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# Master's Degree

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## Theme

# Some qualitative results for a new class of fractional integral equations

**Presented by :**

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**In front of the jury composed of :**

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## Dedication:

All praise is due to Allah, by whose grace good deeds are completed. I thank Him for His countless blessings and for granting me strength along the way.

I dedicate this humble work **to myself**, in recognition of every moment of patience, struggle, and perseverance that brought me here.

To my **beloved parents**,
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# Abstract

In this memory, we focus on studying the issue of the existence, uniqueness and stability of solutions for a coupled systems of nonlinear integral equations under the $\psi$-RiemannLiouville fractional integral in some spaces endowed with vector-valued norms (generalized Banach spaces in the sense of Perov). The desired results are achieved by using a combination of fixed point theorems with vector-valued norms technique as well as convergent to zero matrices. More specifically, we essentially confirm the existence of at least one solution for the suggested problems via Schauder's fixed-point theorem whereas the existence of a unique solution for the underlying systems is proved by Perov's fixed-point theorem. While the concept of the matrices converging to zero is implemented to examine different types of stabilities in the sense of Ulam-Hyers (UH) of the given problems. Finally, some illustrative examples are provided to demonstrate the validity of our theoretical findings.
**Key words and phrases:** Integral Equations, Coupled system, $\psi$-Caputo fractional derivative, fixed-point theorems, existence and uniqueness, Ulam-Hyrs stability, Bielecki norm, Banach space, Generalized Banach space, Vector-Valued Norms.

**AMS Subject Classification :** 26A33, 34A08, 34B15, 45G15.

# Résumé

Dans ce mémoire, nous nous intéressons à l'étude de l'existence, de l'unicité et de la stabilité des solutions pour un système couplé d'équations intégrales non linéaires sous l'intégrale fractionnaire de $\psi$Riemann-Liouville dans certains espaces munis de normes vectorielles (espaces de Banach généralisés au sens de Perov). Les résultats souhaités sont obtenus en utilisant une combinaison de théorèmes du point fixe avec la technique des normes vectorielles ainsi que des matrices convergentes vers zéro. Plus précisément, nous confirmons essentiellement l'existence d'au moins une solution pour les problèmes suggérés via le théorème du point fixe de Schauder, tandis que l'existence d'une solution unique pour les systèmes sous-jacents est prouvée par le théorème du point fixe de Perov. Le concept de matrices convergentes vers zéro est mis en uvre pour examiner différents types de stabilités au sens d'Ulam-Hyers (UH) des problèmes donnés. Enfin, quelques exemples illustratifs sont fournis pour démontrer la validité de nos résultats théoriques.
**Mots clés et expressions :** Équations intégrales, Système couplé, Dérivée fractionnaire de $\psi$-Caputo, théorèmes du point fixe, existence et unicité, stabilité d'Ulam-Hyrs, norme de Bielecki, espace de Banach, espace de Banach généralisé, normes vectorielles.
**Classification AMS :** 26A33, 34A08, 34B15, 45G15.

# تلخيص

في هذه المذكرة, نركز على دراسة مسألة وجود,تفرد واستقرار حلول أنظمة معادلات تكاملية غير خطية مقترنة تحت تكامل $\psi$ ريمان -ليوفيل الكسري في بعض الفضاءات المزودة بمعايير متجهة القيمة (فضاءات باناخ المعممة وفقا لمفهوم بيروف ). يتم تحقيق النتائج المرجوة باستخدام مزيج من نظريات النقطة الثابتة مع تقنية المعايير المتجهة القيمة,بالاضافة الى مصفوفات متقاربة الى الصفر . وبشكل اكثر تحديدا , نا كد وبشكل اساسي وجود حل واحد على الاقل للمسائل المقترحة من خلال نظرية شاودر للنقطة الثابتة ,بينما يثبت وجود حل فريد للانظمة الأساسية من خلال نظرية بيروف للنقطة الثابتة.
بينما يطبق مفهوم تقارب المصفوفات الى الصفر لدراسة أنواع مختلفة من الاستقرار وفقا لمفهوم اولام-هايرز (UH) للمسائل المعطاة .
وأخيرا ,نقدم بعض الأمثلة التوضيحية لاثبات صحة نتائجنا النظرية.

**الكلمات المفتاحية:** المعادلات التكاملية ,النظام المقترن,المشتقة الكسرية,نظريات النقطة الثابتة ,الوجود والتفرد,استقرار اولم-هايرز, معيار بيليكي, فضاء باناخ,فضاء باناخ المعمم,معايير المتجهات.

**تصنيف موضوع AMS :** 45G15، 34B15, 34A08, 26A33,

# Contents

# List of symbols

We use the following notations throughout this memory:

## Acronyms

- FODs: fractional ordinary differential equations.
- FDEs: Fractional differential equations.
- ODE:ordinary differential equation.
- MLF: Mittag-Leffler function.
- GBS: generalized Banach space.
- UH: Ulam–Hyers stability.
- FPT: fixed point theorem.

## Notation

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{R}^n$ : Space of $n$-dimensional real vectors.
- $\sup$ : Supremum
- max : Maximum
- $n!$ : Factorial $(n), n \in \mathbb{N}$ : The product of all the integers froml to $n$.
- $\Gamma(\cdot)$ : Gamma function.
- $B(\cdot)$ : Beta function.
- $\mathbb{E}_{\alpha}(.)$:Mittag-Leffler function
- $I_{a^+}^{\alpha,\psi}$ : The fractional $\psi$-integral of order $\alpha > 0$.
- $f_{a^+}^{\alpha}$ : The Riemann-Liouville fractional integral of order $\alpha > 0$.
- ${}^{RL}D_{a^+}^{\alpha}$ : The Riemann-Liouville fractional derivative of orde $\alpha > 0$.
- ${}^{C}D_{a^+}^{\alpha}$ : The Caputo fractional derivative of orde $\alpha > 0$.
- $C(J, \mathrm{E})$ : Space of continuous functions on $J$.
- $AC(J, \mathrm{E})$ : Space of absolutely continuous functions on $J$.
- $L^1(J, \mathbb{R})$ : space of Lebesgue integrable functions on $J$.
- $L^p(J, \mathbb{E})$ : space of measurable functions $u$ with $|u|^p$ belongs to $L^1(J, \mathbb{R})$.
- $L^{\infty}(J, \mathrm{E})$ : space of functions $u$ that are essentially bounded on $J$.

# Introduction

The origins of fractional calculus trace back to a letter dated September 30, 1695, in which the German mathematician Gottfried Wilhelm Leibniz posed a curious question to the Marquis de l'Hôpital: what would happen if the order of a derivative were a fraction rather than an integer? This question marked the formal beginning of what would later be known as *fractional calculus*. Over the 18th and 19th centuries, mathematicians such as Euler, Laplace, Fourier, Liouville, and Riemann made significant contributions toward defining and understanding fractional integrals and derivatives. In particular, Liouville and Riemann independently developed definitions of fractional integrals in the mid-1800s, leading to what is now known as the **Riemann–Liouville fractional integral and derivative** [23, 22].

Despite its early development, fractional calculus remained a mathematical curiosity for over two centuries. It was not until the mid-20th century that the theory began to gain traction in the context of applied problems. The introduction of the **Caputo fractional derivative** in the 1960s by Michele Caputo [7] marked a turning point, as it aligned better with physical initial conditions in real-world systems, particularly in viscoelasticity and diffusion processes. Since then, fractional calculus has evolved into a rich and active field of research, with applications spanning physics, biology, control theory, signal processing, and finance [27, 17].

In recent decades, *fractional differential equations* (FDEs) have emerged as robust tools for modeling complex systems with **memory and hereditary characteristics**. Unlike classical differential models, FDEs accommodate nonlocality by involving weakly singular convolution kernels of the form $(t-\tau)^{-\alpha}$, where $0 < \alpha < 1$. This feature enables them to capture anomalous diffusion, relaxation, and long-range dependence behaviors more accurately than integer-order models.

Solving fractional differential equations analytically, however, presents significant challenges due to the nonlocal and often nonlinear nature of the fractional operators involved. This has led to the increasing use of functional analysis tools, particularly those based on fixed point theory, to establish existence, uniqueness, and stability results for solutions of FDEs.

At the heart of these analytical tools lies the concept of a Banach space, a complete normed vector space that provides a natural framework for discussing convergence and continuity. However, with the increasing complexity of modern problems, especially in nonlinear and infinite-dimensional settings classical Banach spaces may no longer be adequate. To overcome these limitations, mathematicians have developed the concept of generalized Banach spaces, which extend the classical theory through the use of vector valued norms, cone structures, and matrix based convergence criteria.

One important class of such generalizations involves vector-valued norms, where the norm of an element is not a scalar but a vector in an ordered Banach space. This allows for the analysis of problems involving multiple interacting components or anisotropic behavior. Another fruitful approach introduces matrix norms, particularly those based on convergent-to-zero matrices, to define more refined notions of contractivity. These generalized norms enable the extension of classical fixed point theorems such as Banach's contraction principle, Schauder's theorem, and Krasnoselskii's theorem to a broader class of operators that arise naturally in the study of fractional differential and integral equations.

In particular, fixed point theory in generalized Banach spaces has proven to be a powerful framework for addressing the question of whether a given nonlinear operator equation, such as those arising from fractional differential equations, has a solution, whether that solution is unique, and how it behaves under small perturbations. These questions are central to ensuring that the mathematical models are not only solvable but also well-posed and stable.

A particularly important notion in this regard is the concept of Ulam–Hyers stability, which investigates whether a function that approximately satisfies a functional equation is necessarily close to a true solution. In the context of fractional differential equations, such stability results help ensure that numerical or approximate solutions remain reliable, even under uncertain data or modeling imperfections.

In this work, we focus on the existence and uniqueness of solutions for certain systems of nonlinear fractional integral equations in some spaces endowed with vector-valued norms (generalized Banach spaces in the sense of Perov). While the concept of the matrices converging to zero is implemented to examine different types of stabilities in the sense of Ulam-Hyers (UH) of the given problems. The organization of our work is outlined below:

- The first chapter summarizes some basic definitions, helpful lemmas, and theorems that are necessary for proving our main outcomes.

- The second chapter deals with some existence, uniqueness and stability results for the following problem:

$$\begin{cases} u(t)-\theta_1 &= \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} F_1(s,u(s),v(s))ds \\ &\quad + \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_2(s,u(s),v(s))ds \\ v(t)-\theta_2 &= \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1-1}}{\Gamma(\beta_1)} G_1(s,u(s),v(s))ds \\ &\quad + \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} G_2(s,u(s),v(s))ds \end{cases}, \quad t \in \mathrm{J}. \tag{1}$$

  where $T > 0, \alpha_i, \beta_i \in (0,1], \theta_1, \theta_2 \in \mathbb{R}, i = 1,2,$ and the nonlinear functions involved in the above system satisfy certain conditions that will be specified hereafter.

- In the third chapter, we give a generalization of the previous system. More precisely, we focus on the problem of the existence and uniqueness of solutions for the following system:

$$\begin{cases} u(x,y) = a_1(x,y) + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f_1(s,t,u(s,t),v(s,t))\mathrm{dtds} \\ \qquad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} f_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ v(x,y) = a_2(x,y) + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\gamma_1-1}(y-t)^{\gamma_2-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} g_1(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ \qquad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\gamma_1+\delta_1-1}(y-t)^{\gamma_2+\delta_2-1}}{\Gamma(\gamma_1+\delta_1)\Gamma(\gamma_2+\delta_2)} g_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \end{cases} (x,y)\in\tilde{\mathrm{I}} \tag{2}$$

  where $\tilde{I} := [0,\ \mathrm{T}_1]\times[0,\ \mathrm{T}_2]\ ,\ \mathrm{T}_1,\ \mathrm{T}_2 > 0, (\alpha_1,\alpha_2), (\beta_1,\beta_2), (\gamma_1,\gamma_2), (\delta_1,\delta_2) \in (0,1]\times(0,1],$ and $a_1, a_2 : \tilde{\mathrm{I}} \to \mathbb{R}, f_1, f_2, g_1, g_2 : \tilde{\mathrm{I}}\times\ \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

- Finally, our work closes with conclusions and some possible future work.

# Chapter 1

# Preliminaries

## 1.1 Functional spaces

### 1.1.1 introdiction

functional spaces are used to study different types of functions. These spaces allow us to understand how functions behave, especially when we deal with equations involving functions like differential or integral equations.

We will now introduce three important functional spaces:

Let $\mathbb{E}$ be a Banach space endowed with the norm $\|.\|_{\mathbb{E}}$ and let $\mathrm{J} := [a, b]$ be a compact interval of $\mathbb{R}$. We present some functional spaces:

### 1.1.2 Space of Continuous Functions

**Definition 1** *Let $C(\mathrm{J}, \mathbb{E})$ be the Banach space of vector-valued continuous functions $u :$ $J \longrightarrow \mathbb{E}$, equipped with the norm*

$$\|u\|_{\infty} = \sup\{\|u(t)\| \, / t \in J\}$$

*Analogously, $C^n(\ \mathrm{J}, )$ is the Banach space of functions $u : \mathrm{J} \longrightarrow \mathbb{E}$, where $u$ is $n$ time continuously differentiable on J.*

$$\|f\|_{C^n} := \sum_{k=0}^{n} \| \, f^{(k)} \big\|_{C} := \sum_{k=0}^{n} \max_{t \in J} \left| f^{(k)}(t) \right|, n \in \mathbb{N}$$

*In particulier if $n = 0, \quad C^{\circ}(\mathrm{J}, \mathbb{E}) \equiv C(\mathrm{J}, \mathbb{E})$.*

### 1.1.3 Spaces of Lebesgue's Integrable Functions $L^p$

Denote by $L^1(\mathrm{J}, \mathbb{R})$ the Banach space of functions $u$ Lebesgues integrable with the norm

$$\|x\|_{L^1} = \int_{\mathrm{J}} |x(r)| dt$$

while $L^p(\mathrm{J}, \mathbb{R})$ denote the space of Lebesgue integrable functions on J where $|u|^p$ belongs to $L^1(\mathrm{J}, \mathbb{R})$, endowed with the norm

$$\|u\|_{L^p} = \left[ \int_0^T |u(t)|^p \mathrm{dt} \right]^{\frac{1}{p}}, \quad 1 < p < \infty$$

In particular, if $p = \infty, L^{\infty}(\mathrm{J}, \mathbb{R})$ is the space of all functions $u$ that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty} = \operatorname{ess\,sup}_{t\in \mathrm{J}} |u(t)| = \inf\{c \geq 0 : |u(t)| \leq c \text{ for a.e. } t\}.$$

.

