |y(t)|.$$

It is clear that the fixed points of the operator F are the solutions to problem 1.50. F is well defined, indeed : if $y(t) \in C([0,T],\mathbb{R})$, Then $Fy(t) \in C([0,T],\mathbb{R})$.
To show that F has a fixed point, it suffices to demonstrate that F is a contraction ; indeed, if $y_1, y_2 \in C([0,T],\mathbb{R}), t \in [0,T]$ By using the Lipschitz condition, we obtain :

$$\begin{aligned}
|Fy_1 - Fy_2| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (f(s, y_1(s))) - (f(s, y_2(s)))(t-s)^{\alpha-1} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (|f(s, y_1(s))) - (f(s, y_2(s)))|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t |y_1(s) - y_2(s)|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} ||y_1 - y_2|| \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{kT^\alpha}{\Gamma(\alpha+1)} ||y_1 - y_2||
\end{aligned}$$

It states that due to the property $\dfrac{kT^\alpha}{\Gamma(\alpha+1)} < 1$, F is a contraction, and according to Banach's Fixed Point Theorem, F has a unique fixed point, which is the solution to problem 1.50.

CHAPITRE 2

STOCHASTIC CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS

Introduction

Probability theory constitutes the foundational building block of the mathematical and statistical sciences, as it enables us to handle phenomena characterized by uncertainty and ambiguity. Initially, the theory emerged in the 17th century to study games of chance and gambling. However, it quickly evolved into a powerful tool with significant roles across various fields, including the natural sciences, engineering, economics, and social sciences.

Probability provides a measure of the likelihood that a specific event will occur when performing a given experiment. Rather than offering definitive answers about expected outcomes, probability equips us with tools to assess the plausibility of each possible result. For example, when tossing a fair coin, the probability of obtaining heads is 50%, as is the probability of obtaining tails.

With the advancement of sciences and their applications, and the increasing complexity of phenomena under study, it became necessary to develop analytical tools capable of offering a deeper understanding of complex and uncertain systems, particularly in representing systems that evolve randomly over time. This led to the emergence of stochastic process theory as a natural extension of classical probability theory.

Stochastic processes are families of random variables indexed by time or by a particular space, focusing on the study of sequences of events that evolve in a random manner. These processes consist of collections of random variables, where the current state of the process depends on the previous state and certain probabilities governing future changes. A classical example of stochastic processes is the Brownian motion or Wiener process, and the study of such processes helps in predicting the future behavior of dynamic systems.

Famous examples of stochastic processes include the motion of particles suspended in a fluid (Brownian motion) and the fluctuation of stock prices in financial markets.

In this chapter, we focus on establishing the theoretical foundations necessary for understanding and applying stochastic differential equations (SDE). This study will provide a comprehensive review of basic probability concepts, including probability spaces, random variables, and mathematical expectation. We will then move on to the study of stochastic processes, covering concepts such as filtration, martingales, and Brownian motion, which serve as the fundamental building blocks for constructing stochastic models.

Subsequently, we will discuss stochastic integration, with particular emphasis on the Itô integral, which is an essential tool for solving stochastic differential equations. We will also examine Itô's formula and its important properties, such as the isometry property.

In the final part of the chapter, we will present stochastic differential equations and their applications, offering prominent practical examples such as the Ornstein–Uhlenbeck process, the geometric Brownian motion widely used in financial modeling, and the Black–Scholes model, which revolutionized the theory of option pricing.

2.1 probability Basics

2.1.1 probability space

In this section, we have revisited some definitions and theorems from [35].

Definition 2.1.1
A sigma-algebra (or σ-algebra) the probability space Ω is defined as a family $\mathcal{F}$ of subsets of Ω (called events) satisfying the following properties :

1. *the empty set $\varnothing$ belongs to$\mathcal{F}$.*
2. *if an event A is in $\mathcal{F}$, then its complement A^c is also in $\mathcal{F}$.*
3. *if $(A_n)_{n=1}^{\infty}$ is a sequence of events belonging to $\mathcal{F}$, then the union of all these events, $\bigcup_{n=1}^{\infty} A_n$, is also in $\mathcal{F}$.*

Definition 2.1.2
the probability measure on the probability space $(\Omega, \mathcal{F})$ is defined as a function $\mathbb{P}$ de $\mathcal{F}$to the interval $[0, 1]$, satisfying the following conditions :

1. *the probability of the certain event, $\mathbb{P}(\Omega)$, is equal to 1.*
2. *for any sequence of events A_n belonging to $\mathcal{F}$ and pairwise disjoint, the probability of the union of these events, $\mathbb{P}\left(\bigcup_{n=0}^{\infty} A_n\right)$, is equal to the infinite sum of individual probabilities, $\sum_{n=0}^{\infty} \mathbb{P}(A_n)$.*

Definition 2.1.3
A probability space is defined as a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where : - Ω is a set, - $\mathcal{F}$ is a sigma-algebra (or σ-tribe) on Ω, - $\mathbb{P}$ is a probability measure defined on $(\Omega, \mathcal{F})$.

2.1.2 Random variable

Definition 2.1.4
let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. A random variable on $(\Omega, \mathcal{F}, \mathrm{P})$, is any function $X : \Omega \to \mathbb{R}$ such that :

$$\{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\} \in \mathcal{F}, \forall B \in \mathbb{B}(\mathbb{R}) \tag{2.1}$$

2.1.3 Expectation of a Random variable

Definition 2.1.5 *(cumulative distribution function)[35]*
the cumulative distribution function of a random variable X defined on $(\Omega, \mathcal{F}, \mathrm{P})$ is the function $F_{\boldsymbol{X}}(x)$ defined on $\mathbb{R}$ by :

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) \tag{2.2}$$

Definition 2.1.6 *[35]*
If the cumulative distribution function $F_X(x)$ is differentiable, the derivative of this function, denoted $f_X(x)$, is called the probability density function of the random variable X :

$$\frac{\partial F_X(x)}{\partial x} = f_X(x) \tag{2.3}$$

Definition 2.1.7 *[35]*
the mathematical expectation or mean, denoted $\mathbb{E}(X)$,*is defined as follows :*

1. ***Discrete case**, when the random variable X takes discrete values (i.e., integers) in a given interval, whether bounded or unbounded.*

$$\mathbb{E}(X) = \sum_{K=1}^{\infty} x_K \mathbb{P}(X = x_K) \tag{2.4}$$

2. ***continuous case** Si X is a real-valued random variable (absolutely continuous)*

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x) dx \tag{2.5}$$

Definition 2.1.8
let X and Y defined :

$$\begin{aligned} \operatorname{Var}(X,Y) &= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) \\ &= \mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2 \geq 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \operatorname{Cov}(X,Y) &= \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \end{aligned} \tag{2.7}$$

Conditional Expectation

1. **Conditioning with respect to an event** $B \in \mathcal{F}$ **:**

   let $A \in \mathcal{F}$:

$$\mathbb{P}(A/B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \tag{2.8}$$

   let X be an integrable random variable defined $\mathbb{E}(|X|) < \infty$) :

$$\mathbb{E}(X/B) = \frac{\mathbb{P}(X 1_B)}{\mathbb{P}(B)} \text{ si } \mathbb{P}(B) \neq 0 \tag{2.9}$$

2. **Conditioning for a random variable (taking values in the countable set) :**
   let X be an integrable random variable :

$$\mathbb{E}(X/Y) = \psi(Y) \tag{2.10}$$

   where

$$\psi(y) = \mathbb{E}(X/Y = y), y \in D \tag{2.11}$$

3. **Conditioning with respect to a sigma-algebra** $\mathcal{F}_1$
   let X be an integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_1$ be a sub-sigma-algebra of $\mathcal{F}$

Definition 2.1.9
the conditional expectation of X with respect to $\mathcal{F}_1$*. denoted* $\mathbb{E}(X/\mathcal{F}_1)$ *is any random variable Z such that* $\mathbb{E}(|z|) < \infty$ *that satisfies :*
i) Z is a random variable $\mathcal{F}_1$*-measurable.*
ii) $\mathbb{E}(XU) = \mathbb{E}(ZU)$,*for all bounded* $\forall U$ *measurable random variables* $\mathcal{F}_1$.

Proposition 2.1.1
let X and Y be to integrable random variables and $\mathcal{F}_1 \subset \mathcal{F}$*, then :*

1. $\mathbb{E}(aX + Y/\mathcal{F}_1) = a\mathbb{E}(X/\mathcal{F}_1) + \mathbb{E}(Y/\mathcal{F}_1)$.

2. *If $X \leq Y$ then $\mathbb{E}(X/\mathcal{F}_1) \leq \mathbb{E}(Y/\mathcal{F}_1)$.*
3. *$\mathbb{E}(\mathbb{E}(X/\mathcal{F}_1)) = \mathbb{E}(X)$ (taking $A = \Omega$ in the definition).*
4. *If X is independent of $\mathcal{F}_1$ then $\mathbb{E}(X/\mathcal{F}_1) = \mathbb{E}(X)$, meaning that in the absence of any information about X,the best estimate of X is its expectation.*
5. *If X is $\mathcal{F}_1$ measurable, then $\mathbb{E}(X/\mathcal{F}_1) = X$. this expresses the fact that $\mathcal{F}_1$ already contains all the information about X.*
6. *If X is $\mathcal{F}_1-$ measurable and $\mathbb{E}(|XY|) < +\infty$, then $\mathbb{E}(XY/\mathcal{F}_1) = X\mathbb{E}(Y/\mathcal{F}_1)$.*
7. *If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X/\mathcal{F}_2)/\mathcal{F}_1) = \mathbb{E}(X/\mathcal{F}_1)$.*

2.1.4 Convergence of sequences of random variables

let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables and X another random variable, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$. there are several ways to the sequence (X_n) to X.

— **Convergence in probability** :

$$X_n \xrightarrow[n\to\infty]{\overset{P}{\longrightarrow}} X \quad \text{si} \quad : \forall \epsilon > 0, \lim_{n\to\infty} \mathbb{P}(\omega \in \Omega : X_n(\omega) - X(\omega) > \epsilon) = 0 \tag{2.12}$$

— **Almost sure convergence** :

$$X_n \xrightarrow[n\to\infty]{\longrightarrow} X \text{ p.s si } : \mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1 \tag{2.13}$$

— **Convergence in mean** (or convergence in L^1) :

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^1\right) = 0 \tag{2.14}$$

— **Quadratic convergence** (or convergence in L^2) :

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^2\right) = 0 \tag{2.15}$$

2.2 Filtration and Stochastic Processes

2.2.1 Filtration

Definition 2.2.1 *[28]*
A filtration the context of a probability space $(\Omega, B, \mathbb{P})$, is defined as an increasing sequence $(\mathcal{F}_n)_{n\geq 0}$ sub-sigma algebras of B, i.e., $\mathcal{F}_t$ is contained in $\mathcal{F}_s$ for all $t \leq s$.

Definition 2.2.2
Given a measurable space $(\Omega, \mathcal{F})$,a real-valued random variable X is said to be a measurable function from $(\Omega, \mathcal{F})$ to $\mathbb{R}$ if :

$$X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathbb{B}(\mathbb{R}) \tag{2.16}$$

Definition 2.2.3
the sigma algebra generated by a family of random variables $(X_t, t \in [0, T])$ is the smallest sigma algebra containing the sets $X_t^{-1}(B)$ for all $t \in [0, T]$ and $B \in \mathbb{B}(\mathbb{R})$. It is denoted as $\sigma(X_t, t \leq T)$

Definition 2.2.4
let $(\mathcal{F}_t)_{t\geq 0}$ is said to be right continuous if :

$$\mathcal{F}_t = \bigcap_{e>0} \mathcal{F}_{t+e} \forall t \geq 0 \tag{2.17}$$

It is left-continuous if :

$$\mathcal{F}_t = \sigma\left(\bigcup_{0<s<t} \mathcal{F}_s\right) \forall t > 0 \tag{2.18}$$

the same sequence of filtration is termed complete with respect to a probability measure $\mathbb{P}$ when $\mathcal{F}_0$ includes all subsets of $\mathcal{F}$ with probability measure zero according to $\mathbb{P}$.

Definition 2.2.5
A filtrated probability space,denoted as $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$,is the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the compatible filtration $\{\mathcal{F}_t, t \geq 0\}$.

Definition 2.2.6
A filtration $(\mathcal{F}t)\, t \geq 0$ is said to satisfy the usual conditions if it is both right-continuous and complete.

2.2.2 stochastic process

this section, we explore some fundamental concepts related to stochastic processes and begin by defining them.[35]

Definition 2.2.7
let T be anon-empty subset of $\mathbb{R}$. A stochastic process $(X_t)_{t\in T}$ in $\mathbb{R}^n$ is a family of random variables taking values in $\mathbb{R}^n$ indexed by T. for fixed $\omega \in \Omega$ $t \longmapsto X_t(\omega)$ is called trajectory.