### 1.1.4 Spaces of Absolutely Continuous Functions

**Definition 2** *A function $u : \mathrm{J} \to \mathbb{E}$ is said to be absolutely continuous on J if for all $\varepsilon > 0$ there exists a number $\delta > 0$ such that ; for all finite partitions $[a_i, b_i]_{i=1}^n \subset \mathrm{J}$ then $\sum_{k=1}^n (b_k - a_k) < \delta$ implies that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$*

*We denote by $AC(\mathrm{J}, \mathbb{E})$ ( or $AC^1(\mathrm{J}, \mathbb{E})$) the space of all absolutely continuous functions defined on J . It is known that $AC(\mathrm{J}, \mathbb{E})$ coincides with the space of primitives of Lebesgue summable functions:*

$$u \in AC(\mathrm{J}, \mathbb{E}) \Leftrightarrow u(t) = c + \int_a^t \phi(s)ds, \quad \phi \in L^1(\mathrm{J}, \mathbb{R}) \tag{1.1}$$

*and therefore an absolutely continuous function $u$ has a summable derivative $u'(t) = \phi(t)$ almost everywhere on $\mathrm{J}$. Thus ((1.1) ) yields*

$$u'(t) = \phi(t) \text{ and } c = u(0).$$

.

**Definition 3** *For $n \in \mathbb{N}^*$ we denote by $AC^n(\mathbf{J}, \mathbb{E})$ the space of functions $u : \mathrm{J} \longrightarrow \mathbb{E}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$ belongs to $AC(\mathrm{J}, \mathrm{E})$ :*

$$\begin{aligned} AC^n(\mathrm{J}, \mathbb{E}) &= \left\{u \in C^{n-1}(\mathrm{J}, \mathbb{E}) : u^{(n-1)} \in AC(\mathrm{J}, \mathbb{E})\right\} \\ &= \left\{u \in C^{n-1}(\mathrm{J}, \mathbb{E}) : u^{(n)} \in L^1(\mathrm{J}, \mathbb{E})\right\} \end{aligned}$$

*The space $AC^n(\mathrm{J}, \mathbb{E})$ consists of those and only those functions $u$ which can be represented in the form*

$$u(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} \phi(s)ds + \sum_{k=0}^{n-1} c_k t^k \tag{1.2}$$

*where $\phi \in L^1(J, \mathbb{R}), c_j(k = 1, \ldots, n-1) \in \mathbb{R}$. It follows from (1.2) that*

$$\phi(t) = u^{(n)}(t) \text{ and } c_k = \frac{u^{(k)}(0)}{k!}, (k = 1, \ldots, n-1)$$

*and*

$$AC^n_\delta([a;b], \mathbb{E}) = h : [a;b] \to \mathbb{E} : \delta^{n-1}h(t) \in AC([a;b], \mathbb{E}).$$

*where $\delta = t\dfrac{d}{dt}$ is the Hadamard derivative.*
*Interested reader can find more details in [16, 17].*

## 1.2 Special Functions

Before introducing the basic facts on fractional operators, we recall two types of functions that are important in Fractional Calculus: the Gamma and Beta functions. Some properties of these functions are also recalled. More details about these functions can be found in [26, 14, 27].

### 1.2.1 Gamma function

The Euler Gamma function is an extension of the factorial function to real numbers and is considered the most important Eulerian function used in fractional calculus because it appears in almost every fractional integral and derivative definitions.

**Definition 4 ([26, 14])** *The Gamma function, or second order Euler integral, denoted* $\Gamma(\cdot)$ *is defined as:*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \mathrm{dt}, \quad \alpha > 0.. \tag{1.3}$$

*For positive integer values* $n$*, the Gamma function becomes* $\Gamma(n) = (n-1)!$ *and thus can be seen as an extension of the factorial function to real values.*

***Proposition 1*** *The basic properties of the Gamma function are:*

1. *The function* $\Gamma(\alpha)$ *is continuos for* $\alpha > 0$.
2. *The integral* (1.3) *is convergent for* $\alpha > 0$ *and divergent for* $\alpha \leq 0$.
3. *An important property of the gamma function* $\Gamma(\alpha)$ *is that it satisfies :*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0.$$

4. *The following relations are also valid:*

$$\begin{aligned} \Gamma(n+1) &= n!, \quad \forall n \in \mathbb{N}, \\ \Gamma(1) &= 1, \\ \Gamma(0) &= +\infty. \end{aligned}$$

5. *Taking account that the* $\Gamma$ *function can be written as* $\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$*, it results that the* $\Gamma$ *function can be defined also for negative values of* $\alpha$*, in the interval* $-1 < \alpha < 0$.
6. *The following particular values for* $\Gamma$ *function can be useful for calculation purposes:*

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}, \\ \Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi}, \\ \Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}, \\ \Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}. \end{aligned}$$

### 1.2.2 Beta Function

**Definition 5 ([26, 14])** *The Beta function, or the first order Euler function, can be defined as*

$$\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \mathrm{dt}, \quad p,q > 0.$$

In the following we will enumerate the basic properties of the Beta function:

**Proposition 2** *1. The following formula which expresses the Beta function in terms of the Gamma function:*

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q > 0.$$

2. For every $p > 0$ and $q > 0$, we have:

$$\mathrm{B}(p, q) = \mathrm{B}(q, p).$$

3. For every $p > 0$ and $q > 1$, the Beta function $\mathrm{B}$ satisfies the property:

$$\mathrm{B}(p, q) = \frac{q-1}{p+q-1}\mathrm{B}(p, q-1).$$

4. For any natural numbers $m, n$ we obtain:

$$\mathrm{B}(m, n) = \frac{(n-1)!(m-1)!}{(n+m-1)!}.$$

### 1.2.3 Mittag-Leffler Function

**Definition 6** *[14] For $\alpha > 0$ and $z \in \mathbb{R}$, the one-parameter Mittag-Leffler function (MLF) is defined as follows:*

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

*Especially, when $\alpha = 1$ the one-parameter Mittag-Leffler function coincides with the exponential function, that is*

$$\mathbb{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

*Moreover, the one-parameter MLF plays an important role in solving fractional ordinary differential equations (FODEs). Indeed, as $u(t) = u_a e^{-\lambda t}$ is the unique solution of the ordinary differential equation (ODE)*

$$\begin{cases} u'(t) + \lambda u(t) = 0, t > a, \\ u(a) = u_a, \end{cases}$$

*so the MLF*

$$u(t) = u_a \mathbb{E}_\alpha \left(-\lambda(\psi(t) - \psi(a))^\alpha\right), t > a, \alpha \in (0, 1),$$

*solves the homogeneous linear FODE with constant coefficients*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} u(t) + \lambda u(t) = 0, t > a, \\ u(a) = u_a, \end{cases}$$

*where ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}$ represents the $\psi$-Caputo fractional derivative. The previous equation was studied by Almeida [3]. It has been used to model some population growth and the proof of its solution is obtained by using the standard technique of successive approximation.*

**Remark 1** *It is important to note that several generalizations of the one-parameter MLF are available in the literature; such as a two-parameter Mittag-Leffler function, three-parameter Mittag-Leffler function, and the Mittag-Leffler function for matrix arguments. They have been used to solve some FODEs. But here we are not interested in them.*

**Lemma 1** *[14] Let $\alpha \in (0, 1)$ and $z \in \mathbb{C}$. Among the numerous properties of the $MLF$, we mention that*

*(1) The function $\mathbb{E}_\alpha$ is nonnegative,*

*(2) $\mathbb{E}_\alpha(0) = 1$*

*(3) $\mathbb{E}_\alpha(\cdot)$ is an increasing function on $\mathbb{R}_+$.*

## 1.3 Fractional Calculus

Fractional calculus is a generalization of classical calculus, where the concepts of differentiation and integration are extended to non-integer (fractional) orders. Instead of taking the first, second, or $n$-th derivative of a function, we can define derivatives of arbitrary real or complex order.

This theory dates back to the $17^{\text{th}}$ century and has gained much attention in recent decades due to its applications in various fields such as physics, control theory, biology, and engineering. Fractional differential equations, which involve derivatives of fractional order, are particularly useful in modeling processes with memory and hereditary properties.

Several definitions of fractional derivatives exist, such as the Riemann–Liouville and Caputo derivatives. Among them, the Caputo derivative is commonly used in physical models because it allows for initial conditions to be expressed in the same form as in classical differential equations.

### 1.3.1 $\psi$ Riemann–Liouville Fractional Integral

We denote by $C(\mathbb{J},\mathbb{R})$ the set of all real-valued continuous functions defined on the interval $\mathbb{J}$, and by $AC(\mathbb{J},\mathbb{R})$ the space of absolutely continuous functions on $\mathbb{J}$. The space $C^1(\mathbb{J},\mathbb{R})$ consists of all functions $u:\mathbb{J}\to\mathbb{R}$ that have a continuous first derivative on $\mathbb{J}$.

Moreover, $C([a,b],\mathbb{R}^n)$ refers to the Banach space of all continuous mappings $\varpi:[a,b]\to\mathbb{R}^n$, equipped with the norm:

$$\|\varpi\| = \sup_{\zeta\in[a,b]} \|\varpi(\zeta)\|.$$

For vectors $\varpi=(\varpi_1,\dots,\varpi_n)$ and $\omega=(\omega_1,\dots,\omega_n)$ in $\mathbb{R}^n$, we write $\varpi\le\omega$ to mean $\varpi_i\le\omega_i$ for all $i=1,\dots,n$, and $\varpi\le c$ (for $c\in\mathbb{R}$) means $\varpi_i\le c$ for all components.

The non-negative cone in $\mathbb{R}^n$ is defined by:

$$\mathbb{R}^n_+ = \{\varpi\in\mathbb{R}^n : \varpi_i\ge 0,\ i=1,\dots,n\}.$$

We also use the notations:

$$|\varpi| = (|\varpi_1|,\dots,|\varpi_n|),\quad \max(\varpi,\omega) = (\max(\varpi_1,\omega_1),\dots,\max(\varpi_n,\omega_n)).$$

The space $AC(\mathbb{J},\mathbb{R})$ is precisely the collection of functions that are primitives of Lebesgue integrable functions. That is,

$$u\in AC(\mathbb{J},\mathbb{R}) \iff u'(t) \text{ exists a.e. on } \mathbb{J},\ u'\in L^1(\mathbb{J},\mathbb{R}),\ \text{and } u(t)-u(a)=\int_a^t u'(s)\,ds \text{ for all } t\in\mathbb{J}.$$

From now on, let $\psi\in C^1(\mathbb{J},\mathbb{R})$ be a strictly increasing, positive function such that $\psi'(t)>0$ for all $t\in\mathbb{J}$.

We now define certain functional spaces associated with fractional calculus in terms of $\psi$. The space $L^1_\psi(\mathbb{J},\mathbb{R})$ is given by:

$$L^1_\psi(\mathbb{J},\mathbb{R}) = \left\{u:\mathbb{J}\to\mathbb{R}\,\middle|\, u \text{ is measurable and } \int_a^b |u(t)|\psi'(t)\,dt<\infty\right\},$$

and

$$C^1_\psi(\mathbb{J},\mathbb{R}) = \{u\in C(\mathbb{J},\mathbb{R})\,|\,\mathcal{D}_\psi u\in C(\mathbb{J},\mathbb{R})\},$$

where the $\psi$-derivative is defined by:

$$\mathcal{D}_\psi u(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)u(t) = \lim_{h\to 0}\frac{u(t+h)-u(t)}{\psi(t+h)-\psi(t)}.$$

The space $AC_\psi(\mathbb{J},\mathbb{R})$ contains all functions $u$ such that $\mathcal{D}_\psi u$ exists almost everywhere on $\mathbb{J}$, belongs to $L^1_\psi(\mathbb{J},\mathbb{R})$, and satisfies:

$$u(t)-u(a)=\int_a^t \mathcal{D}_\psi u(s)\psi'(s)\,ds \quad \text{for all } t\in\mathbb{J}.$$

For $\alpha\in(0,1)$, we define the space:

$$C^\alpha_\psi(\mathbb{J},\mathbb{R})=\left\{u\in C(\mathbb{J},\mathbb{R})\,\middle|\,{}^c\mathbb{D}^{\alpha;\psi}_{a^+}u\in C(\mathbb{J},\mathbb{R})\right\},$$

where ${}^c\mathbb{D}^{\alpha;\psi}_{a^+}$ stands for the Caputo-type fractional derivative with respect to $\psi$.