Definition 2.2.8 *(natural filtration)*
the natural filtration of a stochastic process $X = \{X_t, t \geq 0\}$, denote by F^X, is the increasing family of generated sigma-algebras generatedby$\{X(s), 0 \leq s \leq t\}$. $t \geq 0$ that is :

$$F^X = \left\{F_t^X = \sigma(\{X(s), 0 \leq s \leq t\}), t \geq 0\right\} \tag{2.19}$$

Definition 2.2.9
A process $X = (X)_{t\geq 0}$ is measurable if the mapping :

$$X : \mathbb{R} \times \Omega \to \mathbb{R}^n$$
$$(t, \omega) \mapsto X_t(\omega)$$

is measurable with respect $\mathbb{B}(\mathbb{R}^+) \otimes \mathcal{F}$ and $\mathbb{B}(\mathbb{R}^n)$

Definition 2.2.10
A process $(X_t)_{t\in T}$ is said to be continuous if for almost every $w \in \Omega, t \to X_t(w)$ is continuous (i.e., the trajectories are continuous).

Definition 2.2.11
Let (X_t) be a process and $(\mathcal{F}t)$ a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $X = (X)t \geq 0$ is adapted to the filtration $(\mathcal{F}t)\, t \geq 0$ if, for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

Definition 2.2.12
Let $X = (X)t \in T$ be a stochastic process. The finite-dimensional laws of the process X are the laws of vectors of the type $(Xt_1, \ldots, X_{t_n})$ where $n \geq 1$ and $t_1, \ldots, t_n \in T$. Two processes $(X)t \geq 0$ and $(Y)t \geq 0$ are said to have the same law if they have the same finite-dimensional laws.

Definition 2.2.13
The process $X = (X)t \geq 0$ has independent increments if, for all $n \geq 1$ and for all $t_1 < t_2 < \ldots < t_n \in T$, the vectors $\left(Xt_1, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}\right)$ are independent.

Definition 2.2.14
A progressive process $X_t, t \in T$ (with respect to $\mathcal{F}$) is a process such that, for all $t \in T$, the function $(s, \omega) \in [0, t] \times \Omega \mapsto X_s(\omega)$ is measurable.

Definition 2.2.15
For all $p \geq 1$ and a stochastic process $X = (X_t)\, t \geq 0$, X is said to be bounded in L^p if its L^p norm is finite, that is, if $\sup t \geq 0 \mathbb{E}\left[|X_t|^p\right] < \infty$.

Definition 2.2.16
Given a filtration $(\mathcal{F}t)$ and a function $T : \Omega \to \mathbb{R}+ \cup \infty$, T is said to be a stopping time with respect to $(\mathcal{F}t)\, t \geq 0$ if, for all $t \in \mathbb{R}_+$, the event $T \leq t$ is measurable by $\mathcal{F}_t$.

Theorem 2.2.1
Let $(X_t, t \geq 0)$ be an adapted process with continuous trajectories, and let T be a stopping time. Then, we have the following equality :

$$\int_0^T \mathbb{E}\,|X_t|\, dt = \mathbb{E}\left(\int_0^T |X_t|\, dt\right)$$

Moreover, if this quantity is finite, we also have :

$$\int_0^T \mathbb{E}X_t dt = \mathbb{E}\left(\int_0^T X_t dt\right)$$

Definition 2.2.17
Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}t)\, t \geq 0, \mathbb{P})$, an adapted and integrable process $X = (X_t)_{t \geq 0}$ is :

Martingale

Definition 2.2.18

- *A martingale if, for all $0 \leq s \leq t$, $\mathbb{E}\left(X_t/\mathcal{F}_s\right) = X_s$.*
- *A supermartingale if, for all $0 \leq s \leq t$, $\mathbb{E}\left(X_t/\mathcal{F}_s\right) \leq X_s$.*
- *A submartingale if, for all $0 \leq s \leq t$, $\mathbb{E}\left(X_t/\mathcal{F}_s\right) \geq X_s$.*

Proposition 2.2.1

- *Any continuous martingale is a local martingale.*
- *A positive local martingale is a supermartingale.*
- *A bounded local martingale is a martingale.*

Some inequalities

Theorem 2.2.2 *[33] (Hölder's inequality)*
If $X \in L^q, Y \in L^p$, such that $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$, then :

$$\mathbb{E}[|XY|] \leq \mathbb{E}\left[|X|^q\right]^{\frac{1}{q}} \mathbb{E}\left[|Y|^p\right]^{\frac{1}{p}}$$

Theorem 2.2.3 *(Cauchy-Schwarz inequality)*
Let X and Y be two square-integrable random variables. Then :

1. *XY is integrable.*
2. $$(\mathbb{E}(|XY|))^2 \leq \mathbb{E}\left(X^2\right) \mathbb{E}\left(Y^2\right)$$

Theorem 2.2.4 *(Doob's inequality)*
Let $(M_n, n \in N)$ be a real square-integrable martingale. Then :

$$\mathbb{E}\left[\max_{0\leq k\leq n} M_k^2\right] \leq 4\mathbb{E}\left[M_n^2\right]$$

In particular,

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} M_n^2\right] \leq 4\sup_{n\in\mathbb{N}}\mathbb{E}\left[M_n^2\right]$$

2.3 Brownian motion

2.3.1 Gaussian vector

In all that follows, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space.[35]

Definition 2.3.1
A random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Gaussian or normal random variable with parameters $(m, \sigma^2), \left(m \in \mathbb{R}, \sigma \in \mathbb{R}_+^*\right)$ if its density function f_X is given by :

$$f_X = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)}$$

In this case, its law $\mathbb{P}_X$ is given by

$$\forall A \in \mathbb{B}(\mathbb{R}) \quad \mathbb{P}_X(A) = \int_{\mathcal{F}} f_X(x)dx$$

And it is noted

$$X \sim \mathcal{N}\left(m, \sigma^2\right)$$

If $m = 0$, the vector X is said to be centered.

Remark 2.3.1
When the standard deviation σ is zero, the random variable X is constant, meaning that X is almost surely equal to the mean m, i.e., $\mathbb{P}$.

Proposition 2.3.1
A random variable X following the normal distribution $\mathcal{N}(m, \sigma^2)$ has :
- *Expected value : $\mathbb{E}[X] = m$.*
- *Variance : $\mathrm{Var}(X) = \sigma^2$.*
- $\mathrm{Cov}(X_s, X_t) = \min(s,t) \quad \forall 0 \leq s, t < T$.

Definition 2.3.2
$X = (X_1, X_2, \ldots, X_n)$ is a Gaussian random vector if all linear combinations of its components are Gaussian, that is, for any choice of coefficients $a_1, \ldots, a_n \in \mathbb{R}$, the random variable $\sum_{i=1}^n a_i X_i$ is Gaussian.

Definition 2.3.3
A process $X = (X_t)_{t\in T}$ is a Gaussian process if all its finite-dimensional distributions are Gaussian, i.e., for all $n \geq 1$ and for any choice of times $t_1 < t_2 < \ldots < t_n \in T$, the vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian.

Proposition 2.3.2
If the random vector (X_1, X_2) is Gaussian, then the random variables X_1 and X_2 are independent if and only if their covariance $Cov(X_1, X_2)$ is zero.

Proposition 2.3.3
Any vector of independent Gaussian random variables is a Gaussian vector.

2.3.2 Brownian motion

Historical Remarks and Basic Definitions. In 1828, Robert Brown published a brief account of the microscopical observations made in the months of June, July and August, 1827 on the particles contained in the pollen of plants [9].
In 1900, Bachelier [3] postulated that stock prices execute Brownian motion, and he developed a mathematical theory which was similar to the theory which Einstein [21] developed.In 1923, Norbert Wiener proved the existence of Brownian motion and made significant contributions to related mathematical theories, so Brownian motion is often called a Wiener process [40].

Definition 2.3.4 *(Standard Brownian Motion)*
A standard Brownian motion in dimension d over a time interval $T = [0, T]$ or over the set of positive real numbers $\mathbb{R}^+$ is a continuous process with values in $\mathbb{R}^d$, denoted by $(W_t)_{t\in T} = \left(W_t^1, \ldots, W_t^d\right)_{t\in T}$, which satisfies the following properties :

$W_0 = 0$ almost surely. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ is independent of the information up to time s, $\sigma\left(W_u, u \leq s\right)$. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ follows a centered normal distribution, with a variance-covariance matrix $(t - s)I_d$, where I_d is the identity matrix of size d.

Definition 2.3.5 *(Brownian motion with respect to a filtration)*
A vectorial Brownian motion in dimension d over a time interval $T = [0, T]$ or over the set of positive real numbers $\mathbb{R}^+$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in T}$ is a continuous process $\mathcal{F}$-adapted taking values in $\mathbb{R}^d$, denoted by $(W_t)_{t\in T} = \left(W_t^1, \ldots, W_t^d\right)_{t\in T}$, which satisfies the following properties :

$W_0 = 0$ almost surely. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ follows a centered normal distribution, with a variance-covariance matrix of $(t - s)I_d$, where I_d is the identity matrix of size d.

Remark 2.3.2
Un mouvement brownien standard est un mouvement brownien par rapport à sa propre filtration naturelle.

Lemma 2.3.1

$$\mathbb{E}[W_t] = 0 \quad \textit{and} \quad \mathbb{E}[W_t^2] = t - t_0 \quad \textit{for each time } t \geq t_0.$$

Proof. *We observe that :*

$$W_t - W_{t_0} \sim N(0, t - t_0)$$

and that :

$$\mathbb{E}[W_t - W_{t_0}] = \mathbb{E}[W_t] = 0.$$

Moreover :

$$\begin{aligned}\mathbb{E}[W^2(t)] &= \mathbb{E}[W^2(t)] - (\mathbb{E}[W_t])^2 \\ &= \mathbb{V}[W_t] \\ &= \mathbb{V}[W_t - W_{t_0}] \\ &= t - t_0.\end{aligned}$$

This concludes the proof.

Proposition 2.3.4
Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then :

1. *Symmetry : The process $(-W) = (-W_t)_{t\geq 0}$ is also a Brownian motion.*
2. *Scale Change : For all $\lambda > 0$, the process $W^\lambda = \left(W_t^\lambda\right)_{t\geq 0}$ defined by $W_t^\lambda = \left(\frac{1}{\lambda}\right) W\lambda^2 t$ is a Brownian motion.*
3. *Simple Markov Property : For all $s \geq 0$, if $\mathcal{F}_s := \sigma\left(W_u, u \leq s\right)$ and $W_t^{(s)} = Wt + s - W_s$, then the process $W^{(s)} = (W_t^s)_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_s$.*

2.3.3 Stochastic Integral

Langevin's 1908 approach to Brownian motion, rooted in mechanical equations with random forces, offers an alternative and more concrete perspective compared to earlier models. This method has since gained wider recognition.
We first define the integral for elementary processes. Then, leveraging a result on complete spaces, we extend this definition to adapted processes with a second-order moment. Finally, we explore Itôs formula and the integral associated with an Itô process.

Definition 2.3.6 *(Wiener Integral)*
The Wiener integral is simply an integral of the form :

$$\int_0^t X_s dW_s$$

Lemma 2.3.2 *[7]*
If $f \in H_0^2$ is an elementary function, and we define $I_t(\omega) = \int_0^t f(\omega, s)dB_s$, then I_t is a martingale with respect to $\mathcal{F}_t$.

Theorem 2.3.1
Let $f \in H^2$. The continuous process $\int_0^t f dB$ is a martingale with respect to $\grave{a}F_t$.
One major advantage of the Itô integral is that it implies :

$$\forall f \in H^2, t \geq 0, \quad E\left[\int_0^t f(\cdot, s)dB_s\right] = 0$$

Itô's Formula

Introduction
When performing a change of variable in an integral, setting $y = f(x)$, we generally use the notation $dy = f'(x)dx$. This relation follows directly from Leibniz's notation, where $dy = \frac{df}{dx}dx$. When dealing with a composite function, for instance when x depends on time t, we can still apply the chain rule in integrals, which gives :

$$dy = \frac{df}{dx}\frac{dx}{dt}dt$$

Since the Itô integral, like the Riemann integral, is defined as an infinite sum of infinitesimal increments dB_t, and because this definition aligns with Leibniz's notation and related manipulations, it seems reasonable to apply the same rules in this context.
However, if we take $f(x) = x^2$ as an example, we would have :

$$Y_t = B_t^2, \quad dY_t = 2B_t dB_t, \quad \text{and} \quad \int_0^t 2B_s dB_s = \int_0^t dY_s = Y_t - Y_0 = B_t^2 \quad \text{(FALSE)}$$

This is incorrect since $\int_0^t 2B_s dB_s$ is a martingale.
Using Itô's formula, an additional term appears :

$$B_t^2 = \int_0^t 2B_s dB_s + \int_0^t ds$$

and the rules of variable substitution must be adjusted.

Definition 2.3.7 *[7](Itô's Formula)*
Let B_t be a $\mathcal{F}_t$ Brownian motion on $(\Omega, \mathcal{A}, P)$. An Itô process is a stochastic process X_t of the form :

$$X_t(\omega) = X_0(\omega) + \int_0^t u(\omega, s)ds + \int_0^t v(\omega, s)dB_s(\omega)$$

where u and v are $\mathcal{F}_t$-adapted, and the involved integrals are well-defined a.s., i.e. $\int_0^t |u(\omega, s)|ds < \infty$ a.s., and $v \in H^2$.
We use the following notation :

$$dX_t = udt + vdB_t$$

which leads to Itô's formula in dimension 1.

Theorem 2.3.2 *[7]*
Let X_t be an Itô process defined by :

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dB_s$$

where we denote $u_s = u(\omega, s)$ and $v_s = v(\omega, s)$. Let $f(t, x) : [0, \infty[\times \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then :

$$Y_t = f(t, X_t)$$

is an Itô process satisfying :

$$Y_t = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)(u_s ds + v_s dB_s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, v_s^2 ds$$

Example

We consider the Itô process B_t. It satisfies the integral equation :

$$B_t = \int_0^t dB_s$$

or equivalently, it is the solution of :

$$dX_t = dB_t$$

where $u = 0$ and $v = 1$. We consider the function $f(t, x) = x^2$. Itô's formula gives :

$$d\left(X_t^2\right) = 2X_t dB_t + 1 \cdot dt$$

The last term related to $\frac{\partial^2 f}{\partial x^2} = 1$ is essential. Indeed, when taking expectations, we obtain :

$$E\left[dX_t^2\right] = 2E\left[X_t dB_t\right] + 1 \cdot dt = 1 \cdot dt$$

Since the Itô integral is a martingale, the middle term cancels out. Thus, we get the deterministic equation for $B_t = X_t$:

$$dE\left[B_t^2\right] = 1 \cdot dt, \quad E\left[B_0^2\right] = 1, \quad \text{so} \quad E\left[B_t^2\right] = t$$

Itô Isometry

Theorem 2.3.3 *[7]*
Let $f \in H^2[0, T]$. Then :

$$\|I(f)\|_{L^2(dP)} = \|f\|_{L^2(dP \times dt)}$$

Proof
Let f_n be elementary functions such that $\|f - f_n\|_{L^2(dP\times dt)} \to 0$.
By definition of the integral, $\|I(f_n) - I(f)\|_{L^2(dP)} \to 0$.
Moreover, $\|f_n\|_{L^2(dP\times dt)} = \|I(f_n)\|_{L^2(dP)}\ \forall n \geq 0$.

2.3.4 Stochastic differential equations (SDE)

Stochastic differential equations (SDE) are an extension of traditional deterministic differential equations, making it possible to incorporate uncertainty, noise, or random disturbances into dynamic models. That is, a stochastic element is included to represent uncertain or random phenomena that affect dynamic systems.
SDE first emerged in the twentieth century with the development of stochastic process theory by Robert Brown and were later mathematically formulated by Einstein and Wiener. The modern theory was largely built on Itô calculus, developed by Kiyoshi Itô in the 1940s, which provides a precise mathematical framework for defining stochastic integrals and differential equations driven by Brownian motion.
They were first used in statistical physics to describe the diffusion of particles in fluids.
Stochastic differential equations are applied in various fields, including economics and finance, physics and electronics, engineering, and biology.

Definition 2.3.8
A Stochastic Differential Equation (SDE) is an equation for the process X (with values in $\mathbb{R}^d$) of the form :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.20}$$

Theorem 2.3.4
Let $T > 0$ and $b : [0, T] \times \mathbb{R} \to \mathbb{R}, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying, on $[0, T] \times \mathbb{R}$:

$$|b(t, x) + \sigma(t, x)| \leq C(1 + |x|), \quad |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|$$

where $C > 0$ is a constant, and Z is a random variable in $L^2(dP)$ independent of the -algebra generated by $(B_s)_{s\geq 0}$. Then, the SDE (2.20) admits a unique t-continuous solution in $L^2(dP\times dt)$ adapted to the filtration generated by Z and B_s.

2.3.5 Example

We will now provide some applied examples to help better understand stochastic differential equations.
Let's consider the following SDE :

$$dX_t = a(b - X_t)dt + \sigma dW(t)$$

We need to verify :

1. $|b(x, t) - b(y, t)| + |a(x, t) - a(y, t)| \leq K|x - y|, \forall t \geq 0$
2. $|b(x, t)|^2 + |a(x, t)|^2 \leq K^2 (1 + x^2), \forall t \geq 0$
3. $E[X_0^2] < \infty$

We have :

$$\begin{aligned} |b(x, t) - b(y, t)| + |a(x, t) - a(y, t)| &= |a(b - x) - a(b - y)| + |\sigma - \sigma| \\ &= |a||x - y| \end{aligned}$$

since :

$$|x| \leq \begin{cases} 1 & \text{if } |x| \leq 1 \\ x^2 & \text{if } |x| \geq 1 \end{cases} \leq \begin{cases} 1 + x^2 & \text{if } |x| \leq 1 \\ 1 + x^2 & \text{if } |x| \geq 1 \end{cases}$$

thus

$$\begin{aligned}
|b(x,t)|^2 + |a(x,t)|^2 &= |a(b-x)|^2 + |\sigma|^2 \\
&= a^2(b-x)^2 + \sigma^2 \\
&= a^2\left(b^2 - 2bx + x^2\right) + \sigma^2 \\
&\leq a^2\left(b^2 + 2|b||x| + x^2\right) + \sigma^2 \\
&\leq a^2\left(b^2 + 2|b|\left(1+x^2\right) + x^2\right) + \sigma^2 \\
&= a^2\left(b^2 + 2|b|\right) + \sigma^2 + (2|b|+1)x^2 \\
&\leq \max\left(a^2\left(b^2+2|b|\right) + \sigma^2 (2|b|+1)\right)\left(1+x^2\right)
\end{aligned}$$

So let's set $K = \max\left(|a|, \sqrt{a^2\left(b^2+2|b|\right)+\sigma^2}, \sqrt{2|b|+1}\right)$.
Since the initial condition wasn't specified, we only need to choose X_0 to be square-integrable to fulfill condition (3).

Ornstein-Uhlenbeck Process as a Solution to the Langevin Equation

[7] The Langevin equation $\frac{d}{dt}V = -\gamma V + L(t)$ in the Itô formalism can be written as :

$$\begin{aligned}
&dV_t = -\gamma V_t dt + \sigma dB_t \\
&V(0) = v_0
\end{aligned}$$

which has a solution according to theorem 2.3.4.
Here, dB_t replaces a mathematically ill-defined random force $L(t)$. So we have :

$$dV_t = -\gamma V_t dt + L(t)$$

For each trajectory $V_t(\omega)$, we would use the method of variation of constants. This method is compatible with our formalism. By setting

$$C_t = V_t e^{\gamma t}$$

we have, applying Itô's formula to $f(t,x) = e^{\gamma t} x$:

$$dC_t = \gamma C_t dt + e^{\gamma t}\left(-\gamma V_t dt + \sigma dB_t\right) = e^{\gamma t}\sigma dB_t$$

and thus

$$V_t = e^{-\gamma t} v_0 + \int_0^t e^{-\gamma(t-s)}\sigma dB_s$$

Application to Finance : Geometric Brownian Motion and the Black-Scholes Model

[7] In this model, the price of a stock is governed by the SDE

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

$$S_0 = s_0$$

We set :

$$Y_t = \log(S_t)$$

As we have no guarantee that S_t does not vanish, we will perform a formal calculation without justification. We apply Itô's formula with the function $f(t,x) = \log x$. We get

$$d\log\left(S_t\right) = \left(\mu dt + \sigma dB_t\right) - \sigma^2 dt = \left(\mu - \sigma^2/2\right)dt + \sigma dB_t$$

By integrating both sides, we obtain

$$Y_t = \log\left(s_0\right) + \left(\mu - \sigma^2/2\right)t + \sigma B_t \quad \text{or} \quad S_t = s_0 e^{\left(\mu-\sigma^2/2\right)t+\sigma B_t}$$

CHAPITRE 3

FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATION

INTRODUCTION

A fractional stochastic differential equation is a concept used to numerically solve this type of equation. Fractional differential equations are generalizations of classical differential equations, where the derivatives are fractional operators. Fractional stochastic differential equations add a stochastic component, i.e., a random element, to the process.

Fractional differential equations are now receiving increasing attention due to their applications in various disciplines such as mechanics, physics, chemistry, biology, electrical engineering, control theory, heat, etc. There have been only a few articles dealing with stochastic differential equations of fractional Caputo order, and most of these articles have attempted to establish a result on the existence and uniqueness of solutions. Here, we distinguish two types of solutions : The first concerns mild solutions ; for more details, see [6] on the existence and uniqueness of this type of solution. The other type of solution is defined as a solution to a stochastic problem associated with integral equations. We have the following fractional equation :

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha}x(t) = Ax(t) + f(t), \qquad 0 < \alpha < 1, t \geq 0, \\ &x(0) = \eta, \end{aligned} \tag{3.1}$$

To solve 3.1, we use the Laplace transform.
Equation 3.1 is equivalent to :

$$L\{{}^{C}\mathcal{D}^{\alpha}x(t)\} = L\{Ax(t) + f(t)\}$$

Thus :

$$S^{\alpha}\hat{x}(s) - S^{\alpha-1}\eta = L\{Ax(t) + f(t)\}$$

$\Rightarrow S^{\alpha}\hat{x}(s) = A\hat{x}(s) + \hat{f}(s) + S^{\alpha-1}\eta$ (by linearity)

$\Rightarrow (S^{\alpha} - A)\hat{x} = S^{\alpha-1}\eta + \hat{f}(s)$

$\Rightarrow \hat{x}(s) = (S^{\alpha} - A)^{-1}[S^{\alpha-1}\eta + \hat{f}(s)]$

Now, we apply L^{-1}

$L^{-1}(\hat{x}(s)) = L^{-1}\{(S^{\alpha} - A)^{-1}[S^{\alpha-1}\eta + \hat{f}(s)]\}$

$\Rightarrow$

$$x(t) = L^{-1}\{(S^{\alpha} - A)^{-1}[S^{\alpha}\eta + \hat{f}(s)\} \tag{3.2}$$

Lemma 3.0.1 *[29]*
Let C be the complex plane, for all $\alpha > 0$, $\beta > 0$ and $A \in \mathbb{C}^{n\times n}$. The Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})$ is given by $s^{\alpha-\beta}(s^{\alpha}-A)^{-1}$, valid for $\Re(s) > |A|^{1/\alpha}$, where $\Re s$ represents the real part of the complex number s.

Proof : For $\Re(s) > \|A\|^{1/\alpha}$, we have $\sum_{k=0}^{\infty} A^k S^{-(k+1)\alpha} = (s^{\alpha}-A)^{-1}$. Then

$$\begin{aligned}\mathcal{I}\left[t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})\right] &= \mathcal{I}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(At^{\alpha})^k}{\Gamma(\alpha k+\beta)}\right]\\ &= \sum_{k=0}^{\infty}\frac{A^k\mathcal{I}\left[t^{\alpha k+\beta-1}\right]}{\Gamma(\alpha k+\beta)}\\ &= s^{\alpha-\beta}\sum_{k=0}^{\infty}A^k s^{-(k+1)\alpha}\\ &= s^{\alpha-\beta}(s^{\alpha}-A)^{-1}.\end{aligned}$$

We have :
$0 < \alpha < 1$ *et* $\beta = 1$
Thus : $s^{\alpha-1}(s^{\alpha}-A)^{-1} = \mathcal{I}\{t^{1-1}E_{\alpha,1}(At^{\alpha})\}$

$$\begin{aligned}x(t) =&\mathcal{I}^{-1}\{(S^{\alpha}-A)^{-1}[S^{(\alpha-1)}\eta\}\\ =&\mathcal{I}^{-1}\{\eta\mathcal{I}\{t^0E_{\alpha,1}(At^{\alpha})\}\\ =&\eta E_{\alpha,1}(At^{\alpha})\end{aligned}$$

Therefore : $\mathcal{I}\{t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\} = s^{\alpha-\alpha}(s^{\alpha}-A)^{-1}$
$\Leftrightarrow \mathcal{I}\{t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\} = (s^{\alpha}-A)^{-1}$
We have :

$$\begin{aligned}\mathcal{I}^{-1}\{(S^{\alpha}-A)^{-1}\hat{f}(s)\} =&\mathcal{I}^{-1}\{\mathcal{I}\{t^{\alpha-1}E_{(\alpha,\alpha)}A(t^{\alpha})\}\hat{f}(s)\}\\ =&\mathcal{I}^{-1}\{\mathcal{I}\{t^{\alpha-1}E_{(\alpha,\alpha)}A(t^{\alpha})\}\mathcal{I}\{f(t)\}\\ =&\mathcal{I}^{-1}\{\mathcal{I}\{t^{\alpha-1}E_{(\alpha,\alpha)}A(t^{\alpha})\star f(t)\}\}\\ =&t^{\alpha-1}E_{(\alpha,\alpha)}(At^{\alpha})\star f(t)\end{aligned}$$