**Definition 7** *[2, 17] Let $\alpha>0$. The $\psi$-Riemann-Liouville fractional integral of order $\alpha$ of a function $u\in L^1_\psi(\mathbb{J},\mathbb{R})$ with respect to $\psi$ is defined is defined for a.e. $t$ by*

$$\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)\mathrm{ds}$$

*Moreover, for $\alpha=0$, we set $\mathbb{I}^{\alpha;\psi}_{a^+}u:=u$.*

**Remark 2** *Notice that, for suitably chosen $\psi$, we obtain some well-known definitions of fractional integrals, for example,*

- *The Riemann-Liouville integral [17] when $\psi(t)=t$*
- *the Hadamard integral [17] when $\psi(t)=\ln t$,*
- *the fractional integral with Sigmoid function [20] when $\psi(t)=\frac{1}{1+e^{-t}}$,*
- *the fractional integral with exponential memory [12] when $\psi(t)=e^{-\sigma t}$.*

**Lemma 2** *[2, 3] Let $\alpha,\beta>0$, and $u\in C(\mathbb{J},\mathbb{R})$. Then for each $t\in\mathbb{J}$ we have*

*(1) $\mathbb{I}^{\alpha;\psi}_{a^+}:C(\mathbb{I},\mathbb{R})\to C(\mathbb{J},\mathbb{R})$ is a continuous operator.*

*(2) $\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(a)=\lim_{t\to a^+}\left(\mathbb{I}^{\alpha;\psi}_{a^+}u\right)(t)=0$,*

*(3) $\mathbb{I}^{\alpha;\psi}_{a^+}$ is a linear bounded operator from $C(\mathbb{J},\mathbb{R})$ into $C(\mathbb{J},\mathbb{R})$ and*

$$\left\|\mathbb{I}^{\alpha;\psi}_{a^+}u\right\|_\infty\le\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\|u\|_\infty$$

*(4) $\mathbb{I}^{\alpha;\psi}_{a^+}\mathbb{I}^{\beta;\psi}_{a^+}u(t)=\mathbb{I}^{\beta;\psi}_{a^+}\mathbb{I}^{\alpha;\psi}_{a^+}u(t)=\mathbb{I}^{\alpha+\beta;\psi}_{a^+}u(t)$,*

*(5) $\mathbb{I}^{\alpha;\psi}_{a^+}(\psi(t)-\psi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\psi(t)-\psi(a))^{\alpha+\beta-1}$,*

*(6) $\mathbb{I}^{\alpha;\psi}_{a^+}\left(\mathbb{E}_\alpha\left(\lambda(\psi(t)-\psi(a))^\alpha\right)=\frac{1}{\lambda}\left(\mathbb{E}_\alpha\left(\lambda(\psi(t)-\psi(a))^\alpha-1\right),\quad\lambda>0\right.\right.$.*

Let us now present a special case of the previous integral where $\psi(t)=t$.

**Definition 8** *[2] Let $\alpha>0,\ \mathrm{J}:=[0,\ \mathrm{T}];\mathrm{T}>0$. The Riemann-Liouville (R-L) fractional integral of order $\alpha$ of a function $u\in L^1(\mathbf{J},\mathbb{R})$ is defined for almost everywhere $\mathrm{x}\in\mathrm{J}$ by*

$$\left(\mathbb{I}^\alpha_{0^+}\mathrm{u}\right)(\mathrm{x})=\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathrm{u}(\mathrm{t})\mathrm{dt}.$$

Analogously, we define the mixed R-L fractional integral as follows:

**Definition 9** *[17] Let* $\beta = (\beta_1, \beta_2) \in (0, +\infty) \times (0, +\infty), \tilde{\mathrm{I}} := [0,\ \mathrm{T}_1] \times [0,\ \mathrm{T}_2], \mathrm{T}_1,\ \mathrm{T}_2 > 0$. *The mixed R-L fractional integral of order* $\beta$ *of a function* $u \in L^1(\tilde{\mathrm{I}}, \mathbb{R})$ *is defined for almost everywhere (* $\mathrm{x}, \mathrm{y}$ *)* $\in \tilde{I}$ *by*

$$\left(\mathrm{I}_{0+}^{\beta}\mathrm{u}\right)(\mathrm{x}, \mathrm{y}) = \int_0^{\mathrm{x}} \int_0^{\mathrm{y}} \frac{(\mathrm{x}-\mathrm{s})^{\beta_1 - 1}(\mathrm{y}-\mathrm{t})^{\beta_2 - 1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \mathrm{u}(\ \mathrm{s}, \mathrm{t}) \mathrm{dtd}.$$

The following inequality has an important tool in the forthcoming analysis.

**Lemma 3** *[17] Let* $\alpha, \lambda > 0$. *Then for all* $\mathrm{x} \in \mathrm{J}$ *we have*

$$\frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{x}} (\mathrm{x}-\mathrm{t})^{\alpha-1} e^{\lambda \mathrm{t}} \mathrm{dt} \leq \frac{e^{\lambda \mathrm{x}}}{\lambda^{\alpha}}.$$

**Proof 1** *Actually, using the change of variable* $\tau = \mathrm{x} - \mathrm{t}$ *in the above integral expression we can get*

$$\frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{x}} (\mathrm{x}-\mathrm{t})^{\alpha-1} e^{\lambda \mathrm{t}} \mathrm{dt} = \frac{e^{\lambda \mathrm{x}}}{\Gamma(\alpha)} \int_0^{\mathrm{x}} \tau^{\alpha-1} e^{-\lambda\tau} \mathrm{d}\tau$$

*Using now the variable substitution* $\omega = \lambda\tau$ *in the just-above equation, we can acquire*

$$\begin{aligned}
\mathbb{I}_{0+}^{\alpha} e^{\lambda \mathrm{x}} &= \frac{e^{\lambda \mathrm{t}}}{\Gamma(\alpha)\lambda^{\alpha}} \int_0^{\mathrm{x}} \omega^{\alpha-1} e^{-\omega} \mathrm{d}\omega \\
&\leq \frac{e^{\lambda \mathrm{x}}}{\Gamma(\alpha)\lambda^{\alpha}} \int_0^{\infty} \omega^{\alpha-1} e^{-\mathrm{v}}\ \mathrm{d}\omega \\
&= \frac{e^{\lambda \mathrm{x}}}{\lambda^{\alpha}}
\end{aligned}$$

*As a result, we finish this proof.*
*Repeating the same procedure as in the proof of Lemma* (3), *one gets the following inequality:*

$$\int_0^{\mathrm{x}} \int_0^{\mathrm{y}} \frac{(\mathrm{x}-\mathrm{s})^{\beta_1-1}(\mathrm{y}-\mathrm{t})^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} e^{\lambda(\mathrm{s}+\mathrm{t})} \mathrm{dtd} \leq \frac{e^{\lambda(\mathrm{x}+\mathrm{y})}}{\lambda^{\beta_1+\beta_2}}.$$

### 1.3.2 Caputo-type fractional derivative

**Definition 10** *[2] Let* $\alpha > 0, n \in \mathbb{N}, I$ *is the interval* $-\infty \leq a < b \leq \infty, f, \psi \in C^n(I)$ *two functions such that* $\psi$ *is increasing and* $\psi'(x) \neq 0$, *for all* $x \in I$. *The left* $\psi$*-Caputo fractional derivative of* $f$ *of order* $\alpha$ *is given by*

$${}^{C}D_{a+}^{\alpha,\psi} f(x) := I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x)$$

*and the right* $\psi$*-Caputo fractional derivative of* $f$ *by*

$${}^{C}D_{b-}^{\alpha,\psi} f(x) := I_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x)$$

*where*

$$\mathrm{n} = [\alpha] + 1 \textit{ for } \alpha \notin \mathbb{N}, \quad \mathrm{n} - \alpha \textit{ for } a \in \mathbb{N}.$$

*To simplify notation, we will we the abbreviated symbol*

$$f_{\psi}^{[n]} f(x) := \left(\frac{1}{\psi(x)} \frac{d}{dx}\right)^{\mathrm{n}} f(x)$$

*From the definition, it Is clear that, given* $\alpha = \mathrm{m} \in \mathbb{N}$,

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = f_\psi^{[\mathrm{m}]}(x) \quad \text{and} \quad {}^{C}D_{b-}^{\alpha,\psi} f(x) = (-1)^{\mathrm{m}} f_\psi^{[\mathrm{m}]}(x) $$

*and If* $\alpha \in \mathbb{N}$, *then*

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_\psi^{[n]}(t)dt $$

*and*

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1}(-1)^n f_\psi^{[n]}(t)dt $$

*In particular, when* $\alpha \in (0,1)$, *we have*

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x)-\psi(t))^{-\alpha} f'(t)dt $$

*and*

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (\psi(t)-\psi(x))^{-\alpha} f'(t)dt $$

For some special cases of $\psi$, we obtain the Caputo fractional derivative [31], the Caputo-Hademard fractional derivative [13, 15] and the Caputo–Erdélyi–Kober fractional derivative [21]. From now on, we will restrict to the case $\alpha \notin \mathbb{N}$ and we study some features of this $\psi$-Caputo type fractional derivative. Also, to be concise, we will prove the results only for the left fractional derivative, since the methods are similar for the right fractional derivatives, doing the necessary adjustments.

**Theorem 1** *[2] Suppose that* $f, \psi \in C^{n+1}[a,b]$ . *Then, for all* $\alpha > 0$,

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{(\psi(x)-\psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)} f_\psi^{[n]}(a) + \frac{1}{\Gamma(n+1-\alpha)} \int_a^x (\psi(x)-\psi(t))^{n-\alpha} \frac{d}{dx} f_\psi^{[n]}(t)dt $$

*and*

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = (-1)^n \frac{(\psi(b)-\psi(x))^{n-\alpha}}{\Gamma(n+1-\alpha)} f_\psi^{[n]}(b) - \frac{1}{\Gamma(n+1-\alpha)} \int_x^b (\psi(t)-\psi(x))^{n-\alpha} ({}_1)^n \frac{d}{dx} f_\psi^{[n]}(t)dt $$

**Theorem 2** *[2] The* $\psi$*-Caputo fractional derivatives are bounded operators. For all* $\alpha > 0$,

$$ \left\| {}^{C}D_{a+}^{\alpha,\psi} f \right\|_C \leq K \|f\|_{C_\psi^{[n]}} \quad \text{and} \quad \left\| {}^{C}D_{b-}^{\alpha,\psi} f \right\|_C \leq K \|f\|_{C_\psi^{[n]}} $$

*where*

$$ K = \frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)} $$

**Theorem 3** *[2] If* $f \in C^n[a,b]$ *and* $\alpha > 0$, *then*

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = D_{a+}^{\alpha,\psi} \left[ f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} (\psi(x)-\psi(a))^k f_\psi^{[k]}(a) \right] $$

*and*

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = D_{b-}^{\alpha,\psi} \left[ f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} (\psi(b)-\psi(x))^k f_\psi^{[k]}(b) \right] $$

**Lemma 4** *[2] Given $\beta \in \mathbb{R}$, consider the functions*

$$f(x) = (\psi(x) - \psi(a))^{\beta-1} \quad \text{and} \quad g(x) = (\psi(b) - \psi(x))^{\beta-1}$$

*where $\beta > n$. Then, for $\alpha > 0$,*

$${}^C D_{a+}^{\alpha,\psi} f(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(x) - \psi(a))^{\beta-\alpha-1} \quad \text{and} \quad {}^C D_{b-}^{\alpha,\psi} g(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(b) - \psi(x))^{\beta-\alpha-1}.$$

**Proof 2** *Since*

$$f_\psi^{[\mathrm{n}]}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-n)}(\psi(x) - \psi(a))^{\beta-\mathrm{n}-1}$$

*we have*

$$\begin{aligned} {}^C D_{a+}^{\alpha,\psi} f(x) =& \frac{\Gamma(\beta)}{\Gamma(n-\alpha)\Gamma(\beta-n)}(\psi(x) - \psi(a))^{n-\alpha-1} \\ &\times \int_a^z \psi'(t)\left(1 - \frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}\right)^{n-\alpha-1} (\psi(t) - \psi(a))^{\beta-n-1} dt. \end{aligned}$$

*With the change of variables $u = (\psi(t) - \psi(a))/(\psi(x) - \psi(a))$, and with the help of the Beta function*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0$$

*we obtain*

$${}^c D_{a+}^{\alpha,\psi} f(x) = \frac{\Gamma(\beta)}{\Gamma(n-\alpha)\Gamma(\beta-n)}(\psi(x) - \psi(a))^{\beta-\alpha-1} B(n-\alpha, \beta-n).$$

*Using the following property of the Beta function*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

*we prove the formula.*
*For example, for $f(x) = (\psi(x) - \psi(0))^2$, we have ${}^C D_{0+}^{\alpha,\psi} f(x) = 2/\Gamma(3-\alpha)(\psi(x) - \psi(0))^{2-\alpha}$. Note that, when $\alpha = 1$, we have ${}^C D_{0+}^{1,\psi} = 2(\psi(x) - \psi(0))$.*
*In particular, given $n \leq k \in \mathbb{N}$, we have*

$${}^C D_{a+}^{\alpha,\psi} (\psi(x) - \psi(a))^k = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(x) - \psi(a))^{k-a}$$

*and*

$${}^C D_{b-}^{\alpha,\psi} (\psi(b) - \psi(x))^k = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(b) - \psi(x))^{k-\alpha}.$$

*On the other hand, for $n > k \in \mathbb{N}_0$, we have*

$${}^C D_{a+}^{\alpha,\psi} (\psi(x) - \psi(a))^k = {}^C D_{b-}^{\alpha,\psi} (\psi(b) - \psi(x))^k = 0$$

*since*

$$D_\psi^{\mathrm{n}} (\psi(x) - \psi(a))^k = D_\psi^{\mathrm{n}} (\psi(b) - \psi(x))^k = 0.$$

**Lemma 1** *[17] Let $\alpha, \lambda > 0$. Then for all $t \in [a,b]$ we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} \leq \frac{e^{\lambda(\psi(t)-\psi(a))}}{\lambda^\alpha}.$$

**Proof 1** *In the light of the definition of the $\psi$-Riemann-Liouville fractional integral, we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} e^{\lambda(\psi(s)-\psi(a))} \mathrm{ds}$$

*Using the change of variables $\mathrm{y} = \psi(t) - \psi(s)$ we get*

$$\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} = \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)} \int_0^{\psi(t)-\psi(a)} \mathrm{y}^{\alpha-1} e^{-\lambda \mathrm{y}} \mathrm{dy}$$