Definition 3.0.1
When the product $f(x-t)g(t)$ is integrable over any interval $[0,x]$ of $\mathbb{R}^+$, the convolution product of f and g is defined by :

$$(f*g)(x) = \int_0^x f(x-t)g(t)dt$$

$$\begin{aligned}\mathcal{I}^{-1}\{(S^{\alpha}-A)^{-1}\hat{f}(s)\} =&t^{\alpha-1}E_{(\alpha,\alpha)}(At^{\alpha})\star f(t)\\ =&\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(\tau)d\tau\end{aligned}$$

Finally ;

$$\begin{aligned}&{}^C\mathcal{D}^{\alpha}x(t) = Ax(t)+f(t), \qquad 0<\alpha<1, t\geq 0,\\ &x(0)=\eta,\end{aligned}$$

$\Leftrightarrow$

$$x(t) = E_{\alpha,1}(t)\eta + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(\tau)d\tau$$

Lemma 3.0.2 *[38]*
For all $\alpha > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds :

$$\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}\left(\gamma\tau^{2\alpha-1}\right)d\tau \leq E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right).$$

Proof :
Let $\gamma > 0$ Consider the corresponding linear Caputo fractional differential equation of the following form :

$$ {}^{C}\mathcal{D}_{0+}^{2\alpha-1}x(t) = \gamma x(t) \tag{3.3} $$

The Mittag-Leffler function $E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right)$ is a solution of 4.3 (see [18] page 135), hence the following equality :

$$ E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right) = 1 + \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}\left(\gamma \tau^{2\alpha-1}\right) d\tau, $$

Thus, by 3.0.2, we note for all $t > 0$:

$$ \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}\left(\gamma \tau^{2\alpha-1}\right) d\tau \leq E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right) $$

3.1 Caputo Fractional Stochastic Differential Equation

Consider a Caputo FSDE of order $\alpha \in [\frac{1}{2}; 1]$:

$$ \begin{cases} {}^{C}\mathcal{D}_{0+}^{\alpha} X(t) = AX(t) + b(t, X(t)) + \sigma(t, X(t)) \frac{dW(t)}{dt}, \\ X(0) = \eta, \end{cases} \tag{3.4} $$

where $(W)_t$ is a standard scalar Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \mathcal{F}_t, \mathbb{P})$,
$A \in \mathbb{R}^{d\times d}$ and $b, \sigma : [0; T] \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions satisfying the following conditions :

1. $(H1)$ $\exists L > 0$ tel que pour tout $x, y \in \mathbb{R}^d, t \in [0, T]$

$$ \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq L\|x-y\| $$

2. $(H2)$ $\int_0^T \|b(s,0)\|^2 ds < \infty, \quad \underset{s\in[0,T]}{ess\ sup} \|\sigma(s,0)\| < \infty$

For each $t \in [0; \infty)$, let $D_t := L^2(\Omega, \mathcal{F}_t, \mathbb{P})$.
The space of square-integrable functions $f : \Omega \to \mathbb{R}^d$

Definition 3.1.1 *[1] (Classical solution of Caputo FSDE)*
Let $\eta \in D_0$ be an $\mathbb{F}$-adapted process. X is called a solution of 3.4 for $t \in [0, T]$

$$ \begin{aligned} X(t) =& \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(AX(\tau) + b\left(\tau, X\left(\tau\right)\right)\right) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, X(\tau)) dW \end{aligned} \tag{3.5} $$

For all $\forall \eta \in D_0$, there exists a unique solution to 3.5, denoted : $\varphi(t, \eta)$

Proof :
We have the problem 3.4 :

$$ {}^{C}\mathcal{D}_{0+}^{\alpha} X(t) = AX(t) + b(t, X(t)) + \sigma(t, X(t)) \frac{dW(t)}{dt}, \text{ and } X(0) = \eta, $$

Applying $\mathcal{I}^\alpha$ to both sides :

$$\mathcal{I}^\alpha \left({}^C\mathcal{D}^\alpha_{0+}X(t)\right) = \mathcal{I}^\alpha \left(AX(t) + b(t, X(t)) + \sigma(t, X(t))\frac{dW(t)}{dt} \right),$$
$$X(t) - X(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(AX(\tau) + b(\tau, X(\tau))\right) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, X(\tau)) dW$$

becomes

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(AX(\tau) + b(\tau, X(\tau))\right) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, X(\tau)) dW$$

Theorem 3.1.1 *(Variation of constants for Caputo FSDE)*
Let $\eta \in D_0$. We have :

$$\varphi(t, \eta) = E_\alpha\left(t^\alpha A\right)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) b\left(\tau, \Phi(\tau, \eta)\right) d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) \sigma\left(\tau, \Phi(\tau, \eta)\right) dW_\tau. \tag{3.6}$$

$\forall t \in [0, T]$

Remark 3.1.1 *[1]*

1. $\sigma(t, X(t)) = 0$
2. *Note that :* $E_1(M) = E_{1,1}(M) = e^M$ *for* $M \in \mathbb{R}^{d\times d}$

3.6 becomes

$$dX(t) = (AX(t) + b(t, X(t)))dt + \sigma(t, X(t))dW_t \tag{3.7}$$

By applying the previous theorem, we obtain an explicit representation of the solution to the linear inhomogeneous FSDE of the form :

$${}^C\mathcal{D}^\alpha_{0+}X(t) = AX(t) + b(t) + \sigma(t)\frac{dW_t}{dt}, \quad X(0) = \eta \tag{3.8}$$

Corrolaire 3.1.1
Assume that $b \in \mathbb{L}^2\left([0,T], \mathbb{R}^d\right)$ and $\sigma \in \mathbb{L}^\infty\left([0,T], \mathbb{R}^d\right)$.
With $T > 0$; Then, the explicit solution of 3.8 on $[0;T]$ is given by :

$$X(t) = E_\alpha\left(t^\alpha A\right)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) b(\tau) d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) \sigma(\tau) dW_\tau$$

Definition 3.1.2 *(Weak solution of Caputo FSDE)*
An $\mathbb{F}$-adapted process X is called a weak solution of 3.4 with the condition $X(0) = \eta$ if the following equality holds for $t \in [0;T]$:

$$X(t) = E_\alpha\left(t^\alpha A\right)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) b(\tau, X(\tau)) d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) \sigma(\tau, X(\tau)) dW_\tau. \tag{3.9}$$

We will now discuss the existence and uniqueness of weak solutions to 3.4.
In this result, we stipulate that the system coefficients must satisfy both conditions (H1) and (H2).

Theorem 3.1.2 *[1] (Existence and uniqueness of weak solutions) Assume that (H1) and (H2) hold. For all $\eta \in D_0$, there exists a unique weak solution Y of 3.4 satisfying $Y(0) = \eta$, which is denoted $\psi(t, \eta)$ 3.6.*

Proof : Let $\mathbb{H}_\eta^2\left([0,T], \mathbb{R}^d\right) := \left\{\xi \in \mathbb{H}^2\left([0,T], \mathbb{R}^d\right) : \xi(0) = \eta\right\}$ Define the corresponding Lyapunov-Perron operator $\mathcal{T}_\eta : \mathbb{H}_\eta^2\left([0,T], \mathbb{R}^d\right) \to \mathbb{H}_\eta^2\left([0,T], \mathbb{R}^d\right)$ by :

$$\begin{aligned}\mathcal{T}_\eta Y(t) =& E_\alpha\left(t^\alpha A\right)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) b(\tau, Y(\tau)) d\tau \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) \sigma(\tau, Y(\tau)) dW_\tau\end{aligned}$$

We know that the operator $\mathcal{T}_\eta$ is well-defined.
For complete the proof, it suffices to show that $\mathcal{T}_\eta$ is contractive with respect to an appropriate metric on $\mathbb{H}_\eta^2\left([0,T], \mathbb{R}^d\right)$.
For this, we consider $\mathbb{H}^2\left([0,T], \mathbb{R}^d\right)$ with a weighted norm $\|\cdot\|_\gamma$, where $\gamma > 0$, defined as follows :

$$\|\xi\|_\gamma := \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}\left(\|\xi(t)\|^2\right)}{E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right)}} \quad \text{pour tous } \xi \in \mathbb{H}^2\left([0,T], \mathbb{R}^d\right) \tag{3.10}$$

Clearly,
$\|\cdot\|_{\mathbb{H}^2}$ et $\|\cdot\|_\gamma$ are equivalent.
Thus, $\left(\mathbb{H}^2\left([0,T], \mathbb{R}^d\right), \|.\ \|_\gamma\right)$ is also a Banach space.
Then :
The set $\mathbb{H}_\eta^2\left([0,T], \mathbb{R}^d\right)$ with the metric induced by $\| \ cdot\|_\gamma$ is complete. By the compactness of $[0,T]$ and the continuity of the function $t \mapsto E_{\alpha,\alpha}\left(t^\alpha A\right)$,
there exists $M_T := \max_{t\in[0,T]} \|E_{\alpha,\alpha}\left(t^\alpha A\right)\| > 0$.
Now, we choose and fix a positive constant γ such that :

$$2L^2 M_T^2 (T+1) \frac{\Gamma(2\alpha-1)}{\gamma} < 1 \tag{3.11}$$

Therefore, by (H1)

$$\begin{aligned}\| T_X - T_Y \| = \| &\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) b(X) dS \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) \sigma(X) dW_S \\ &- \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) b(Y) dS \\ &- \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) \sigma(Y) dW_S \,\| \\ &\le (L_b^2 + L_\sigma^2) \,\| \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)[X(s) - Y(s)] dW_s\right) \|^2\end{aligned}$$

By the definition of $\mathcal{T}_\eta$, (H1), the Ito isometry andM_T, we have

$$\begin{aligned}\|\mathcal{T}_\eta X(t) - \mathcal{T}_\eta Y(t)\|_{\text{ms}}^2 \le& 2L^2 M_T^2 \left\| \int_0^t (t-\tau)^{\alpha-1} \right\| X(\tau) - Y(\tau)\| d\tau\|_{\text{ms}}^2 \\ &+ 2L^2 M_T^2 \int_0^t (t-\tau)^{2\alpha-2} \|X(\tau) - Y(\tau)\|_{\text{ms}}^2 d\tau\end{aligned}$$

Using Hölder's inequality, we obtain :

$$\|\mathcal{T}_\eta X(t)-\mathcal{T}_\eta Y(t)\|_{\mathrm{ms}}^2 \le 2L^2 M_T^2(T+1)\int_0^t (t-\tau)^{2\alpha-2}\|X(\tau)-Y(\tau)\|_{\mathrm{ms}}^2 d\tau$$

and by the definition of $\|\cdot\|_\gamma$ we have :

$$\begin{aligned}&\frac{\|\mathcal{T}_\eta X(t)-\mathcal{T}_\eta Y(t)\|_{\mathrm{ms}}^2}{E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right)}\\ \le & 2L^2 M_T^2(T+1)\frac{\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}\left(\gamma\tau^{2\alpha-1}\right)d\tau}{E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right)}\|X-Y\|_\gamma^2\end{aligned}$$

Thus

$$\|\mathcal{T}_\eta X-\mathcal{T}_\eta Y\|_\gamma \le \sqrt{2L^2 M_T^2(T+1)\frac{\Gamma(2\alpha-1)}{\gamma}}\|X-Y\|_\gamma$$

3.2 Study of Unique Solutions and Their Stability in Fractional Stochastic Equations

Introduction

Consider a Caputo fractional stochastic differential equation (for short Caputo FSDE) of order $\beta\in(\frac{1}{2},1)$ of the following form :

$$^{C}\mathcal{D}_{0^+}^{\beta}X(t)=b(t,X(t))+\sigma(t,X(t))\frac{dW(t)}{dt}, \tag{3.12}$$

where $b,\sigma:[0,\infty)\mathbb{R}^d\to\mathbb{R}^d$ are measurable and $(W_t)_t\in[0,\infty]$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F}:=\{\mathcal{F}_t\}_{t\in[0,\infty)},\mathbb{P})$. For each $t\in[0,\infty)$, let $\mathcal{X}_t:=\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P})$ denote the space of all $\mathcal{F}_t$ -measurable, mean square integrable functions $f=(f_1,...,f_d)^T:\Omega\to\mathbb{R}^d$ with

$$||f||_{ms}:=\sqrt{\sum_{i=1}^{d}\mathbb{E}(|f_i|^2)}=\sqrt{\mathbb{E}||f||^2},$$

where $\mathbb{R}^d$ is endowed with the standard Euclidean norm.A process $X:[0,\infty)\to\mathbb{L}(\Omega,\mathcal{F},\mathbb{P})$ is said to be $\mathbb{F}-$adapted if $X(t)\in\mathcal{X}_t$ for all $t\ge 0$.For each $\kappa\in\mathcal{X}_0$, a $\mathbb{F}$-adapted process X is called a solution of (3.12) with the initial condition $X(0)=\kappa$ if the following equality holds for $t\in[0,\infty]$

$$X(t)=\kappa+\frac{1}{\Gamma(\beta)}\left(\int_0^t (t-s)^{\beta-1}b(s,X(s))ds+\int_0^t (t-s)^{\beta-1}\sigma(s,X(s))dW_s\right) \tag{3.13}$$

we assume that the coefficients b and σ satisfy the following standard conditions :

(H1) There exists $L>0$ such that for all $x,y\in\mathbb{R}^d$, $t\in[0,\infty)$

$$||b(t,x)-b(t,y)||+||\sigma(t,x)-\sigma(t,y)||\le L||x-y||.$$

(H2) $\sigma(\cdot,0)$ is essentially bounded,i.e.

$$||\sigma(\cdot,0)||_\infty := ess\sup_{s\in[0,\infty)}||\sigma(s,0)||<\infty$$

and $b(\cdot,0)$ is $\mathbb{L}^2$ integrable, i.e.

$$\int_0^\infty ||b(s,0)||^2 ds<\infty$$

Our first result in this article is to show the global existence and uniqueness solutions of (3.12) when $(H1)$ and $(H2)$ hold. Furthermore, we also show the continuity dependence of solutions on the initial values.

We need this lemma : Here, the weight function is the Mittag–Leffler function $E_{2\beta-1}(\cdot)$ defined as :

$$E_{2\beta-1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\beta-1)k+1)} \quad \text{for all } t \in \mathbb{R}.$$

The following result is a technical lemma which is used later to estimate the operator T_κ.

Lemma 3.2.1
For any $\beta > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds :

$$\frac{\gamma}{\Gamma(2\beta-1)} \int_0^t (t-s)^{2\beta-2} E_{2\beta-1}(\gamma t^{2\beta-1}) ds \leq E_{2\beta-1}(\gamma t^{2\beta-1}).$$

Proof.
Let $\gamma > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form :

$${}^C\mathcal{D}_{0^+}^{2\beta-1} x(t) = \gamma x(t). \tag{3.14}$$

The Mittag–Leffler function $E_{2\beta-1}(\gamma t^{2\beta-1})$ is a solution of 3.14, Hence, the following equality holds :

$$E_{2\beta-1}(\gamma t^{2\beta-1}) = 1 + \frac{\gamma}{\Gamma(2\beta-1)} \int_0^t (t-s)^{2\beta-2} E_{2\beta-1}(\gamma t^{2\beta-1}) ds,$$

which completes the proof.

Theorem 3.2.1
(Global existence and uniqueness and Continuity dependence on the initial values of solutions of Caputo FSDE). Suppose that $(H1)$ and $(H2)$ hold. Then

(i) for any $\kappa \in \mathcal{X}_0$, the initial value problem 3.12 with the initial condition $X(0) = \kappa$ has a unique global solution on the whole interval $[0, \infty)$ denoted by $\varphi(\cdot, \kappa)$;

(ii) on any bounded time interval $[0, T]$, where $T > 0$, the solution $\varphi(\cdot, \kappa)$ depends continuously on κ, i.e.

$$\lim_{\zeta \to \kappa} \sup_{t \in [0,T]} ||\varphi(t, \zeta) - \varphi(t, \kappa)||_{ms} = 0$$

we give an application of the main results concerning the mean square Lyapunov exponent of non-trivial solutions to a bounded bilinear Caputo FSDE. To formulate this result, we consider the following equation :

$${}^C\mathcal{D}_{0^+}^{\beta} x(t) = A(t)x(t)) + B(t)x(t)\frac{dW(t)}{dt}, \tag{3.15}$$

where $A, B : [0, \infty) \to \mathbb{R}^{d \times d}$ are measurable and essentially bounded, i.e.,

$$\operatorname*{ess\,sup}_{t \in [0,\infty)]} ||A(t)||, \quad \operatorname*{ess\,sup}_{t \in [0,\infty)]} ||B(t)|| < \infty$$

for each $\kappa \in \mathcal{X}_0 \setminus \{0\}$, there exists a unique solution of 3.15, denoted by $\Phi(\cdot, \kappa)$, satisfying the initial condition $X(0) = \kappa$. The mean square Lyapunov exponent of $\Phi(\cdot, \kappa)$ is defined by

$$\lambda_{ms}(\Phi(\cdot, \kappa)) := \limsup_{t \to \infty} \frac{1}{t} \log ||\Phi(t, \kappa)||_{ms} \tag{3.16}$$

In the following corollary, we show the non-negativity of the mean square Lya- punov exponent of an arbitrary non-trivial solution.

Corrolaire 3.2.1 *(Non-negativity of mean square Lyapunov exponent for solutions of linear Caputo fsde). The mean square Lyapunov exponent of a nontrivial solution of 3.15 is always non-negative, i.e.,*

$$\lambda_{ms}(\Phi(\cdot, \kappa)) \geq 0 \;\; for \;\; all \;\; \kappa \in \mathcal{X}_0 \setminus \{0\}$$

3.2.1 Existence and uniqueness results

Existence, uniqueness, and continuity dependence on the initial values of solutions

Our aim in this subsection is to prove the result on global existence, uniqueness, and continuity dependence on the initial values of solutions to the equation 3.12. In fact, in order to prove Theorem 3.2.1(i) it is equivalent to show the existence and uniqueness solutions on an arbitrary interval $[0, T]$, where $T > 0$ is arbitrary. In what follows, we choose and fix a $T > 0$ arbitrarily. Let $\mathbb{H}^2([0, T])$ be the space of all the processes X which are measurable, $\mathbb{F}_T$-adapted, where $\mathbb{F}_T := \{\mathcal{F}\}_{t\in[0,T]}$, and satisfies that

$$||X||_{\mathbb{H}^2} := \sup_{0\leq t\leq T} ||X(t)||_{ms} < \infty$$

Obviously, $(\mathbb{H}^2([0, T]), ||.||_{\mathbb{H}^2})$ is a Banach space. For any $\kappa \in \mathcal{X}_0$, we define an operator T_κ : $\mathbb{H}^2([0, T]) \to \mathbb{H}^2([0, T])$ by

$$T_\kappa\psi(s) := \kappa + \frac{1}{\Gamma(\beta)} \left(\int_0^t (t-s)^{\beta-1} b(s, \psi(s) ds + \int_0^t (t-s)^{\beta-1} \sigma(s, \psi(s)) dW_s \right). \tag{3.17}$$

The following lemma is devoted to showing that this operator is well-defined.

Lemma 3.2.2
For any $\kappa \in \mathcal{X}_0$, the operator s_κ is well-defined.

Proof.
Let $\psi \in \mathbb{H}^2([0, T])$ be arbitrary. From the definition of $T_\kappa\psi$ as in 3.17 and the inequality $\|x + y + z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, we have for all $t \in [0, T]$

$$\begin{aligned} \|T_\kappa\psi(t)\|^2_{ms} \leq 3\|\kappa\|^2_{ms} + \frac{3}{\Gamma(\beta)^2} \mathbb{E} \left(\left\| \int_0^t (t-s)^{\beta-1} b(s, \psi(s)) ds \right\|^2 \right) \\ + \frac{3}{\Gamma(\beta)^2} \mathbb{E} \left(\left\| \int_0^t (t-s)^{\beta-1-\sigma} (s, \psi(s)) dW_s \right\|^2 \right). \end{aligned} \tag{3.18}$$

By the Hölder inequality, we obtain

$$\begin{aligned} \mathbb{E} \left(\left\| \int_0^t (t-s)^{\beta-1} b(s, \psi(s)) ds \right\|^2 \right) &\leq \int_0^t (t-s)^{2\beta-2} ds \mathbb{E} \left(\int_0^t \|b(s, \psi(s))\|^2 ds \right) \\ &= \frac{t^{2\beta-1}}{2\beta-1} \mathbb{E} \left(\int_0^t \|b(s, \psi(s))\|^2 ds \right). \end{aligned} \tag{3.19}$$

From (H1), we derive

$$\|b(s, \psi(s))\|^2 \leq 2(\|b(s, \psi(s)) - b(s, 0)\|^2 + \|b(s, 0)\|^2)$$

$$\leq 2L^2 \|\psi(s)\|^2 + 2\|b(s, 0)\|^2.$$

Therefore,

$$\begin{aligned} \mathbb{E} \left(\int_0^t \|b(s, \psi(s))\|^2 ds \right) &\leq 2L^2 \mathbb{E} \left(\int_0^t \|\psi(s)\|^2 ds \right) + 2 \int_0^t \|b(s, 0)\|^2 ds. \\ &\leq 2L^2 T \sup_{t\in[0,T]} \mathbb{E}(\|\psi(s)\|^2) + 2 \int_0^T \|b(s, 0)\| \end{aligned}$$

which together with 3.19 implies that

$$\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1} b(s,\psi(s))ds\right\|^2\right) \le \frac{2L^2T^{2\beta}}{2\beta-1}\|\psi\|_{\mathbb{H}^2}^2 + \frac{2T^{2\beta-1}}{2\beta-1}\int_0^T \|b(s,0)\|^2 ds. \tag{3.20}$$

Now, using Itô's isometry , we obtain

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}\sigma(s,\psi(s))dW_s\right\|^2\right) &= \sum_{1\le i\le d}\mathbb{E}\left(\int_0^t (t-s)^{\beta-1}\sigma_i(s,\psi(s))dWs\right)^2\\ &= \sum_{1\le i\le d}\mathbb{E}\left(\int_0^t (t-s)^{2\beta-2}|\sigma_i(s,\psi(s))|^2 ds\right).\\ &= \mathbb{E}\left(\int_0^t (t-s)^{2\beta-2}\|\sigma(s,\psi(s))\|^2 ds\right).\end{aligned}$$

From (H1), we also have

$$\|\sigma(s,\psi(s))\|^2 \le 2L^2\|\psi(s)\|^2 + 2\|\sigma(s,0)\|^2 \le 2L^2\|\psi(s)\|^2 + 2\|\sigma(\cdot,0)\|_\infty^2.$$

Therefore, for all $t\in[0,T]$ we have

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}\sigma(s,\psi(s))dW_s\right\|^2\right)\\ &\le 2L^2\mathbb{E}\left(\int_0^t (t-s)^{2\beta-2}\|\psi(s)\|^2 ds\right) + 2\|\sigma(\cdot,0)\|_\infty^2\int_0^t (t-s)^{2\beta-2}ds\\ &\le 2L^2\frac{T^{2\beta-1}}{2\beta-1}\|\psi(s)\|_{\mathbb{H}^2}^2 + \frac{2T^{2\beta-1}}{2\beta-1}\int_0^T \|\sigma(\cdot,0)\|_\infty^2.\end{aligned}$$

This together with 3.18 and 3.20 implies that $\|T_\kappa\psi\|_{\mathbb{H}^2}<\infty$. Hence, the map T_κ is well-defined. To prove existence and uniqueness of solutions, we will show that the operator T_κ defined as above is contractive under a suitable temporally weighted norm for the same method to prove the existence and uniqueness of solutions of stochastic differential equations). We are now in a position to prove Theorem 3.2.1.

Proof of Theorem. 3.2.1. Let $T>0$ be arbitrary. Choose and fix a positive constant γ such that

$$\gamma > \frac{3L^2(T+1)\Gamma(2\beta-1)}{\Gamma(\beta)^2}. \tag{3.21}$$

On the space $\mathbb{H}^2([0,T])$, we define a weighted norm $\|\cdot\|_\gamma$ as below

$$\|X\|_\gamma := \sup_{t\in[0,T]}\sqrt{\frac{\mathbb{E}(\|X(t)\|^2)}{E_{2\beta-1}(\gamma t^{2\beta-1})}} \quad \text{for all } X\in\mathbb{H}^2([0,T]). \tag{3.22}$$

Obviously, two norms $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^2([0,T]),\|\cdot\|_\gamma)$ is also a Banach space.

(i) Choose and fix $\kappa\in\mathcal{X}_0$. By virtue of Lemma 3.2.2, the operator T_κ is well-defined. We will prove that the map T_κ is contractive with respect to the norm $\|\cdot\|_\gamma$.