*Using now the change of variables $\mathrm{v} = \lambda \mathrm{y}$ in the above equation we get*

$$\begin{aligned}\mathbb{I}_{a^+}^{\alpha;\psi} e^{\lambda(\psi(t)-\psi(a))} &= \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)\lambda^\alpha} \int_0^{\lambda(\psi(t)-\psi(a))} \mathrm{v}^{\alpha-1} e^{-\mathrm{v}} \mathrm{dv} \\ &\leq \frac{e^{\lambda(\psi(t)-\psi(a))}}{\Gamma(\alpha)\lambda^\alpha} \int_0^{\infty} \mathrm{v}^{\alpha-1} e^{-\mathrm{v}} \mathrm{dv} \\ &= \frac{e^{\lambda(\psi(t)-\psi(a))}}{\lambda^\alpha}.\end{aligned}$$

*This completes the proof.*

**Remark 1 ([10, 9])** *On $C(\mathbb{J}_a^b, \mathbb{R}^n)$ we describe a Bielecki type norm $\|\cdot\|_{\mathfrak{D}}$ as follows*

$$\|\varpi\|_{\mathfrak{D}} := \sup_{\varsigma \in \mathbb{J}_a^b} \frac{\|\varpi(\varsigma)\|}{e^{\vartheta(\Theta(\varsigma)-\Theta(a))}}, \quad \vartheta > 0. \tag{1.4}$$

*Therefore, we possess the below characteristics.*

1. *$\Big(C(\mathbb{J}_a^b, \mathbb{R}^n), \|\cdot\|_{\mathfrak{D}}\Big)$ is a Banach space.*

2. *On $C(\mathbb{J}_a^b, \mathbb{R}^n)$, the norms $\|\cdot\|_{\mathfrak{D}}$ and $\|\cdot\|_\infty$ are equivalent, where $\|\cdot\|_\infty$ represented the Chebyshev norm on $C(\mathbb{J}_a^b, \mathbb{R}^n)$, i.e;*

$$\iota_1 \|\cdot\|_{\mathfrak{D}} \leq \|\cdot\|_\infty \leq \iota_2 \|\cdot\|_{\mathfrak{D}},$$

*where*

$$\iota_1 = 1, \quad \iota_2 = e^{\vartheta(\Theta(b)-\Theta(a))}.$$

*Additional informations of Bielecki type norms can be found in [8, 10, 9].*

### 1.3.3 The relation between derivatives and integrals.

**Lemma 5** *[2] Let $\tau, \rho > 0$, $\varpi \in C([a,b],\mathbb{R})$, $f \in C^n[a,b]$. Then for $\varsigma \in \mathbb{J}_a^b$ we get*

1. *${}^C D_{a^+}^{\tau;\varphi} I_{a^+}^{\tau;\varphi} f(\varsigma) = f(\varsigma)$,*

2. *$I_{a^+}^{\tau;\varphi C} D_{a^+}^{\tau;\varphi} f(\varsigma) = f(\varsigma) - f(a), \quad 0 < \tau \leq 1$,*

3. *$I_{a^+}^{\tau;\varphi} (\varphi(\varsigma) - \varphi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+\tau)} (\varphi(\varsigma) - \varphi(a))^{\rho+\tau-1}$,*

4. ${}^{C}D^{\tau;\varphi}_{a^+}(\varphi(\varsigma)-\varphi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-\tau)}(\varphi(\varsigma)-\varphi(a))^{\rho-\tau-1}$,

5. ${}^{C}D^{\tau\varphi}_{a^+}(\varphi(\varsigma)-\varphi(a))^{k} = 0, \forall k \in \{0,\ldots,n-1\}$.

**Proof 3** *We shall provide a proof for the first and second equations:*

1. *Using the semigroup law and the integration by parts formula repeatedly, we get*

$$\begin{aligned}
I^{\alpha,\psi C}_{a+}D^{\alpha,\psi}_{a+}f(x) &= I^{a,\psi}_{a+}I^{n-\alpha,\psi}_{a+}f^{[n]}_{\psi}(x) = I^{n,\psi}_{a+}f^{[n]}_{\psi}(x)\\
&= \frac{1}{(n-1)!}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-1}f^{[n]}_{\psi}(t)dt\\
&= \frac{1}{(n-1)!}\int_a^x (\psi(x)-\psi(t))^{n-1}\cdot\frac{d}{dt}f^{[n-1]}_{\psi}(t)dt\\
&= \frac{1}{(n-2)!}\int_a^x (\psi(x)-\psi(t))^{n-2}\cdot\frac{d}{dt}f^{[n-2]}_{\psi}(t)dt - \frac{f^{[n-1]}_{\psi}(a)}{(n-1)!}(\psi(x)-\psi(a))^{n-1}\\
&= \frac{1}{(n-3)!}\int_a^x (\psi(x)-\psi(t))^{n-3}\cdot\frac{d}{dt}f^{[n-3]}_{\psi}(t)dt - \sum_{k=n-2}^{n-1}\frac{f^{[k]}_{\psi}(a)}{k!}(\psi(x)-\psi(a))^{k}\\
&= \ldots = \int_a^x \frac{d}{dt}f(t)dt - \sum_{k=1}^{n-1}\frac{f^{[k]}_{\psi}(a)}{k!}(\psi(x)-\psi(a))^{k}\\
&= f(x) - \sum_{k=0}^{n-1}\frac{f^{[k]}_{\psi}(a)}{k!}(\psi(x)-\psi(a))^{k}
\end{aligned}$$

*In particular, given $\alpha \in (0,1)$, we have*

$$I^{\alpha,\psi C}_{a+}D^{\alpha,\psi}_{a+}f(x) = f(x)-f(a) \quad \textit{and} \quad I^{\alpha,\psi C}_{b-}D^{\alpha,\psi}_{b-}f(x) = f(x)-f(b).$$

2. *By definition,*

$$^{C}D^{\alpha,\psi}_{a+}I^{\alpha,\psi}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^z \psi'(t)(\psi(x)-\psi(t))^{\mathrm{n}-\alpha-1}F^{[\mathrm{n}]}_{\psi}(t)dt \tag{1.5}$$

*where $F(x) = I^{\alpha,\psi}_{a+}f(x)$. By direct computations, we get*

$$\begin{aligned}
F^{[n-1]}_{\psi}(x) &= \frac{1}{\Gamma(\alpha-n+1)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-n}f(t)dt\\
&= \frac{f(a)}{\Gamma(\alpha-n+2)}(\psi(x)-\psi(a))^{\alpha-n+1} + \frac{1}{\Gamma(\alpha-n+2)}\int_a^z (\psi(x)-\psi(t))^{\alpha-n+1}f'(t)dt,
\end{aligned}$$

*and thus*

$$F^{[n]}_{\psi}(x) = \frac{f(a)}{\Gamma(\alpha-n+1)}(\psi(x)-\psi(a))^{\alpha-n} + \frac{1}{\Gamma(\alpha-n+1)}\int_a^x (\psi(x)-\psi(t))^{\alpha-n}f'(t)dt.$$

*Replacing this last formula into Eq. ((1.5)), using the change of variables $u = (\psi(t) - \psi(a))/(\psi(x) - \psi(a))$ and the Dirichlet's formula, we deduce*

$$\begin{aligned}
{}^{C}D_{a+}^{\alpha,\psi} I_{a+}^{\alpha,\psi} f(x) &= \frac{f(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^z \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}(\psi(t)-\psi(a))^{\alpha-n}dt \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^x \int_a^t \psi'(t)(\psi(x)-\psi(t))^{n-a-1}(\psi(t)-\psi(\tau))^{\alpha-n} f'(\tau)d\tau dt \\
&= \frac{f(a)(\psi(x)-\psi(a))^{n-\alpha-1}}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^z \psi'(t)\left(1-\frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}\right)^{n-\alpha-1}(\psi(t)-\psi(a))^{\alpha-n}dt \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^x \int_z^x \psi(\tau)(\psi(x)-\psi(\tau))^{n-\alpha-1}(\psi(\tau)-\psi(t))^{\alpha-n} f'(t)d\tau dt \\
&= \frac{f(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_0^1 (1-u)^{n-\alpha-1}u^{\alpha-n}du \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^x f'(t) \int_t^z \psi'(\tau)(\psi(x)-\psi(\tau))^{n-\alpha-1}(\psi(\tau)-\psi(t))^{\alpha-n}d\tau dt \\
&= \frac{f(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \cdot \Gamma(n-\alpha)\Gamma(\alpha-n+1) \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} \int_a^x f'(t)dt - \Gamma(n-\alpha)\Gamma(\alpha-n+1) \\
&= f(x).
\end{aligned}$$

## 1.4 Some Classical Results in Matrix Analysis

In order to talk about the contribution of Perov, we need to fix some notations:
Let $m$ be a fixed natural number such that $m \geq 2$. By $M_{m\times m}(\mathbb{R}_+)$we denote the set of all $m \times m$ matrices with nonnegative elements. The unit matrix of $M_{m\times m}(\mathbb{R})$ will be denoted by $\mathbb{I}$. In addition, we use the symbol $\Theta$ to denote the zero $m \times m$ matrix.

Now, let $u, v \in \mathbb{R}^m$ with $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m)$. By $u \preceq v$ we mean $u_i \leq v_i, i = 1, \ldots, m$ and

$$\mathbb{R}_+^m = \{u \in \mathbb{R}^m : u_i \in \mathbb{R}_+, i = 1, \ldots, m\}$$

If $c \in \mathbb{R}$, then $x \preceq c$ means $x_i \leq c, i = 1, \ldots, m$.

**Definition 11** *([28]) Let $\mathbb{E}$ be a nonempty set. By a vector-valued metric (a generalized metric in the sense of Perov) on $\mathbb{E}$ we mean a map $\mathrm{d} : \mathbb{E} \times \mathbb{E} \to \mathbb{R}_+^m$ with the following axioms:*

**(i)** $\mathrm{d}(x, y) = 0_{\mathbb{R}^m}$ *if and only $x = y$ for all $x, y \in \mathbb{E}$;*

**(ii)** $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ *for all $x, y \in \mathbb{E}$;*

**(iii)** $\mathrm{d}(x, z) \preceq \mathrm{d}(x, y) + \mathrm{d}(y, z)$ *for all $x, y, z \in \mathbb{E}$.*

*We call the pair $(\mathbb{E}, \mathrm{d})$ a generalized metric space with*

$$\mathrm{d}(x, y) := \begin{pmatrix} \mathrm{d}_1(x, y) \\ \mathrm{d}_2(x, y) \\ \vdots \\ \mathrm{d}_m(x, y) \end{pmatrix}$$

*Notice that d is a generalized metric space on $\mathbb{E}$ if and only if $\mathrm{d}_i, i = 1, \ldots, m$, are metrics on $\mathbb{E}$.*

**Remark 3** *([28]) Notice that for a generalized metric space in the sense of Perov, the notions of convergence sequence, Cauchy sequence, completeness, open subset, and closed subset are similar to those for usual metric spaces.*

Similarly, we can define a vector-valued norm on a linear space $\mathbb{X}$, as in the following defintion:

**Definition 12** *([28]) Let $\mathbb{X}$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm (also known as the generalized norm in the sense of Perov) on $\mathbb{X}$, we mean a map*

$$\| \cdot \| : \mathbb{X} \to \mathbb{R}^m_+$$

*satisfying the following statements:*

**(i)** $\|x\| = 0_{\mathbb{R}^m}$ *if and only* $x = 0$, *for all* $x \in \mathbb{X}$;

**(ii)** $\|\lambda x\| = |\lambda| \|x\|$ *for all* $x \in \mathbb{X}$ *and* $\lambda \in \mathbb{K}$;

**(iii)** $\|x + y\| \preceq \|x\| + \|y\|$ *for all* $x, y \in \mathbb{X}$.

**Remark 4** *([28]) To any vector-valued norm $\| \cdot \|$ one can associate the vector valued metric $\mathrm{d}(x, y) := \|x - y\|$, and one says that ( $\mathbb{X}, \| \cdot \|$ ) is a generalized Banach space if $\mathbb{X}$ is complete with respect to the d .*

**Definition 13** *([28]) A square matrix $\mathbb{A}$ of nonnegative numbers is said to be convergent to zero if $\mathbb{A}^n$ tend to the zero matrix $\Theta$ as $n \to +\infty$.*

Convergent to zero matrices can be characterized as follows:

**Theorem 4** *([28]) For any nonnegative square matrix $\mathbb{A}$, the following statements are equivalent:*

**(i)** $\mathbb{A}$ *is convergent to zero;*

**(ii)** *the spectral radius $\rho(\mathbb{A})$ is strictly less than 1 . In other words, this means that all the eigenvalues of $\mathbb{A}$ are in the open unit disc, i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$,*

**(iii)** *the matrix $\mathbb{I} - \mathbb{A}$ is nonsingular and $(\mathbb{I} - \mathbb{A})^{-1} = \mathbb{I} + \mathbb{A} + \cdots + \mathbb{A}^n + \cdots$;*

**(iv)** $\mathbb{I} - \mathbb{A}$ *is nonsingular and $(\mathbb{I} - \mathbb{A})^{-1}$ is a nonnegative matrix.*

**Remark 5** *([29]) Some examples of matrices convergent to zero are:*

1. *Any matrix $\mathbb{A} \in M_{2\times 2}(\mathbb{R}_+)$of the form $\mathbb{A} := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$ or $\mathbb{A} := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, with $a + b <$ 1.*

2. *Any matrix $\mathbb{A} \in M_{2\times 2}(\mathbb{R}_+)$of the form $\mathbb{A} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $\max\{a, b\} < 1$.*

3. *Any matrix $\mathbb{A} \in M_{2\times 2}(\mathbb{R}_+)$of the form $\mathbb{A} := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, with $\max\{a, c\} < 1$.*

**Remark 6** *([29]) Any matrix $\mathbb{A} \in M_{2\times 2}(\mathbb{R}_+)$of the form $\mathbb{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a + b \geq 1$ and $c + d \geq 1$ does not converges to zero.*

## 1.5 Some Fixed Point Theorems on Spaces Endowed with Vector Valued Norms

The concept of fixed points has played a crucial role in mathematical analysis, topology, and functional analysis for centuries. The earliest fixed point results can be traced back to Brouwer's Fixed Point Theorem (1912), which states that any continuous function mapping a convex compact subset of Euclidean space to itself has at least one fixed point. Later, Banach introduced the famous Banach Contraction Principle (1922), which provided a powerful tool for proving the existence and uniqueness of fixed points for contractive mappings. Over time, mathematicians such as Schauder, Perov, and Krasnoselskii extended and generalized these results to broader contexts, making fixed point theory a fundamental tool in solving differential equations, optimization problems, and game theory.

**Definition 14** *([28]) A subset $\Omega$ of $C(\mathrm{J}, \mathbb{R})$ is uniformly bounded if there exists a constant $\mathrm{k} > 0$ such that $|\varpi(\mathrm{x})| \leq \mathrm{k}$ for each $\mathrm{x} \in \mathrm{J}$ and each $\varpi \in \Omega$.*

**Definition 15** *([28]) A subset $\Omega$ of $C(\mathrm{J}, \mathbb{R})$ is equicontinuous if for every $\varepsilon > 0$ there exists some $\delta > 0$ (which depends only on $\varepsilon$ ) such that for all $\varpi \in \Omega$ and all $\mathrm{x}_1, \mathrm{x}_2 \in \mathrm{J}$ with $|\mathrm{x}_1 - \mathrm{x}_2| < \delta$, we have $|\varpi(\mathrm{x}_1) - \varpi(\mathrm{x}_2)| < \varepsilon$.*

In the following, we state the Ascoli-Arzela theorem.

**Theorem 5** *([28]) A subset $\Omega$ of $C(\mathrm{J}, \mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous.*

**Definition 16** *([28]) Let ( $\mathbb{E}, \mathrm{d}$ ) be a generalized metric space. An operator $\mathbb{T} : \mathbb{E} \to \mathbb{E}$ is said to be Perov contraction if there exists a matrix $\mathbb{A} \in M_{m \times m}(\mathbb{R}_+)$which converges to zero such that*

$$\mathrm{d}(\mathbb{T}(x), \mathbb{T}(y)) \preceq \mathbb{A}\mathrm{d}(x, y), \qquad \text{for all } x, y \in \mathbb{E}$$

We end up this section by introducing the following fixed-point theorems whose involvements assist us in achieving the desired results successfully.

**Theorem 6** *(Perov's fixed point theorem [28]) Let ( $\mathbb{E}, \mathrm{d}$ ) be a complete generalized metric space and $\mathbb{T} : \mathbb{E} \to \mathbb{E}$ be a Perov contraction operator with Lipschitz matrix $\mathbb{A}$. Then $\mathbb{T}$ has a unique fixed point $x_0$, and for each $x \in \mathbb{E}$, we have*

$$\mathrm{d}\left(\mathbb{T}^k(x), x_0\right) \preceq \mathbb{A}^k(\mathbb{I} - \mathbb{A})^{-1}\, \mathrm{d}(x, \mathbb{T}(x)) \qquad \text{for all } k \in \mathbb{N}$$

**Theorem 7** *( Schauder's fixed point theorem in generalized Banach space [25] ) Let $\mathbb{X}$ be a generalized Banach space, $D \subset \mathbb{X}$ be a nonempty closed convex subset of $\mathbb{X}$, and $\mathbb{K} : D \to D$ be a continuous operator with relatively compact range. Then $\mathbb{K}$ has at least a fixed point in $D$.*

**Theorem 8** *(Krasnoselskii's fixed point theorem in generalized Banach space [25]) Let $\Pi$ be a nonempty $(\Pi \neq \emptyset)$ convex closed subset of a GBS $\varpi$. Let $\mathbb{V}$ and $\mathbb{U}$ map $\Pi$ into $\varpi$ and that*

- $(i)$ $\forall \varpi, w \in \Pi, \quad \mathbb{U}\varpi + \mathbb{V}w \in \Pi$;
- $(ii)$ $\mathbb{V}$ *is an $\mathbb{A}$-contraction mapping.*
- $(iii)$ $\mathbb{U}$ *is continuous and compact;*

*Then the operator $\mathbb{U}\varpi + \mathbb{V}\varpi = \varpi$ possess at least one solution on $\Pi$.*

# Chapter 2

# Uniqueness and $\mathcal{UH}$ stability of solutions for the coupled system of nonlinear fractional $\psi$-integral equations

## 2.1 Qualitative Results

Throughout this section, we prove the existence, uniqueness and $\mathcal{UH}$ stability of solutions for the coupled system of nonlinear fractional $\psi$-integral equations (1).

Now, in order to establish qualitative results of the mentioned system (1), we need to provide the following lemma.

**Lemma 2** *If the solution of the coupled system of nonlinear fractional $\psi$-integral equations given by*

$$\begin{cases} \mathrm{D}^{\alpha_1,\varphi}u(t) = F_1(t,u(t),v(t)) + I^{\alpha_2,\varphi}F_2(t,u(t),v(t)), \\ \quad t \in \mathrm{J} := [a,b], \quad \alpha \in [0,1] \end{cases} \tag{2.1}$$

*exists, then it's equivalent to the integral equation*

$$u(t) = \theta_1 + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} F_1(t,u(t),v(t))ds \tag{2.2}$$

$$+ \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_2(t,u(t),v(t))ds. \tag{2.3}$$

In view of Lemma 2, we need to present the following lemma, which plays a key role in the main theorems.

**Lemma 3** *Let $(\alpha_i,\beta_i) \in (0,1]$ and $F_i, G_i \in C(\mathrm{J}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\quad i=1,2$. Then, the coupled system of nonlinear fractional $\psi$-integral equations is given by*

$$\begin{cases} u(t)-\theta_1 &= \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} F_1(s,u(s),v(s))ds \\ &\quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_2(s,u(s),v(s))ds \\ v(t)-\theta_2 &= \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1-1}}{\Gamma(\beta_1)} G_1(s,u(s),v(s))ds \\ &\quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} G_2(s,u(s),v(s))ds \end{cases}, \quad t \in \mathrm{J}. \tag{2.4}$$

Now, the product space $\mathbb{X} := C(\mathrm{J}, \mathbb{R}^n) \times C(\mathrm{J}, \mathbb{R}^n)$ is a $\mathcal{GBS}$ with the following norm:

$$\|(u,v)\|_{\mathbb{X}} = \begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix}.$$

Also, let the operator $\mathbb{T} = \Big(\mathbb{T}_1, \mathbb{T}_2\Big) \colon \mathbb{X} \to \mathbb{X}$ defines

$$\mathbb{T}(u,v) = \Big(\mathbb{T}_1(u,v), \mathbb{T}_2(u,v)\Big), \tag{2.5}$$

with

$$\begin{aligned}(\mathbb{T}_1(u,v))(t) = \theta_1 &+ \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} F_1(s,u(s),v(s))ds \\ &+ \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_2(s,u(s),v(s))ds \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(\mathbb{T}_2(u,v))(t) = \theta_2 &+ \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1-1}}{\Gamma(\beta_1)} G_1(s,u(s),v(s))ds \\ &+ \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} G_2(s,u(s),v(s))ds \end{aligned} \tag{2.7}$$

Let's list the following hypotheses:

$(\mathrm{HP}_1)$ For $i = 1, 2$, the functions $F_i$ and $G_i$ are continuous on $\mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n$

$(\mathrm{HP}_2)$ There exist $\mathrm{L}_{Z_i} > 0$ and $\bar{\mathrm{L}}_{Z_i} > 0$, $Z = (F, G)$, $i = 1, 2$, where

$$\|F_i(s,u_1,v_1) - F_i(s,u_2,v_2)\| \le \mathrm{L}_{F_i}\|u_1-u_2\| + \bar{\mathrm{L}}_{F_i}\|v_1-v_2\|,$$

$$\|G_i(s,u_1,v_1) - G_i(s,u_2,v_2)\| \le \mathrm{L}_{G_i}\|u_1-u_2\| + \bar{\mathrm{L}}_{G_i}\|v_1-v_2\|,$$

for all $t \in \mathrm{J}$ and each $u_1, v_1, u_2, v_2 \in \mathbb{R}^n$.

$(\mathrm{HP}_3)$ $\bar{\mathbb{A}}_i, \bar{\mathbb{B}}_i < 1$, where

$$\bar{\mathbb{A}}_1 = \mathbb{A}_1\mathrm{L}_{F_1} + \mathbb{B}_1\mathrm{L}_{F_2}, \qquad \bar{\mathbb{A}}_2 = \mathbb{A}_2\mathrm{L}_{G_1} + \mathbb{B}_2\mathrm{L}_{G_2}$$

$$\bar{\mathbb{B}}_1 = \mathbb{A}_1\bar{\mathrm{L}}_{F_1} + \mathbb{B}_1\bar{\mathrm{L}}_{F_2} \qquad \bar{\mathbb{B}}_2 = \mathbb{A}_2\bar{\mathrm{L}}_{G_1} + \mathbb{B}_2\bar{\mathrm{L}}_{G_2}.$$

For computational convenience, we introduce the following notations:

$$\begin{aligned}
\mathbb{A}_1 &:= \frac{(\varphi(t)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \quad \mathbb{B}_1 := \frac{(\varphi(t)-\varphi(0))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}, \\
\mathbb{A}_2 &:= \frac{(\varphi(t)-\varphi(0))^{\beta_1}}{\Gamma(\beta_1+1)}, \quad \mathbb{B}_2 := \frac{(\varphi(t)-\varphi(0))^{\beta_1+\beta_2}}{\Gamma(\beta_1+\beta_2+1)}, \\
\bar{\Delta}_1 &:= \frac{\left[\Big(\mathrm{L}_{F_2}\|u\| + \bar{\mathrm{L}}_{F_2}\|v\|\Big) + F_{2.\max}\right]}{\Gamma(\alpha_1+\alpha_2+1)}, \\
\bar{\Delta}_2 &:= \frac{\left[\Big(\mathrm{L}_{G_2}\|u\| + \bar{\mathrm{L}}_{G_2}\|v\|\Big) + G_{2.\max}\right]}{\Gamma(\alpha_1+\alpha_2+1)}.
\end{aligned}$$

Next, we are in a position to investigate and prove the uniqueness result by using Perov's fixed point theorem.

**Theorem 9** *Let the hypotheses* $(\mathrm{HP}_1)$–$(\mathrm{HP}_3)$ *hold. Then, the coupled system of nonlinear fractional* $\psi$*-integral equations* (1) *possesses one and only one solution.*

**Proof 2** *In order to show that* $\mathbb{T}$ *has an exactly one fixed point, we will use the Perov's fixed point theorem. Indeed, we prove that the mapping* $\mathbb{T}$ *is* $\mathbb{A}_{\mathbb{MAT}}$*-contraction on* $\mathbb{X}$*.*

*Now, for given* $(u_1, v_1), (u_2, v_2) \in \mathbb{X}$ *and* $t \in \mathrm{J}$*, using* $(\mathrm{HP}_1)$ *and* $(\mathrm{HP}_2)$*, we can get*

$$
\begin{aligned}
&\Big\|\Big(\mathbb{T}_1(u_1, v_1)\Big)(t) - \Big(\mathbb{T}_1(u_2, v_2)\Big)(t)\Big\| \\
&\quad \leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} \Big\|F_1(s, u_1(s), v_1(s)) - F_1(s, u_2(s), v_2(s))\Big\| ds \\
&\quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \Big\|F_2(s, u_1(s), v_1(s)) - F_2(s, u_2(s), v_2(s))\Big\| ds \\
&\quad \leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} \Big(\mathrm{L}_{F_1}\|u_1(s)-u_2(s)\| + \bar{\mathrm{L}}_{F_1}\|v_1(s)-v_2(s)\|\Big) ds \\
&\quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \Big(\mathrm{L}_{F_2}\|u_1(s)-u_2(s)\| + \bar{\mathrm{L}}_{F_2}\|v_1(s)-v_2(s)\|\Big) ds \\
&\quad \leq \frac{(\varphi(t)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)} \Big(\mathrm{L}_{F_1}\|u_1-u_2\| + \bar{\mathrm{L}}_{F_1}\|v_1-v_2\|\Big) \\
&\quad + \frac{(\varphi(t)-\varphi(0))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \Big(\mathrm{L}_{F_2}\|u_1-u_2\| + \bar{\mathrm{L}}_{F_2}\|v_1-v_2\|\Big) \\
&\quad \leq \mathbb{A}_1 \Big(\mathrm{L}_{F_1}\|u_1-u_2\| + \bar{\mathrm{L}}_{F_1}\|v_1-v_2\|\Big) \\
&\quad + \mathbb{B}_1 \Big(\mathrm{L}_{F_2}\|u_1-u_2\| + \bar{\mathrm{L}}_{F_2}\|v_1-v_2\|\Big)
\end{aligned}
$$

*Hence,*

$$
\begin{aligned}
\Big\|\mathbb{T}_1(u_1, v_1) - \mathbb{T}_1(u_2, v_2)\Big\| &\leq \Big[\mathbb{A}_1\mathrm{L}_{F_1} + \mathbb{B}_1\mathrm{L}_{F_2}\Big]\|u_1-u_2\| \\
&+ \Big[\mathbb{A}_1\bar{\mathrm{L}}_{F_1} + \mathbb{B}_1\bar{\mathrm{L}}_{F_2}\Big]\|v_1-v_2\| \\
&:= \bar{\mathbb{A}}_1\|u_1-u_2\| + \bar{\mathbb{B}}_1\|v_1-v_2\|.
\end{aligned}
$$

*By the same technique, we can also get*

$$
\begin{aligned}
\Big\|\mathbb{T}_2(u_1, v_1) - \mathbb{T}_2(u_2, v_2)\Big\| &\leq \Big[\mathbb{A}_2\mathrm{L}_{G_1} + \mathbb{B}_2\mathrm{L}_{G_2}\Big]\|u_1-u_2\| \\
&+ \Big[\mathbb{A}_2\bar{\mathrm{L}}_{G_1} + \mathbb{B}_2\bar{\mathrm{L}}_{G_2}\Big]\|v_1-v_2\| \\
&:= \bar{\mathbb{A}}_2\|u_1-u_2\| + \bar{\mathbb{B}}_2\|v_1-v_2\|.
\end{aligned}
$$

*This implies that*

$$
\Big\|\mathbb{T}(u_1, v_1) - \mathbb{T}(u_2, v_2)\Big\|_{\mathbb{X}} \leq \mathbb{A}_{\mathbb{MAT}}\|(u_1, v_1) - (u_2, v_2)\|_{\mathbb{X}},
$$

*where*

$$
\mathbb{A}_{\mathbb{MAT}} = \begin{pmatrix} \bar{\mathbb{A}}_1 & \bar{\mathbb{B}}_1 \\ \bar{\mathbb{A}}_2 & \bar{\mathbb{B}}_2 \end{pmatrix}. \tag{2.8}
$$

*According to* $(\mathrm{HP}_3)$*, we have* $\mathbb{A}^n_{\mathbb{MAT}} \to 0$ *as* $n \to \infty$*. Thus* $\mathbb{T}$ *is contractive and due to the Perov's theorem,* $\mathbb{T}$ *has exactly one fixed point. Thus, the coupled system of nonlinear fractional* $\psi$*-integral equations* (1) *possesses a unique solution in* $\mathbb{X}$*.*

The following result is achieved based on the Krasnoselskii's Theorem 8 in $\mathcal{GBS}$.