For this purpose, let $\psi,\hat\psi\in\mathbb{H}^2([0,T])$ be arbitrary. From 3.17 and the inequality $\|x+y\|^2\le 2(\|x\|^2+\|y\|^2)$ for all $x,y\in\mathbb{R}^d$, we derive the following inequalities for all $t\in[0,T]$:

$$\begin{aligned}\mathbb{E}\left(\|T_\kappa\psi(t)-T_\kappa\hat\psi(t)\|^2\right) \le{}& \frac{2}{\Gamma(\beta)^2}\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}(b(s,\psi(s))-b(s,\hat\psi(s)))ds\right\|^2\right)\\ &+\frac{2}{\Gamma(\beta)^2}\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}(\sigma(s,\psi(s))-\sigma(s,\hat\psi(s)))dW_s\right\|^2\right).\end{aligned}$$

Using the Hölder inequality and (H1), we obtain

$$\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}(b(s,\psi(s))-b(s,\hat{\psi}(s)))ds\right\|^2\right) \le L^2 t\int_0^t (t-s)^{2\beta-2}\mathbb{E}(\|\psi(s)-\hat{\psi}(s)\|^2)ds.$$

On the other hand, by Itô's isometry and (H1), we have

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t (t-s)^{\beta-1}(\sigma(s,\psi(s))-\sigma(s,\hat{\psi}(s)))dW_s\right\|^2\right)\\
&=\mathbb{E}\left(\int_0^t (t-s)^{2\beta-2}\|\sigma(s,\psi(s))-\sigma(s,\hat{\psi}(s))\|^2 ds\right)\\
&\le L^2\int_0^t (t-s)^{2\beta-2}\mathbb{E}(\|\psi(s)-\hat{\psi}(s)\|^2)ds.
\end{aligned}$$

Thus, for all $t\in[0,T]$ we have

$$\mathbb{E}\left(\|T_\kappa\psi(t)-T_\kappa\hat{\psi}(t)\|^2\right)\le \frac{2L^2(t+1)}{\Gamma(\beta)^2}\int_0^t (t-s)^{2\beta-2}\mathbb{E}(\|\psi(s)-\hat{\psi}(s)\|^2)ds.$$

which together with the definition of $\|\cdot\|_\gamma$ as in (3.22) implies that

$$\frac{\mathbb{E}\left(||T_\kappa\psi(t)-T_\kappa\hat{\psi}(t)||^2\right)}{E_{2\beta-1}(\gamma t^{2\beta-1})}\le \frac{2L^2(t+1)}{\Gamma(\beta)^2}\frac{\int_0^t (t-s)^{2\beta-2}E_{2\beta-1}(\gamma s^{2\beta-1})\,ds}{E_{2\beta-1}(\gamma t^{2\beta-1})}\|\psi-\hat{\psi}\|_\gamma^2.$$

In light of Lemma 3.2.2, we have for all $t\in[0,T]$

$$\frac{\mathbb{E}\left(||T_\kappa\psi(t)-T_\kappa\hat{\psi}(t)||^2\right)}{E_{2\beta-1}(\gamma t^{2\beta-1})}\le \frac{2\Gamma(2\beta-1)L^2(T+1)}{\Gamma(\beta)^2\gamma^2}\|\psi-\hat{\psi}\|_\gamma^2.$$

Consequently,

$$\|T_\kappa\psi-T_\kappa\hat{\psi}\|_\gamma\le\kappa\|\psi-\hat{\psi}\|_\gamma,\quad\text{where}\quad \kappa:=\sqrt{\frac{2\Gamma(2\beta-1)L^2(T+1)}{\Gamma(\beta)^2\gamma^2}}.$$

By (3.21), we have $\kappa<1$ and therefore the operator T_κ is a contractive map on $H^2([0,T]),\|\cdot\|_\gamma$. Using the Banach fixed point theorem, there exists a unique fixed point of this map in $H^2([0,T])$. This fixed point is also the unique solution of (3.12) with the initial condition $X(0)=\kappa$. The proof of this part is complete.

(ii) Choose and fix $T>0$ and $\kappa,\zeta\in\mathcal{X}_0$. Since $\varphi(\cdot,\kappa)$ and $\varphi(\cdot,\zeta)$ are solutions of (3.12) it follows that

$$\begin{aligned}
\varphi(t,\kappa)-\varphi(t,\zeta)&=\kappa-\zeta+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(b(s,\varphi(s,\kappa))-b(s,\varphi(s,\zeta)))\,ds\\
&\quad+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(\sigma(s,\varphi(s,\kappa))-\sigma(s,\varphi(s,\zeta)))\,dW_s.
\end{aligned}$$

Hence, using the inequality $\|x+y+z\|^2\le 3(\|x\|^2+\|y\|^2+\|z\|^2)$ for all $x,y,z\in\mathbb{R}^d$, (H1), the Hölder inequality and Itô's isometry (see Part (i)), we obtain

$$\begin{aligned}
\mathbb{E}\left(\|\varphi(t,\kappa)-\varphi(t,\zeta)\|^2\right)&\le\frac{3L^2(t+1)}{\Gamma(\beta)^2}\int_0^t (t-s)^{2\beta-2}\mathbb{E}(\|\varphi(s,\kappa)-\varphi(s,\zeta)\|^2)\,ds\\
&\quad+3\mathbb{E}(\|\kappa-\zeta\|^2).
\end{aligned}$$

By definition of $\|\cdot\|_\gamma$, we have

$$\mathbb{E}\left(\|\varphi(t,\kappa)-\varphi(t,\zeta)\|^2\right)\frac{E_{2\beta-1}(\gamma t^{2\beta-1})}{\Gamma(\beta)^2\gamma^2} \leq \frac{3L^2(t+1)}{\Gamma(\beta)^2}\int_0^t (t-s)^{2\beta-2}E_{2\beta-1}(\gamma s^{2\beta-1})\,ds \times \|\varphi(\cdot,\kappa)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\mathbb{E}(\|\kappa-\zeta\|^2).$$

By virtue of Lemma 3.2.2, we have

$$\|\varphi(\cdot,\kappa)-\varphi(\cdot,\zeta)\|_\gamma^2 \leq \frac{3L^2(T+1)\Gamma(2\beta-1)}{\gamma\Gamma(\beta)^2}\|\varphi(\cdot,\kappa)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\|\kappa-\zeta\|_{\text{ms}}^2.$$

Thus, by (3.21) we have

$$\|\varphi(\cdot,\kappa)-\varphi(\cdot,\zeta)\|_\gamma^2 \leq \frac{3L^2(T+1)\Gamma(2\beta-1)}{\gamma\Gamma(\beta)^2}\|\varphi(\cdot,\kappa)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\|\kappa-\zeta\|_{\text{ms}}^2.$$

Hence,

$$\lim_{\kappa\to\zeta}\sup_{t\in[0,T]}\|\varphi(t,\kappa)-\varphi(t,\zeta)\|_{\text{ms}} = 0.$$

The proof is complete.
We conclude this section with a discussion on the gap in the proof of global existence of solutions for Caputo fractional stochastic differential equation in 3.14.

3.2.2 Exemple

Consider the following caputo fractional stochastic differential equation :

$$\begin{aligned} {}^C D_{0+}^{\frac{4}{5}} X(t) &= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} X(t) + \begin{bmatrix} \sin X_1 \\ X_2+3 \end{bmatrix} + \begin{bmatrix} \cos X_1 \\ \tan X_2 \end{bmatrix} \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t\in(0,1], \\ X(0) &= \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \end{aligned} \tag{3.23}$$

is a Brownian motion with

- $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$
- $W(t)$ is a Brownian motion
- $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$
- $b(t,X(t)) = \begin{bmatrix} \sin X_1 \\ X_2+3 \end{bmatrix}$, $\sigma(t,X(t)) = \begin{bmatrix} \cos X_1 \\ \tan X_2 \end{bmatrix}$. are measurable functions

Then,
by 3.6, the unique solution of 3.23 is :

$$\begin{aligned} X(t) =& E_{\frac{4}{5}} t^{\frac{4}{5}} A\eta + \int_0^t (t-\tau)^{-\frac{1}{5}} E_{\frac{4}{5},\frac{4}{5}}\left((t-\tau)^{\frac{4}{5}}A\right) b(\tau,X(\tau))\mathrm{d}\tau \\ &+ \int_0^t (t-\tau)^{-\frac{1}{5}} E_{\frac{4}{5},\frac{4}{5}}\left((t-\tau)^{\frac{4}{5}}A\right) \sigma(\tau,X(\tau))\mathrm{d}W(\tau). \end{aligned}$$

3.2.3 Exemple

Consider the following caputo fractional stochastic differential equation :

$$\begin{aligned} {}^C\mathcal{D}_{0+}^{\frac{3}{2}} X(t) &= \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} X(t) + \begin{bmatrix} e^{-X_1} \\ X_2^{1/3} \end{bmatrix} + \begin{bmatrix} \sqrt{1+X_1^2} \\ \arctan X_2 \end{bmatrix} \frac{dW(t)}{dt}, \quad t\in(0,1], \\ X(0) &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \end{aligned} \tag{3.24}$$

where :

— $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$

— $W(t)$ is a standard Brownian motion

— $B = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$

— $b(t, X(t)) = \begin{bmatrix} e^{-X_1} \\ X_2^{1/3} \end{bmatrix}$, $\sigma(t, X(t)) = \begin{bmatrix} \sqrt{1+X_1^2} \\ \arctan X_2 \end{bmatrix}$ are measurable functions

Then,
by 3.6, the unique solution of 3.24s :

$$\begin{aligned} X(t) &= E_{\frac{3}{2},1}(t^{\frac{3}{2}}B)\eta \\ &+ \int_0^t (t-\tau)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}\left((t-\tau)^{\frac{3}{2}}B\right) f(\tau, X(\tau))d\tau \\ &+ \int_0^t (t-\tau)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}\left((t-\tau)^{\frac{3}{2}}B\right) g(\tau, X(\tau))dW(\tau). \end{aligned}$$

3.3 Analysis of stochastic neutral fractional functional differential equations

3.3.1 Problem formulation

Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a complete filtered probability space with a family $\{\mathscr{F}_t, t \in [0,T]\}$ of increasing sub- σ-algebras called filtration. The filtration is stated as right continuous if $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all $t \in [0,T]$. Let $\mathbb{X}$ and $\mathbb{H}$ be separable Hilbert spaces. Let $\mathrm{L}(\mathbb{X})$ be the space of all bounded linear operators and $W(t)$ be an $\mathbb{H}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Furthermore, consider the Hilbert space $\mathscr{H}_0 = Q^{1/2}\mathbb{H}$ with the inner product $(X,Y)_0 = \left(Q^{-1/2}X, Q^{-1/2}Y\right)$ for all $X, Y \in \mathscr{H}_0$, and the corresponding norm is denoted by $\|\cdot\|_0$. Let $\mathrm{L_Q}$ be the space of all Hilbert-Schmidt operators from $\mathscr{H}_0$ to $\mathbb{X}$. Also, we denote the expectation with respect to probability $\mathcal{P}$ by $\mathbb{E}$. Consider the stochastic neutral fractional functional differential equation of the form

$$\left\{ \begin{array}{l} {}^C D^\alpha \left(x(t) - g\left(t, x(t)\right)\right) = Ax(t) + f\left(t, x(t)\right) + \sigma\left(t, x(t)\right) \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in [0,T], \\ x(0) = x_0, \quad t \in \mathbb{X} \end{array} \right. \tag{3.25}$$

where $1/2 < \alpha \leq 1$. The solution $x(t,\omega), t \in [0,T], \omega \in \Omega$, represented as $x(t) : \Omega \to \mathbb{X}$, takes values in a real separable Hilbert space $\mathbb{X}$. We represent $x(t) : \Omega \to C_\tau, t \in [0,T]$ by defining $x(t) = \{x(t+\theta) : \theta \in [0,T]\}$, Further, the initial condition x_0. denote the Borel measurable functions which are continuous and satisfy the Lipschitz condition : for all $x_1, x_2 \in \mathbb{X}$ and $t \in [0,T]$, there exist $L_1, L_2 > 0$ such that

$$\|f\left(t, x_1\right) - f\left(t, x_2\right)\|_{\mathbb{X}} \leq L_1\left(\|x_1 - x_2\|_{\mathbb{X}}\right) \tag{3.26}$$

$$\|\sigma\left(t, x_1\right) - \sigma\left(t, x_2\right)\|_{\mathrm{L}_Q} \leq L_2\left(\|x_1 - x_2\|_{\mathbb{X}}\right) \tag{3.27}$$

Also, f and σ satisfy the linear growth condition : for all $x \in C_\tau$ and $t \in [0,T]$, there exist positive constants $L_3, L_4 > 0$ such that

$$\|f(t,x)\|_{\mathbb{X}}^2 \leq L_3\left(1 + \|x\|_{\mathbb{X}}^2\right) \tag{3.28}$$

$$\|\sigma(t,x)\|_{\mathrm{L_Q}}^2 \leq L_4\left(1 + \|x\|_{\mathbb{X}^2}^2\right) \tag{3.29}$$

We impose some hypothesis on the continuous function g as follows : Assume there is a constant $\gamma > 0$ such that, for all $x \in C_\tau$ and $t \in [0,T]$,

$$\|g(t,x)\|_{\mathbb{X}}^2 \leq \gamma^2\left(1 + \|x\|_{\mathbb{X}}^2\right) \tag{3.30}$$

Also, let the function g be a contraction, that is, there exists a constant $\eta \in (0,1)$ such that, for all $x_1, x_2 \in C_\tau$ and $t \in [0,T]$,

$$\|g(t,x_1) - g(t,x_2)\|_{\mathbb{X}} \leq \eta \|x_1 - x_2\|_{\mathbb{X}}. \tag{3.31}$$

We now present some well-known standard definitions in fractional calculus that are used frequently in establishing our results. For $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, we state the following well-known definitions.
For simplicity, the bounds of Mittag-Leffler functions with one and two parameters when acting on the bounded linear operator A of 3.25 are represented as follows :

$$S_1 = \sup_{t\in[0,T]} \|E_\alpha(At^\alpha)\|_{\mathrm{L}(\mathbb{X})}, \quad S_2 = \sup_{t\in[0,T]} \|E_{\alpha,\alpha}(At^\alpha)\|_{\mathrm{L}(\mathbb{X})} \tag{3.32}$$

Our next intention is to find a solution representation of 3.25 based on the approach followed in [36]. In order to find the solution representation, we need the following hypothesis.
(H1) The operator $A \in \mathrm{L}(\mathbb{X})$ commutes with the fractional integral operator I^α on $\mathbb{X}$ and $\|A\|^2_{\mathrm{L}(\mathbb{X})} < \frac{(2\alpha-1)(\Gamma(\alpha))^2}{T^{2\alpha}}$.