**Theorem 10** *Let* $(\mathrm{HP}_1)$ *and* $(\mathrm{HP}_2)$ *hold. Then, the coupled system of nonlinear fractional* $\psi$*-integral equations* (1) *admits at least one solution.*

**Proof 3** *In order to use Theorem* (8)*, we need to take a set* $\mathbb{Q}_\xi \subseteq \mathbb{X}$ *such that* $\mathbb{Q}_\xi$ *is closed, convex, bounded and define it as*

$$\mathbb{Q}_\xi = \{(u,v) \in \mathbb{X} : \|(u,v)\|_{\mathbb{X}} \leq \xi\},$$

*with* $\xi := (\xi_1, \xi_2) \in \mathbb{R}_+^2$ *such that*

$$\begin{cases} \xi_1 \geq \rho_1 M_1 + \rho_2 M_2, \\ \xi_2 \geq \rho_3 M_1 + \rho_4 M_2, \end{cases}$$

*where* $M_1, M_2$ *and* $\rho_i, i = \overline{1,4}$ *are non-negative real numbers that will be specified later.*

*Now, consider the mappings* $\mathbb{U} = (\mathbb{U}_1, \mathbb{U}_2)$ *and* $\mathbb{V} = (\mathbb{V}_1, \mathbb{V}_2)$ *on* $\mathbb{Q}_\xi$ *as*

$$\begin{cases} \mathbb{U}_1(u,v)(t) = \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_2(s,u(s),v(s))ds, \\ \mathbb{U}_2(u,v)(t) = \displaystyle\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} G_2(s,u(s),v(s))ds, \end{cases}$$

*and*

$$\begin{cases} \mathbb{V}_1(u,v)(t) = \theta_1 \quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)} F_1(s,u(s),v(s))ds, \\ \mathbb{V}_2(u,v)(t) = \theta_2 \quad + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta_1-1}}{\Gamma(\beta_1)} G_1(s,u(s),v(s))ds. \end{cases}$$

*It is obvious that both* $\mathbb{U}$ *and* $\mathbb{V}$ *are well-defined. Moreover, by Lemma 3 the mappings form the system* (2.4) *as*

$$\mathbb{T}(u,v) := (\mathbb{U}_1(u,v), \mathbb{U}_2(u,v)) + (\mathbb{V}_1(u,v), \mathbb{V}_2(u,v)). \tag{2.9}$$

*Our purpose is to confirm this fact that* $\mathbb{U}$ *and* $\mathbb{V}$ *fulfill all properties of Theorem 8. For simplicity, we set*

$$F_{i.\max} := \sup_{t\in \mathrm{J}} \|F_i(t,0,0)\|, \quad G_{i.\max} := \sup_{t\in \mathrm{J}} \|G_i(t,0,0)\|,$$

*and for better clarity, the proof is broken down into three steps.*

***Step 1 :*** $\mathbb{U}(u,v) + \mathbb{V}(\bar{u},\bar{v}) \in \mathbb{Q}_\xi$, $\forall\,(u,v),(\bar{u},\bar{v}) \in \mathbb{Q}_\xi$.

*In fact, from* $(\mathrm{HP}_2)$*, for* $(u,v),(\bar{u},\bar{v}) \in \mathbb{X}$, $\forall t \in \mathrm{J}$*, we can obtain*

$$\begin{aligned}
&\|\mathbb{U}_1(u,v)(t)\| \\
&\leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \Big(\Big\|F_1(s,u(s),v(s)) - F_1(s,0,0)\Big\| + \Big\|F_1(s,0,0)\Big\|\Big)\mathrm{ds} \\
&\leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \Big[\Big(\mathrm{L}_{F_1}\|u(s)\| + \bar{\mathrm{L}}_{F_1}\|v(s)\|\Big) + F_{1.\max}\Big]\mathrm{ds} \\
&\leq \mathbb{B}_1\Big[\Big(\mathrm{L}_{F_1}\|u\| + \bar{\mathrm{L}}_{F_1}\|v\|\Big) + F_{1.\max}\Big] \\
&\leq \mathbb{B}_1\mathrm{L}_{F_1}\|u\| + \mathbb{B}_1\bar{\mathrm{L}}_{F_1}\|v\| + \mathbb{B}_1 F_{1.\max}.
\end{aligned}$$

*Hence,*

$$\|\mathbb{U}_1(u,v)\| \leq \mathcal{A}_1\|u\| + \mathcal{B}_1\|v\| + \mathcal{C}_1. \tag{2.10}$$

*By similar procedure, we get*

$$\|\mathbb{U}_2(u,v)\| \leq \mathcal{A}_2\|u\| + \mathcal{B}_2\|v\| + \mathcal{C}_2, \tag{2.11}$$

*with*

$$\begin{aligned} \mathcal{A}_1 &= \mathbb{B}_1 \mathrm{L}_{F_1}, \quad \mathcal{B}_1 = \mathbb{B}_1 \bar{\mathrm{L}}_{F_1}, \quad \mathcal{C}_1 = \mathbb{B}_1 F_{1.\max}, \\ \mathcal{A}_2 &= \mathbb{B}_2 \mathrm{L}_{G_2}, \quad \mathcal{B}_2 = \mathbb{B}_2 \bar{\mathrm{L}}_{G_1}, \quad \mathcal{C}_2 = \mathbb{B}_2 G_{2.\max}. \end{aligned}$$

*Thus, the inequalities* (2.10) *and* (2.10), *implies that*

$$\left\|\mathbb{U}(u,v)\right\|_{\mathbb{X}} := \begin{pmatrix} \left\|\begin{matrix}\mathbb{U}_1(u,v)\\ \mathbb{U}_2(u,v)\end{matrix}\right\| \end{pmatrix} \leq \mathbb{B}_{\mathbb{MAT}} \begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}, \tag{2.12}$$

*where*

$$\mathbb{B}_{\mathbb{MAT}} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{pmatrix}.$$

*In a similar way, we get*

$$\left\|\mathbb{V}(\bar{u},\bar{v})\right\|_{\mathbb{X}} := \begin{pmatrix} \left\|\begin{matrix}\mathbb{V}_1(\bar{u},\bar{v})\\ \mathbb{V}_2(\bar{u},\bar{v})\end{matrix}\right\| \end{pmatrix} \leq \mathbb{D}_{\mathbb{MAT}} \begin{pmatrix} \|\bar{u}\| \\ \|\bar{v}\| \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{2.13}$$

*where*

$$\mathbb{D}_{\mathbb{MAT}} = \begin{pmatrix} \mathbb{A}_1 & \bar{\mathbb{A}}_1 \\ \mathbb{A}_2 & \bar{\mathbb{A}}_2 \end{pmatrix}.$$

*Recombine* (2.12) *and* (2.13), *it implies that*

$$\left\|\mathbb{U}(u,v)\right\|_{\mathbb{X}} + \left\|\mathbb{V}(\bar{u},\bar{v})\right\|_{\mathbb{X}} \leq \mathbb{B}_{\mathbb{MAT}} \begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} + \mathbb{D}_{\mathbb{MAT}} \begin{pmatrix} \|\bar{u}\| \\ \|\bar{v}\| \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1 + \theta_1 \\ \mathcal{C}_2 + \theta_2 \end{pmatrix}. \tag{2.14}$$

*Therefore, we check for* $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2_+$ *such that* $\mathbb{U}(u,v) + \mathbb{V}(\bar{u},\bar{v}) \in \mathbb{Q}_\xi$. *Regarding to this, in view of* (2.14), *it is sufficient to verify that*

$$\mathbb{C}_{\mathbb{MAT}} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \leq \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

*where* $\mathbb{C}_{\mathbb{MAT}} = \mathbb{B}_{\mathbb{MAT}} + \mathbb{D}_{\mathbb{MAT}}$, *and*

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_1 + \theta_1 \\ \mathcal{C}_2 + \theta_2 \end{pmatrix}.$$

*Equivalently*

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \leq (\mathbb{I} - \mathbb{C}_{\mathbb{MAT}}) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{2.15}$$

*Since the spectral radius of* $\mathbb{C}_{\mathbb{MAT}}$ *is* $< 1$. *According to Theorem 4, we have the matrix* $(\mathbb{I} - \mathbb{C}_{\mathbb{MAT}})$ *is non-singular and* $(\mathbb{I} - \mathbb{C}_{\mathbb{MAT}})^{-1}$ *has positive elements. So,* (2.15) *is equal to*

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \geq (\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

*In addition, if we take*

$$(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1} = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix},$$

*thus, we find*

$$\begin{cases} \xi_1 \geq \rho_1 M_1 + \rho_2 M_2, \\ \xi_2 \geq \rho_3 M_1 + \rho_4 M_2. \end{cases}$$

*Therefore,* $\mathbb{U}(u,v)+\mathbb{V}(\bar{u},\bar{v}) \in \mathbb{Q}_\xi$.

***Step 2 :*** *The mapping* $\mathbb{V}$ *is* $\mathbb{D}_{\mathbb{MAT}}$*-contraction on* $\mathbb{Q}_\xi$.
*Indeed,* $\forall t \in \mathrm{J}$ *and for any* $(u_1,v_1),(u_2,v_2) \in \mathbb{Q}_\xi$. *By similar procedure in the proof of Theorem 9, it is not difficult to verify that*

$$\left\|\mathbb{V}(u_1,v_1)-\mathbb{V}(u_2,v_2)\right\|_{\mathbb{X},\mathfrak{B}} \leq \mathbb{D}_{\mathbb{MAT}}\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}}.$$

*Since the spectral radius of* $\mathbb{D}_{\mathbb{MAT}}$ *is* $<1$. *Hence, the mapping* $\mathbb{V}$ *is an* $\mathbb{D}_{\mathbb{MAT}}$*-contraction on* $\mathbb{Q}_\xi$.

***Step 3 :*** *The mapping* $\mathbb{U}$ *is continuous and compact.*
*By the continuity of* $\mathbb{G}_1$ *and* $\mathbb{G}_2$, *we deduce that* $\mathbb{U}$ *is continuous. Moreover, we show that* $\mathbb{U}$ *is uniformly bounded on* $\mathbb{Q}_\xi$. *From* (2.12), *and* $\forall(u,v)\in\mathbb{Q}_\xi$, *we find that*

$$\left\|\mathbb{U}(u,v)\right\|_{\mathbb{X}} := \begin{pmatrix} \left\|\begin{matrix}\mathbb{U}_1(u,v)\\ \mathbb{U}_2(u,v)\end{matrix}\right\| \end{pmatrix} \leq \mathbb{B}_{\mathbb{MAT}} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} < \infty.$$

*This means that the mapping* $\mathbb{U}$ *is uniformly bounded on* $\mathbb{Q}_\xi$.
*At the last step, we are going to prove that* $\mathbb{U}(\mathbb{Q}_\xi)$ *is equicontinuous. From the hypotheses* $(\mathrm{HP}_1)$ *and* $(\mathrm{HP}_2)$, *for* $(u,v)\in\mathbb{Q}_\xi$, *and* $t_1 \leq t_2$ *for any* $t_1,t_2\in\mathrm{J}$, *we obtain*

$$\begin{aligned}
&\|\mathbb{U}_1(u,v)(t_2)-\mathbb{U}_1(u,v)(t_1)\| \\
=&\| \int_0^{t_2} \frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_1(s,u(s),v(s))ds \\
&-\int_0^{t_1} \frac{\varphi'(s)(\varphi(t_1)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_1(s,u(s),v(s))ds\| \\
=&\| \int_0^{t_1} \frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_1(s,u(s),v(s))\,\mathrm{ds} \\
&+\int_{t_1}^{t_2} \frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_1(s,u(s),v(s))\,\mathrm{ds} \\
&-\int_0^{t_1} \frac{\varphi'(s)(\varphi(t_1)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} F_1(s,u(s),v(s))\,\mathrm{ds}\| \\
\leq& \int_0^{t_1} \frac{\varphi'(s)[(\varphi(t_2)-\varphi(s))^{\alpha_1+\alpha_2-1}-(\varphi(t_1)-\varphi(s))^{\alpha_1+\alpha_2-1}]}{\Gamma(\alpha_1+\alpha_2)} \|F_1(s,u(s),v(s))\|\mathrm{ds} \\
&+\int_{t_1}^{t_2} \frac{\varphi'(s)(\varphi(t_2)-\varphi(s))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \|F_1(s,u(s),v(s))\|\mathrm{ds} \\
&\leq \frac{\left[\left(\mathrm{L}_{F_2}\|u\|+\bar{\mathrm{L}}_{F_2}\|v\|\right)+F_{2.\max}\right]}{\Gamma(\alpha_1+\alpha_2+1)} \left[\left|(\varphi(t_2)-\varphi(a))^{\alpha_1+\alpha_2}-(\varphi(t_2)-\varphi(t_1))^{\alpha_1+\alpha_2}-(\varphi(t_1)\varphi(a))^{\alpha_1+\alpha_2}\right|\right] \\
&+\frac{\left[\left(\mathrm{L}_{F_2}\|u\|+\bar{\mathrm{L}}_{F_2}\|v\|\right)+F_{2.\max}\right]}{\Gamma(\alpha_1+\alpha_2+1)} (\varphi(t_2)-\varphi(t_1))^{\alpha_1+\alpha_2} \\
&\leq \frac{\left[\left(\mathrm{L}_{F_2}\|u\|+\bar{\mathrm{L}}_{F_2}\|v\|\right)+F_{2.\max}\right]}{\Gamma(\alpha_1+\alpha_2+1)} \left[2(\varphi(t_2)-\varphi(t_1))^{\alpha_1+\alpha_2}+\left|(\varphi(t_2)-\varphi(a))^{\alpha_1+\alpha_2}-(\varphi(t_1)\varphi(a))^{\alpha_1+\alpha_2}\right|\right].
\end{aligned}$$

*Then*

$$\|\mathbb{U}_1(u,v)(t_2)-\mathbb{U}_1(u,v)(t_1)\| \leq \bar{\Delta}_1\left[2\left(\varphi\left(t_2\right)-\varphi\left(t_1\right)\right)^{\alpha_1+\alpha_2}+\left|\left(\varphi\left(t_2\right)-\varphi(a)\right)^{\alpha_1+\alpha_2}-\left(\varphi\left(t_1\right)\varphi(a)\right)^{\alpha_1+\alpha_2}\right|\right].$$

*Similarly,*

$$\|\mathbb{U}_2(u,v)(t_2)-\mathbb{U}_2(u,v)(t_1)\| \leq \bar{\Delta}_2\left[2\left(\varphi\left(t_2\right)-\varphi\left(t_1\right)\right)^{\beta_1+\beta_2}+\left|\left(\varphi\left(t_2\right)-\varphi(a)\right)^{\beta_1+\beta_2}-\left(\varphi\left(t_1\right)\varphi(a)\right)^{\beta_1+\beta_2}\right|\right].$$

*Therefore,*

$$\begin{aligned}\|\mathbb{U}(u,v)(t_2)-\mathbb{U}(u,v)(t_1)\| &:= \begin{pmatrix} \|\mathbb{U}_1(u,v)(t_2)-\mathbb{U}_1(u,v)(t_1)\| \\ \|\mathbb{U}_2(u,v)(t_2)-\mathbb{U}_2(u,v)(t_1)\| \end{pmatrix} \\ &\leq \begin{pmatrix} \bar{\Delta}_1\left[2\left(\varphi\left(t_2\right)-\varphi\left(t_1\right)\right)^{\alpha_1+\alpha_2}+\left|\left(\varphi\left(t_2\right)-\varphi(a)\right)^{\alpha_1+\alpha_2}-\left(\varphi\left(t_1\right)\varphi(a)\right)^{\alpha_1+\alpha_2}\right|\right]. \\ \bar{\Delta}_2\left[2\left(\varphi\left(t_2\right)-\varphi\left(t_1\right)\right)^{\beta_1+\beta_2}+\left|\left(\varphi\left(t_2\right)-\varphi(a)\right)^{\beta_1+\beta_2}-\left(\varphi\left(t_1\right)\varphi(a)\right)^{\beta_1+\beta_2}\right|\right]. \end{pmatrix}.\end{aligned}$$

*Thus, we deduce that $\mathbb{T}(\mathbb{Q}_\xi)$ is equicontinuous. Due to Arzelà–Ascoli's theorem, we conclude that the mapping $\mathbb{U}$ is compact. Hence, the requirements of Theorem 8 are fulfilled. Thus, in view of the Krasnoselskii's FPT, we derive that the mapping $\mathbb{T}=\mathbb{U}+\mathbb{V}$ defined by* (2.9) *possesses at least one fixed point $(u,v)\in\mathbb{Q}_\xi$, which is the solution of the coupled system of nonlinear fractional $\psi$-integral equations* (1)*.*

Now, we end this section by discussing the $\mathcal{UH}$ stability of the coupled system of nonlinear fractional $\psi$-integral equations (1) by utilizing its solution in the sense of integral form given as

$$u(\tau)=\mathbb{T}_1(u,v)(\tau), \quad v(\tau)=\mathbb{T}_2(u,v)(\tau),$$

such that $\mathbb{T}_1$ and $\mathbb{T}_2$ are given in (2.6) and (2.7).

Let us define the following mappings $\mathbb{S}_1,\mathbb{S}_2:\mathbb{X}\to C(\mathrm{J},\mathbb{R})$ as:

$$\begin{cases} \mathrm{D}^{\alpha_1,\varphi}\tilde{u}(t)-F_1(t,\tilde{u}(t),\tilde{v}(t))-I^{\alpha_2,\varphi}F_2(t,\tilde{u}(t),\tilde{v}(t))=\mathbb{S}_1(\tilde{u},\tilde{v})(t), & , \\ \mathrm{D}^{\beta_1,\varphi}\tilde{v}(t)-G_1(t,\tilde{u}(t),\tilde{v}(t))-I^{\beta_2,\varphi}G_2(t,\tilde{u}(t),\tilde{v}(t))=\mathbb{S}_2(\tilde{u},\tilde{v})(t), & \end{cases} \quad t\in\mathrm{J}.$$

In addition, we assume that the next inequalities

$$\begin{cases} \left\|\mathbb{S}_1(\tilde{u},\tilde{v})(\tau)\right\|\leq\epsilon_1, \\ \left\|\mathbb{S}_2(\tilde{u},\tilde{v})(\tau)\right\|\leq\epsilon_2, \end{cases} \quad \tau\in\mathrm{J}, \tag{2.16}$$

for some $\epsilon_1,\epsilon_2>0$, are to be held.

**Definition 17** *[4] The coupled system of nonlinear fractional $\psi$-integral equations* (1) *is $\mathcal{UH}$ stable if there are constants $\omega_i>0, i=\overline{1,4}$ such that $\forall\epsilon_1,\epsilon_1>0$ and for all solution $(\tilde{u},\tilde{v})\in\mathbb{X}$ of inequality* (2.16)*, $\exists$ a solution $(u,v)\in\mathbb{X}$ of* (1) *such that*

$$\begin{cases} \left\|\tilde{u}(\tau)-u(\tau)\right\|\leq\omega_1\epsilon_1+\omega_2\epsilon_2, \\ \left\|\tilde{v}(\tau)-v(\tau)\right\|\leq\omega_3\epsilon_1+\omega_4\epsilon_2, \end{cases} \quad \tau\in\mathrm{J}.$$

**Theorem 11** *Consider the hypotheses of Theorem 9 to be held. Then the coupled system of nonlinear fractional $\psi$-integral equations* (1) *is $\mathcal{UH}$ stable.*

**Proof 4** *Let $(u, v) \in \mathbb{X}$ bee the solution of the coupled system of nonlinear fractional $\psi$-integral equations* (1) *satisfying* (2.6) *and* (2.7). *Assume that $(\tilde{u}, \tilde{v})$ is any solution verifying* (2.16):

$$\begin{cases} \mathrm{D}^{\alpha_1,\varphi}\tilde{u}(t) = F_1(t,\tilde{u}(t),\tilde{v}(t)) + I^{\alpha_2,\varphi}F_2(t,\tilde{u}(t),\tilde{v}(t)) + \mathbb{S}_1(\tilde{u},\tilde{v})(t), & , \\ \mathrm{D}^{\beta_1,\varphi}\tilde{v}(t) = G_1(t,\tilde{u}(t),\tilde{v}(t)) + I^{\beta_2,\varphi}G_2(t,\tilde{u}(t),\tilde{v}(t)) + \mathbb{S}_2(\tilde{u},\tilde{v})(t), & \end{cases} \quad t \in \mathrm{J}.$$

*So*

$$\tilde{u}(\tau) = \mathbb{T}_1(\tilde{u},\tilde{v})(t) + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathbb{S}_1(\tilde{u},\tilde{v})(s)\mathrm{ds}, \tag{2.17}$$

*and*

$$\tilde{v}(\tau) = \mathbb{T}_2(\tilde{u},\tilde{v})(t) + \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\mathbb{S}_2(\tilde{u},\tilde{v})(s)\mathrm{ds}. \tag{2.18}$$

*Now,* (2.17) *and* (2.18) *give*

$$\begin{aligned} \left\|\tilde{u}(\tau) - \mathbb{T}_1(\tilde{u},\tilde{v})(\tau)\right\| &\leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\left\|\mathbb{S}_1(\tilde{u},\tilde{v})(s)\right\|\mathrm{ds} \\ &\leq \mathbb{A}_1\epsilon_1, \end{aligned} \tag{2.19}$$

*and*

$$\begin{aligned} \left\|\tilde{v}(t) - \mathbb{T}_2(\tilde{u},\tilde{v})(t)\right\| &\leq \int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\left\|\mathbb{S}_2(\tilde{u},\tilde{v})(s)\right\|\mathrm{ds} \\ &\leq \mathbb{A}_2\epsilon_2. \end{aligned} \tag{2.20}$$

*Thus, by (*$\mathrm{H2}$) *and inequalities* (2.19), (2.20), *we get*

$$\begin{aligned} \left\|\tilde{u}(t) - u(t)\right\| &= \left\|\tilde{u}(t) - \mathbb{T}_1(\tilde{u},\tilde{v})(t) + \mathbb{T}_1(\tilde{u},\tilde{v})(t) - u(t)\right\| \\ &\leq \left\|\tilde{u}(t) - \mathbb{T}_1(\tilde{u},\tilde{v})(t)\right\| + \left\|\mathbb{T}_1(\tilde{u},\tilde{v})(t) - \mathbb{T}_1(u,v)(t)\right\| \\ &\leq \mathbb{A}_1\epsilon_1 + \left(\bar{\mathbb{A}}_1\|\tilde{u}-u\| + \bar{\mathbb{B}}_1\|\tilde{v}-v\|\right). \end{aligned}$$

*Hence we get*

$$\|\tilde{u}-u\| \leq \mathbb{A}_1\epsilon_1 + \left(\bar{\mathbb{A}}_1\|\tilde{u}-u\| + \bar{\mathbb{B}}_1\|\tilde{v}-v\|\right). \tag{2.21}$$

*Similarly, we have*

$$\|\tilde{v}-v\| \leq \mathbb{A}_2\epsilon_2 + \left(\bar{\mathbb{A}}_2\|\tilde{u}-u\| + \bar{\mathbb{B}}_2\|\tilde{v}-v\|\right). \tag{2.22}$$

*Inequalities* (2.21) *and*(2.21) *can be rewritten in a matrix form as*

$$(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})\begin{pmatrix} \|\tilde{u}-u\| \\ \|\tilde{v}-v\| \end{pmatrix} \leq \begin{pmatrix} \mathbb{A}_1\epsilon_1 \\ \mathbb{A}_2\epsilon_2 \end{pmatrix}, \tag{2.23}$$

*where $\mathbb{A}_{\mathbb{MAT}}$ is the matrix given by* (2.8). *Since the spectral radius of $\mathbb{A}_{\mathbb{MAT}}$ is $< 1$; by Theorem 4, we deduce that $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})$ is non-singular and $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}$ possesses positive elements. Hence,* (2.23) *is equivalent to the form*

$$\begin{pmatrix} \|\tilde{u}-u\| \\ \|\tilde{v}-v\| \end{pmatrix} \leq (\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}\begin{pmatrix} \mathbb{A}_1\epsilon_1 \\ \mathbb{A}_2\epsilon_2 \end{pmatrix},$$

*which yields that*

$$\begin{cases} \|\tilde{u}-u\| \leq \rho_1\mathbb{A}_1\epsilon_1 + \rho_2\mathbb{A}_2\epsilon_2, \\ \|\tilde{v}-v\| \leq \rho_3\mathbb{A}_1\epsilon_1 + \rho_4\mathbb{A}_2\epsilon_2, \end{cases}$$

*where $\rho_i, i = \overline{1,4}$ are the elements of $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}$. Consequently, the coupled system of nonlinear fractional $\psi$-integral equations* (1) *is $\mathcal{UH}$ stable.*

## 2.2 Applications

We provide an example in this part to investigate and guarantee the validity of the results.