Lemma 3.3.1 *[39]*
Suppose that A is a linear bounded operator defined on $\mathbb{X}$, and assume that $\|A\|_{L(\mathbb{X})} < 1$. Then $(I-A)^{-1}$ is linear bounded and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$$

The convergence of the above series is in the operator norm and $\|(I-A)^{-1}\|_{L(\mathbb{X})} \leq (1 - \|A\|_{\mathrm{L}(\mathbb{X})})^{-1}$. We now validate the inequality $\|I^\alpha A\|_{L(\mathbb{X})} < 1$. Then, by the above lemma, we could reach the conclusion : $(I - I^\alpha A)^{-1}$ is bounded and linear. Let $x \in \mathbb{X}$; we have

$$\begin{aligned}
\mathbb{E}\left[\|(I^\alpha A)x\|^2_{C([0,T];L^2(\Omega,\mathbb{X}))}\right] &\leq \frac{T}{(\Gamma(\alpha))^2}\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t (t-s)^{2\alpha-2}\|Ax(s)\|^2_{\mathbb{X}}\,\mathrm{d}s\right] \\
&\leq \frac{T^{2\alpha}}{(2\alpha-1)(\Gamma(\alpha))^2}\mathbb{E}\left[\sup_{t\in[0,T]}\|Ax(t)\|^2_{\mathbb{X}}\right] \\
&< \mathbb{E}\|x\|^2_{C([0,T];L^2(\Omega,\mathbb{X}))}
\end{aligned}$$

by (H1), and hence we get the required inequality. Operating by I^α on both sides of 3.25, we have

$$x(t) = x(0) + g(t,x(t)) - g(0,x_0) + I^\alpha Ax(t) + I^\alpha f(t,x(t)) + I^\alpha \sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}$$

and so

$$x(t) = (I - I^\alpha A)^{-1}\left\{x_0 + g(t,x(t)) - g(0,x_0) + I^\alpha f(t,x(t)) + I^\alpha \sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right\}$$

By means of lemma 3.3.1, we obtain

$$\begin{aligned}
x(t) =& \sum_{k=0}^{\infty} (I^{\alpha}A)^{k} \left\{ x_0 + g(t,x(t)) - g(0,x_0) + I^{\alpha} f(t,x(t)) + I^{\alpha}\sigma(t,x(t)) \frac{\mathrm{d}W(t)}{\mathrm{d}t} \right\} \\
=& \sum_{k=0}^{\infty} I^{k\alpha}A^{k}[x_0 - g(0,x_0)] + g(t,x(t)) + \sum_{k=0}^{\infty} I^{k\alpha+\alpha}A^{k} \left\{ f(t,x(t)) + \sigma(t,x(t)) \frac{\mathrm{d}W(t)}{\mathrm{d}t} \right\} \\
=& \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha)} \int_0^t (t-s)^{k\alpha-1} A^k [x_0 - g(0,x_0)]\mathrm{d}s + g(t,x(t)) \\
&+ \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+\alpha)} \int_0^t (t-s)^{k\alpha+\alpha-1} A^k f(s,x(s))\,\mathrm{d}s \\
&+ \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+\alpha)} \int_0^t (t-s)^{k\alpha+\alpha-1} A^k \sigma(s,x(s))\,\mathrm{d}W(s), \\
x(t) =& \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} [x_0 - g(0,x_0)] + g(t,x(t)) + \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k (t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} f(s,x(s))\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k (t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} \sigma(s,x(s))\,\mathrm{d}W(s).
\end{aligned}$$

The solution representation is

$$\begin{aligned}
x(t) =& E_{\alpha}(At^{\alpha})[x_0 - g(0,x_0)] + g(t,x(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s,x(s))\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \sigma(s,x(s))\,\mathrm{d}W(s).
\end{aligned}$$

Since

$$\mathbb{E} \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \sigma(s,x(s)) \right\|_{\mathrm{L_Q}}^2 \, \mathrm{d}s < \infty \tag{3.33}$$

we can say that the stochastic integral is well defined by (H1) and the Hilbert-Schmidt operator (see, Prato and Zabczyk [15]).

Theorem 3.3.1 *(Gronwall's inequality [33])*
Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded non negative function on $[0,T]$, and let $v(\cdot)$ be a non negative integrable function on $[0,T]$. If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \quad \text{for all } 0 \leq t \leq T$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds \right) \quad \text{for all } 0 \leq t \leq T.$$

Theorem 3.3.2
(Holder's inequality) Assume Υ to be a domain in $\mathbb{R}^n$. Also let $1 < p < \infty$ and p' denote the conjugate exponent defined by

$$p' = \frac{p}{p-1}, \quad \text{that is, } \frac{1}{p} + \frac{1}{p'} = 1$$

which also satisfies $1 < p' < \infty$. If $u \in L^p(\Upsilon)$ and $v \in L^{p'}(\Upsilon)$, then $uv \in L^1(\Upsilon)$ and

$$\int_{\Upsilon} |u(x)v(x)|\mathrm{d}x \leq \left(\int_{\Upsilon} |u(x)|^p \, \mathrm{d}x \right)^{1/p} \left(\int_{\Upsilon} |v(x)|^{p'} \mathrm{d}x \right)^{1/p'}$$

The equality holds if and only if $|u(x)|^p$ and $|v(x)|^{p'}$ are proportional a.e. in Υ.

Lemma 3.3.2 *[33]*
For any $a, b \geq 0$ and $0 < \gamma < 1$, we have

$$(a+b)^2 \leq \frac{a^2}{\gamma} + \frac{b^2}{1-\gamma} \tag{3.34}$$

The following inequality is the generalization of Doob's martingale inequality, which will be useful in our proofs to bound the stochastic integrals.

Theorem 3.3.3 *[33]*
Let $p \geq 2$. Let $v \in L^p(\Omega \times [0,T]; \mathbb{R})$ such that

$$\mathbb{E} \int_0^T |v(s)|^p \, \mathrm{d}s < \infty$$

Then

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_0^t v(s) \mathrm{d}W(s) \right|^p \right) \leq \left(\frac{p^3}{2(p-1)} \right)^{p/2} T^{\frac{p-2}{2}} \mathbb{E} \int_0^T |v(s)|^p \, \mathrm{d}s. \tag{3.35}$$

3.3.2 Existence and uniqueness of solutions

The next lemma points us in the direction of establishing the solution's existence and uniqueness.

Lemma 3.3.3
Let $x(t)$ be the solution of 3.25 for which assumptions 3.28-3.30, 3.32 and (H1) hold. Then

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \|x(t)\|_{\mathbb{X}}^2 \right] \leq c_1 e^{\frac{3T^{2\alpha} S_2^2 (L_3 + 4L_4)}{(1-\gamma)(2\alpha-1)}} \tag{3.36}$$

Moreover, the solution $x(t)$ belongs to $C\left([0,T]; L^2(\Omega, \mathbb{X})\right)$.

Proof
Let τ^m be the stopping time defined as

$$\tau^m = T \wedge \inf \{ t \in [0,T] : \|x(t)\|_{\mathbb{X}} \geq m \}$$

for any $m \geq 1$. Fix $x^m = x(t \wedge \tau^m)$ for $t \in [0,T]$. Then, for $0 \leq t \leq T$, we have

$$\begin{aligned} x^m(t) =& E_\alpha (At^\alpha) [x_0 - g(0, x_0)] + g(t, x(t)^m) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} (A(t-s)^\alpha) f(s, x(s)^m) I_{[[0,\tau^m]]}(s) \mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} (A(t-s)^\alpha) \sigma(s, x(s)^m) I_{[0,\tau^m]]}(s) \mathrm{d}W(s) \end{aligned}$$

Applying Lemma 3.3.2, assumptions 3.28-3.30, 3.32 and Doob's martingale inequality, one can derive that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0 \leq r \leq t} \|x^m(r)\|_{\mathbb{X}}^2 \right] \\ \leq& \gamma \mathbb{E}\left[\sup_{0 \leq r \leq t} \left(1 + \|x^m(r)\|_{\mathbb{X}}^2\right) \right] + \frac{6S_1^2 (1+\gamma^2)}{1-\gamma} \|x_0\|_{\mathbb{X}}^2 \\ &+ \frac{3T^{2\alpha-1} S_2^2 (L_3 + 4L_4)}{(1-\gamma)(2\alpha-1)} \mathbb{E} \int_0^T \sup_{0 \leq r \leq s} \left(1 + \|x^m(r)\|_{\mathbb{X}}^2\right) \mathrm{d}s \end{aligned}$$

Finally, by means of Gronwall's inequality,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|x^m(t)\|_{\mathbb{X}}^2\right]\leq c_1 e^{\frac{4T^{2\alpha}S_2^2\left(\|A\|_{L\mathbb{X}}^2\right)^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}$$

Thus,

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau^m}\|x(t)\|_{\mathbb{X}}^2\right]\leq c_1 e^{\frac{4T^{2\alpha}S_2^2\left(\|A\|_{L\mathbb{X}}^2\gamma^2+L_3+4L_4\right)}{(1-\gamma)(2\alpha-1)}} \tag{3.37}$$

Hence, the required inequality is obtained by letting $m\to\infty$.

Theorem 3.3.4

Let assumptions 3.26-3.31,3.32 and (H1) hold. Then there exists a unique solution $x(t)\in C\left([0,T],L^2(\Omega,\mathbb{X})\right)$ *to system 3.25.*

Proof Uniqueness :

Let $x(t)$ and $\bar{x}(t)$ be the solutions of 3.25 with the initial data $x(t)=x_0(t)$ and $\bar{x}(t)=x_0(t)$ for $t\in[0,T]$. Both the solutions belong to the solution space $C\left([0,T],L^2(\Omega,\mathbb{X})\right)$ by Lemma 3.3.3 . Note that the difference in the solutions is

$$x(t)-\bar{x}(t)=g\left(t,x(t)\right)-g\left(t,\bar{x}_t\right)+\mathcal{J}(t)$$

where

$$\begin{aligned}\mathcal{J}(t)=&\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\left(f\left(s,x(s)\right)-f\left(s,\bar{x}_s\right)\right)\mathrm{d}s\\&+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\left(\sigma\left(s,x(s)\right)-\sigma\left(s,\bar{x}_s\right)\right)\mathrm{d}W(s)\end{aligned}$$

Applying Lemmas 3.3.2 and 3.31, we get

$$\|x(t)-\bar{x}(t)\|_{\mathbb{X}}^2\leq\eta\left\|x(t)-\bar{x}_t\right\|_{\mathbb{X}}^2+\frac{1}{1-\eta}\|\mathcal{J}(t)\|_{\mathbb{X}}^2$$

Therefore,

$$\mathbb{E}\left[\sup_{0\leq u\leq t}\|x(u)-\bar{x}(u)\|_{\mathbb{X}}^2\right]\leq\frac{1}{(1-\eta)^2}\mathbb{E}\left[\sup_{0\leq u\leq t}\|\mathcal{J}(u)\|_{\mathbb{X}}^2\right] \tag{3.38}$$

And one can easily derive that

$$\mathbb{E}\left[\sup_{0\leq u\leq t}\|\mathcal{J}(u)\|_{\mathbb{X}}^2\right]\leq 2\frac{T^{2\alpha-1}}{2\alpha-1}S_2^2\left(L_1^2+4L_2^2\right)\int_0^t\mathbb{E}\left[\sup_{0\leq u\leq s}\|x(s)-\bar{x}(s)\|_{\mathbb{X}}^2\right]\mathrm{d}s$$

Consequently, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq u\leq t}\|x(u)-\bar{x}(u)\|_{\mathbb{X}}^2\right]\\&\quad\leq\frac{3T^{2\alpha-1}S_2^2\left(L_1^2+4L_2^2\right)}{2\alpha-1}\int_0^t\mathbb{E}\left[\sup_{0\leq u\leq s}\|x(u)-\bar{x}(u)\|_{\mathbb{X}}^2\right]\mathrm{d}s\end{aligned}$$

Gronwall's inequality implies

$$\mathbb{E}\left[\sup_{0\leq u\leq t}\|x(u)-\bar{x}(u)\|_{\mathbb{X}}^2\right]=0 \tag{3.39}$$

Therefore, the solutions $x(t)$ and $\bar{x}(t)$ are equal for $0\leq t\leq T$, hence for all $0\leq t\leq T$, almost surely.

Existence : Let us split the existence of the solution into the following two steps.
Step 1 : We consider T is sufficiently small so that it satisfies

$$\rho := \gamma + \frac{3T^{2\alpha} S_2^2 \left(\|A\|_{L(\mathbb{X}}^2 \gamma^2 + L_1^2 + 4L_2^2 \right)}{(1-\gamma)(2\alpha - 1)} < 1. \tag{3.40}$$

Set $x_0^0 = x_0$ and $x^0 = x_0$ for $0 \leq t \leq T$. In addition, let $x_0^n = x_0$ for each $n = 1, 2, 3, \ldots$, and define the Picard iterations as follows :

$$\begin{aligned} x^n(t) =& E_\alpha \left(At^\alpha\right) \left[x_0 - g(0, x_0)\right] + g\left(t, x(t)^{n-1}\right) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^\alpha\right) f\left(s, x(s)^{n-1}\right) \mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-s)^\alpha\right) \sigma\left(s, x(s)^{n-1}\right) \mathrm{d}W(s) \end{aligned} \tag{3.41}$$

It is self-evident that $x^0(t)$ is $\mathscr{F}_t$-measurable and belongs to $C\left([0,T]; L^2(\Omega, \mathbb{X})\right)$. Then, by induction, it is easy to say $x^n(t) \in C\left([0,T]; L^2(\Omega, \mathbb{X})\right)$. Consequently, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}\left\{ \left\|x^0(t)\right\|_{\mathbb{X}}^2 \right\} < \infty$$

Applying Lemma 3.3.2, linear growth conditions 3.28 3.29, 3.32, and Doob's martingale inequality on 3.41, one can derive that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0 \leq t \leq T} \|x^n(t)\|_{\mathbb{X}}^2\right] \\ &\leq \gamma \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(1 + \left\|x(t)^{n-1}\right\|_{\mathbb{X}}^2\right)\right] + \frac{3S_1^2}{(1-\gamma)} \|x_0 - g(0, x_0)\|_{\mathbb{X}}^2 \\ &+ 3 \frac{T^{2\alpha-1}}{(1-\gamma)(2\alpha-1)} S_2^2 L_3 \mathbb{E} \int_0^T \sup_{0 \leq s \leq T} \left(1 + \left\|x(s)^{n-1}\right\|_{\mathbb{X}}^2\right) \mathrm{d}s \\ &+ 12 \frac{T^{2\alpha-1}}{(1-\gamma)(2\alpha-1)} S_2^2 L_4 \mathbb{E} \int_0^T \sup_{0 \leq s \leq T} \left(1 + \left\|x(s)^{n-1}\right\|_{\mathbb{X}}^2\right) \mathrm{d}s \\ &\leq \gamma \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(1 + \left\|x^{n-1}(t)\right\|_{\mathbb{X}}^2\right)\right] + \frac{6S_1^2\left(1+\gamma^2\right)}{1-\gamma} \|x_0\|_{\mathbb{X}}^2 \\ &+ \frac{3T^{2\alpha-1} S_2^2 \left(L_3 + 4L_4\right)}{(1-\gamma)(2\alpha-1)} \mathbb{E} \int_0^T \sup_{0 \leq s \leq T} \left(1 + \left\|x^{n-1}(s)\right\|_{\mathbb{X}}^2\right) \mathrm{d}s. \end{aligned}$$

Note that, for $0 \leq t \leq T$,

$$\begin{aligned} &\mathbb{E}\left[\sup_{0 \leq t \leq T} \left\|x^1(t) - x^0(t)\right\|_{\mathbb{X}}^2\right] \\ &\leq 2\gamma \mathbb{E}\|x_0\|_{\mathbb{X}}^2 + \frac{2T^{2\alpha-1} S_2^2 \left(L_1^2 + 4L_2^2\right)}{(1-\gamma)(2\alpha-1)} \mathbb{E} \int_0^T \left(1 + \left\|x^0(s)\right\|_{\mathbb{X}}^2\right) \mathrm{d}s \\ &\leq 2\gamma \mathbb{E}\|x_0\|_{\mathbb{X}}^2 + \frac{2T^{2\alpha-1} S_2^2 \left(L_1^2 + 4L_2^2\right)}{(1-\gamma)(2\alpha-1)} \left(1 + \mathbb{E}\|x_0\|_{\mathbb{X}}^2\right) T \\ &:= K \end{aligned} \tag{3.42}$$

for $n \geq 1$. Next, we claim that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left\|x^{n+1}(t) - x^n(t)\right\|_{\mathbb{X}}^2\right] \leq K\rho^n \tag{3.43}$$

For any $n \geq 1$,

$$
\begin{aligned}
x^{n+1}(t)-x^n(t)= & g\left(t, x(t)^n\right)-g\left(t, x(t)^{n-1}\right) \\
& +\int_0^t(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(A(t-s)^\alpha\right)\left[f\left(s, x(s)^n\right)-f\left(s, x(s)^{n-1}\right)\right] \mathrm{d} s \\
& +\int_0^t(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(A(t-s)^\alpha\right)\left[\sigma\left(s, x(s)^n\right)-\sigma\left(s, x(s)^{n-1}\right)\right] \mathrm{d} W(s) .
\end{aligned}
$$

Simplifying in the same way as above, we get

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{0 \leq t \leq T}\left\|x^{n+1}(t)-x^n(t)\right\|_{\mathbb{X}}^2\right] \\
& \leq \gamma \mathbb{E}\left[\sup_{0 \leq t \leq T}\left\|x^n(t)-x^{n-1}(t)\right\|_{\mathbb{X}}^2\right] \\
& +\frac{3 T^{2 \alpha-1} S_2^2\left(L_1^2+4 L_2^2\right)}{(1-\gamma)(2 \alpha-1)} \int_0^T \mathbb{E}\left[\sup_{0 \leq s \leq T}\left\|x^n(s)-x^{n-1}(s)\right\|_{\mathbb{X}}^2\right] \\
& \leq \rho \mathbb{E}\left[\sup_{0 \leq t \leq T}\left\|x^n(t)-x^{n-1}(t)\right\|_{\mathbb{X}}^2\right] \\
& \leq \rho^n \mathbb{E}\left[\sup_{0 \leq t \leq T}\left\|x^1(t)-x^0(t)\right\|_{\mathbb{X}}^2\right] \\
& \leq K \rho^n .
\end{aligned} \tag{3.44}
$$

In view of 3.44, we say that 3.43 holds for some $n \geq 0$. Thereupon, by means of Chebyshev's inequality,

$$
\mathcal{P}\left[\sup_{0 \leq t \leq T}\left\|x^{n+1}(t)-x^n(t)\right\|_{\mathbb{X}}^2>\frac{1}{2^n}\right] \leq \frac{1}{\left(1 / 2^n\right)^2} \mathbb{E}\left[\sup_{0 \leq t \leq T}\left\|x^{n+1}(t)-x^n(t)\right\|_{\mathbb{X}}^2\right]
$$

Thus, by applying 3.44 and summing up the resultant inequalities, we get

$$
\sum_{n=0}^{\infty} \mathcal{P}\left[\sup_{0 \leq t \leq T}\left\|x^{n+1}(t)-x^n(t)\right\|_{\mathbb{X}}^2>\frac{1}{2^n}\right] \leq \sum_{n=0}^{\infty} K(4 \rho)^n
$$

Since the sum of series $\sum_{n=0}^{\infty} K(4 \rho)^n$ is finite, using the Borel-Cantelli lemma we can conclude that $\sup_{0 \leq t \leq T}\left\|x^{n+1}(t)-x^n(t)\right\|_{\mathbb{X}}^2$ converges to zero, almost surely. Thus, the Picard iterations $x^n(t)$ converge almost surely to a limit $x(t)$ on $t \in[0, T]$ defined by

$$
\lim_{n \rightarrow \infty}\left[x^0(t)+\sum_{i=0}^{n-1}\left(x^{i+1}(t)-x^i(t)\right)\right]=\lim_{n \rightarrow \infty} x^n(t)=x(t)
$$

From 3.41, we have

$$
\begin{aligned}
x(t)= & E_\alpha\left(A t^\alpha\right)\left[x_0-g\left(0, x_0\right)\right]+g(t, x(t))+\int_0^t(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(A(t-s)^\alpha\right) f(s, x(s)) \mathrm{d} s \\
& +\int_0^t(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(A(t-s)^\alpha\right) \sigma(s, x(s)) \mathrm{d} W(s)
\end{aligned} \tag{3.45}
$$

Step 2 : We now eliminate condition 3.40. Take $\delta>0$ to be sufficiently small for

$$
\gamma+\frac{2 \delta^{2 \alpha} S_2^2\left(L_1^2+4 L_2^2\right)}{(1-\gamma)(2 \alpha-1)}<1 \tag{3.46}
$$

In consequence, there exists a solution on $[0, \delta]$ to system 3.25 by performing step 1 . Let us now consider system 3.25 on $[\delta, 2 \delta]$ with the initial condition x_δ. Again by step 1 , there exists a solution on $[\delta, 2 \delta]$. Subsequently, we repeat step 1 until the existence of solution on the interval $[p \delta, T]$ occurs. Hence, we conclude that there exists a solution on the entire interval $[0, T]$ as desired.

3.3.3 Example

The following examples illustrate the for stochastic neutral fractional differential equation

Example 3.3.1
Consider the following equation :

$$\begin{cases} {}^{C}D^{3/5}(x(t) - \mu x(t)) = -\frac{1}{1+t}x(t) + \frac{1}{1+t}\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in (0,T], \\ x(0) = x_0, \quad t \in [0,T] \end{cases} \tag{3.47}$$

where $W(t)$ is a one-dimensional Brownian motion. Let us take the control to be $v \in L^2([0,T];\mathbb{R})$, and so the corresponding controlled equation is

$$\begin{cases} {}^{C}D^{3/5}(z^v - \mu z^v) = -\frac{1}{1+t}z^v(t) + \frac{1}{1+t}v(t), \quad t \in (0,T], \\ z^v(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{3.48}$$

where

$$z^v(t) = x_0 + \frac{1}{\Gamma(3/5)}\int_0^t \frac{(t-s)^{2/5}}{(1+s)} z^v(s)\mathrm{d}s + \frac{1}{\Gamma(3/5)}\int_0^t \frac{(t-s)^{2/5}}{(1+s)} v(s)\mathrm{d}s$$

is the unique solution of 3.48.

Example 3.3.2
Consider the stochastic neutral fractional differential equation with multiplicative noise

$$\begin{cases} {}^{C}D^{2/3}(x(t) - \lambda x(t)) = -\frac{1}{1+t}x(t) + (3 + \sin x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in (0,T], \\ x(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{3.49}$$

where infimum over an empty set is taken as ∞ and where $z^\nu(t)$, the solution of the equation

$$z^\nu(t) = x_0 + \frac{1}{\Gamma(2/3)}\int_0^t (t-s)^{1/3}\frac{1}{1+s} z^\nu(s)\mathrm{d}s + \frac{1}{\Gamma(2/3)}\int_0^t (t-s)^{1/3}\left(3 + \sin z^\nu(s)\right) v(s)\mathrm{d}s,$$

is the unique solution of an appropriate controlled system of 3.49.

CONCLUSION

In this paper, we studied some fractional differential equations of the caputo type. we then extended them by adding a stochastic term, transforming them into fractional stochastic differential equations of the caputo type. We proved the existence and uniqueness of solutions using different methods. additionally, we examined the solutions of neutral fractional stochastic functional differential equations and established the existence of a unique solution under specific conditions.
As future prospects, these equations can be further developed and explored using numerical methods.

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شهادة ترخيص بالتصحيح والإيداع

أنا الأستاذ: زيتوني محمد

الرتبة: استاذ مساعد قسم ب

بصفتي المشرف المسؤول عن تصحيح **مذكرة التخرج ماستر** المعنونة بـ :

A fractional stochastic differential problems with applications

من إنجاز الطالب :
عمر داودي

التي نوقشت بتاريخ: الاثنين 23/06/2025

أشهد أن الطالب قد قام بالتعديلات والتصحيحات المطلوبة من طرف لجنة المناقشة وقد تم التحقق من ذلك من طرفنا وقد استوفت جميع الشروط المطلوبة.

إمضاء مسؤول عن التصحيح

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