**Exemple 1** *Consider the following coupled system of nonlinear fractional $\psi$-integral equations*

$$\begin{cases} \mathrm{D}^{\frac{3}{4},\varphi}u_1(t) = F_1\left(t, u(t), v(t)\right) + I^{\frac{1}{3},\varphi}F_2(t, u(t), v(t)), \\ \mathrm{D}^{\frac{1}{3},\varphi}u_2(t) = G_1\left(t, u(t), v(t)\right) + I^{\frac{1}{4},\varphi}G_2(t, u(t), v(t)), \end{cases} \quad t \in \mathrm{J} := [0,1], \tag{2.24}$$

*with* $(\varphi)$*–Caputo fractional integrals conditions*

$$\begin{cases} u(0) = 1, \\ v(0) = 2, \end{cases} \tag{2.25}$$

*Here,* $\alpha_1 = \frac{3}{4}, \alpha_2 = \frac{1}{3}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{4}, \theta_1 = 1, \theta_2 = 2, \varphi(t) = t^2$, $J := [0,1]$, *and the functions*

$$\begin{aligned} F_1\left(t, u(t), v(t)\right) &= \frac{t|u(t)|}{100(1+|u(t)|)} + \frac{t^3}{49}\sin(v(t)) + 3t; \\ F_2\left(t, u(t), v(t)\right) &= \frac{1}{10e^t}\frac{|u(t)|}{4+|u(t)|} + \frac{1}{9}\cos(v(t)); \\ G_1\left(t, u(t), v(t)\right) &= \cos^{-1}\left(\frac{|u(t)|}{4}\right) + \frac{\frac{1}{3}e^{-t}}{1+|v(t)|}; \\ G_2\left(t, u(t), v(t)\right) &= \frac{\sin\left(|u^3(t)|\right)}{\sqrt{t^2+9}} + \sin^{-1}\left(\frac{t}{2}\right) cos\left(|v^3(t)|\right) + 4t. \end{aligned}$$

*Obviously, the functions* $F_i, G_i, (i = 1, 2)$ *are continuous. Furthermore, for all* $t \in \mathrm{J}$ *and each* $u_1, v_1, u_2, v_2 \in \mathbb{R}^n$, *we have* $(\mathrm{HP}_2)$ *satisfied as follows:*

$$\|F_i(t, u_1, v_1) - F_i(t, u_2, v_2)| \leq \mathrm{L}_{F_i}\|u_1 - u_2\| + \bar{\mathrm{L}}_{F_i}\|v_1 - v_2\|,$$

$$\|G_i(t, u_1, v_1) - G_i(t, u_2, v_2)| \leq \mathrm{L}_{G_i}\|u_1 - u_2\| + \bar{\mathrm{L}}_{G_i}\|v_1 - v_2\|,$$

*where* $\mathrm{L}_{F_1} = \dfrac{1}{100}, \mathrm{L}_{F_2} = \dfrac{1}{10}, \bar{\mathrm{L}}_{F_1} = \dfrac{1}{49}, \bar{\mathrm{L}}_{F_2} = \dfrac{1}{9}$,
$\mathrm{L}_{G_1} = \dfrac{1}{4}, \mathrm{L}_{G_2} = \dfrac{1}{3}, \bar{\mathrm{L}}_{G_1} = \dfrac{1}{3}, \bar{\mathrm{L}}_{G_2} = \dfrac{1}{2}.$
*and we can calculate that*

$$\mathbb{A}_1 = 1.08807, \mathbb{A}_2 = 0.963259, \mathbb{B}_1 = 1.12838, \mathbb{B}_2 = 1.08807.$$

*Thus, we get*

$$\begin{aligned} \bar{\mathbb{A}}_1 &= \left[\mathbb{A}_1\mathrm{L}_{F_1} + \mathbb{B}_1\mathrm{L}_{F_2}\right] < 0.123718, \\ \bar{\mathbb{A}}_2 &= \left[\mathbb{A}_2\mathrm{L}_{G_1} + \mathbb{B}_2\mathrm{L}_{G_2}\right] < 0.60350475, \\ \bar{\mathbb{B}}_1 &= \left[\mathbb{A}_1\bar{\mathrm{L}}_{F_1} + \mathbb{B}_1\bar{\mathrm{L}}_{F_2}\right] < 0.1475796372, \\ \bar{\mathbb{B}}_2 &= \left[\mathbb{A}_2\bar{\mathrm{L}}_{G_1} + \mathbb{B}_2\bar{\mathrm{L}}_{G_2}\right] < 0.8651213333. \end{aligned}$$

*Hence, all conditions of Theorem 9 are satisfied. Therefore the coupled system of nonlinear fractional* $\psi$*-integral equations* (2.24)-(2.25) *has one and only one solution. Consequently, by referring to Theorem 11, we easily conclude that the solution is* $\mathcal{UH}$ *stable.*

# Chapter 3

# Existence and Uniqueness of Solutions for a coupled system of nonlinear fractional integral equations in two Variables

Fractional integral equations have gained significant attention in recent years due to their ability to accurately model processes involving memory and hereditary properties. In this chapter, we investigate the existence and uniqueness of solutions for a coupled system of nonlinear fractional integral equations involving two variables. The system is considered under the framework of the $\psi$-Riemann–Liouville fractional integral, with particular focus on the special case $\psi(t) = t$, which corresponds to the classical Riemann–Liouville integral. By employing tools from fixed point theory in generalized Banach spaces, we establish sufficient conditions ensuring the existence and uniqueness of solutions. These results contribute to the broader understanding of nonlinear fractional systems and provide a foundation for further analytical and numerical studies.

### 3.0.1 Statement of the problem and main results

This chapter discusses the existence as well as uniqueness of solutions for the following system:

$$\begin{cases} u(x,y) = a_1(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f_1(s,t,u(s,t),v(s,t))\mathrm{dtds} \\ \qquad + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} f_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ v(x,y) = a_2(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\gamma_1-1}(y-t)^{\gamma_2-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} g_1(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ \qquad + \int_0^x \int_0^y \frac{(x-s)^{\gamma_1+\delta_1-1}(y-t)^{\gamma_2+\delta_2-1}}{\Gamma(\gamma_1+\delta_1)\Gamma(\gamma_2+\delta_2)} g_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \end{cases} \quad (x,y) \in \tilde{\mathrm{I}} \tag{3.1}$$

where $\tilde{I} := [0,\ \mathrm{T}_1] \times [0,\ \mathrm{T}_2],\ \mathrm{T}_1,\ \mathrm{T}_2 > 0, (\alpha_1,\alpha_2), (\beta_1,\beta_2), (\gamma_1,\gamma_2), (\delta_1,\delta_2) \in (0,1] \times (0,1]$, and $a_1, a_2 : \tilde{\mathrm{I}} \to \mathbb{R}, f_1, f_2, g_1, g_2 : \tilde{\mathrm{I}} \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

Our first result on the uniqueness is based on the Perov's fixed point theorem coupled with the Chebyshev vector-valued norm.

**Theorem 12** *Let the following assumptions hold:*

**(** $H_1'$ **)** *The functions* $f_1, f_2, g_1, g_2 : \tilde{\mathrm{I}} \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous.*

**(** $H_2'$ **)** *There exist constants* $\overline{\mathrm{p}}_i, \overline{\mathrm{q}}_i, \mathring{\mathrm{p}}_i, \mathring{\mathrm{q}}_i, i = 1, 2$ *such that*

$$|f_i(x,y,u_1,v_1) - f_i(x,y,u_2,y_2)| \leq \mathring{\mathrm{p}}_i |u_1 - u_2| + \mathring{\mathrm{q}}_i |v_1 - v_2|$$

$$|g_i(x,y,u_1,v_1) - g_i(x,y,u_2,y_2)| \leq \overline{\mathrm{p}}_i |u_1 - u_2| + \overline{\mathrm{q}}_i |v_1 - v_2|$$

*for all* $(x,y) \in \tilde{\mathrm{I}}$ *and each* $(u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$.

*Then the coupled system (2) possesses a unique solution provided that the spectral radius of the matrix $\mathbb{A}$ is less than one, where the matrix $\mathbb{A}$ is defined as below*

$$\mathbb{A}=\begin{pmatrix}\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathring{\mathrm{p}}_1+\frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathring{\mathrm{p}}_2 & \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathring{\mathrm{q}}_1+\frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathring{\mathrm{q}}_2\\ \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\overline{\mathrm{p}}_1+\frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\overline{\mathrm{p}}_2 & \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\overline{\mathrm{q}}_1+\frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\overline{\mathrm{q}}_2\end{pmatrix} \tag{3.2}$$

**Proof 4** *Consider the Banach space $C(\tilde{\mathrm{I}},\mathbb{R})$ equipped with the norm*

$$\|u\|_\infty=\sup_{(x,y)\in\tilde{\mathrm{I}}}|u(x,y)|.$$

*Consequently, the product space $\mathbb{X}:=C(\tilde{\mathbf{I}},\mathbb{R})\times C(\tilde{\mathbf{I}},\mathbb{R})$ is a generalized Banach space, endowed with the vector-valued norm*

$$\|(u,v)\|_{\mathbb{X},\infty}:=\begin{pmatrix}\|u\|_\infty\\ \|v\|_\infty\end{pmatrix}.$$

*In order to transform the problem (2) into a fixed point problem, we define an operator $\mathfrak{S}=(\mathfrak{S}_1,\mathfrak{S}_2):\mathbb{X}\to\mathbb{X}$ as:*

$$\mathfrak{S}(u,v)=(\mathfrak{S}_1(u,v),\mathfrak{S}_2(u,v)) \tag{3.3}$$

*where*

$$\begin{aligned}(\mathfrak{S}_1(u,v))(x,y)=a_1(x,y)&+\int_0^x\int_0^y\frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}f_1(s,t,u(s,t),v(s,t))\mathrm{dtds}\\&+\int_0^x\int_0^y\frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}f_2(s,t,u(s,t),v(s,t))\mathrm{dtds},\end{aligned}$$

*and*

$$\begin{aligned}(\mathfrak{S}_2(u,v))(x,y)=a_2(x,y)&+\int_0^x\int_0^y\frac{(x-s)^{\gamma_1-1}(y-t)^{\gamma_2-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)}g_1(s,t,u(s,t),v(s,t))\mathrm{dtds},\\&+\int_0^x\int_0^y\frac{(x-s)^{\gamma_1+\delta_1-1}(y-t)^{\gamma_2+\delta_2-1}}{\Gamma(\gamma_1+\delta_1)\Gamma(\gamma_2+\delta_2)}g_2(s,t,u(s,t),v(s,t))\mathrm{dtds},\end{aligned}$$

*We will use Perov's fixed point theorem to demonstrate that $\mathfrak{S}$ has a unique fixed point. To underline this fact, it is enough to show that $\mathfrak{S}$ is $\mathbb{A}$-contraction mapping on $\mathbb{X}$. In fact, for all $(u_1,v_1),(u_2,v_2)\in\mathbb{X}$ and $(x,y)\in\tilde{I}$, keeping in mind the definition of the operator $\mathfrak{S}_1$ together with assumption $(H_2')$, we can write*

$$
\begin{aligned}
&|(\mathfrak{S}_1(u_1,v_1))(x,y)-(\mathfrak{S}_1(u_2,v_2))(x,y)|\\
&\leq \int_0^x\int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\mathring{\mathrm{p}}_1|u_1(s,t)-u_2(s,t)|+\mathring{\mathrm{q}}_1|v_1(s,t)-v_2(s,t)|\right)\mathrm{dtds}\\
&+\int_0^x\int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}\left(\mathring{\mathrm{p}}_2|u_1(s,t)-u_2(s,t)|+\mathring{\mathrm{q}}_2|v_1(s,t)-v_2(s,t)|\right)\mathrm{dtds}\\
&\leq \left(\mathring{\mathrm{p}}_1\|u_1-u_2\|_\infty+\mathring{\mathrm{q}}_1\|v_1-v_2\|_\infty\right)\left(\int_0^x\int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\mathrm{dtds}\right)\\
&+\left(\mathring{\mathrm{p}}_2\|u_1-u_2\|_\infty+\mathring{\mathrm{q}}_2\|v_1-v_2\|_\infty\right)\left(\int_0^x\int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}\mathrm{dtds}\right)\\
&\leq \left(\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\right)\left(\mathring{\mathrm{p}}_1\|u_1-u_2\|_\infty+\mathring{\mathrm{q}}_1\|v_1-v_2\|_\infty\right)\\
&+\left(\frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\right)\left(\mathring{\mathrm{p}}_2\|u_1-u_2\|_\infty+\mathring{\mathrm{q}}_2\|v_1-v_2\|_\infty\right).
\end{aligned}
$$

*Hence*

$$
\begin{aligned}
\|\mathfrak{S}_1(u_1,v_1)-\mathfrak{S}_1(u_2,v_2)\|_\infty \leq &\left(\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathring{\mathrm{p}}_1+\frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathring{\mathrm{p}}_2\right)\|u_1-u_2\|_\infty\\
&+\left(\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathring{\mathrm{q}}_1+\frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathring{\mathrm{q}}_2\right)\|v_1-v_2\|_\infty
\end{aligned}
$$

*In a similar way, we get:*

$$
\begin{aligned}
\|\mathfrak{S}_2(u_1,v_1)-\mathfrak{S}_2(u_2,v_2)\|_\infty \leq &\left(\frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\overline{\mathrm{p}}_1+\frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\overline{\mathrm{p}}_2\right)\|u_1-u_2\|_\infty\\
&+\left(\frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\overline{\mathrm{q}}_1+\frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\overline{\mathrm{q}}_2\right)\|v_1-v_2\|_\infty
\end{aligned}
$$

*By the previous two inequalities we arrive at*

$$
\|\mathfrak{S}(u_1,v_1)-\mathfrak{S}(u_2,v_2)\|_{\mathbb{X},\infty}\preceq \mathbb{A}\,\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X},\infty},
$$

*where $\mathbb{A}$ is the matrix given by (3.2). Since the spectral radius $\rho(\mathbb{A})<1$, then $\mathfrak{S}$ is a Perov contraction. As a result of Perov's fixed point theorem, there exists a unique fixed point for the operator $\mathfrak{S}$, which corresponds to a unique solution for the coupled system (2) $\mathbb{X}$.*

Our second result on the uniqueness is based on the Perov's fixed point theorem combined with the Bielecki vector-valued norm.

**Theorem 13** *Let the assumptions $(H_1')$ and $(H_2')$ are satisfied. Then the the coupled system formulated in 2 has a unique solution.*

**Proof 5** *Before moving further, let us consider on the space $C(\tilde{\mathrm{I}},\mathbb{R})$ the Bielecki norm $\|\cdot\|_{\mathfrak{B}}$ defined as below:*

$$
\|u\|_{\mathfrak{B}} := \sup_{(x,y)\in\tilde{\mathrm{I}}}\frac{|u(x,y)|}{e^{\lambda(x+y)}},\quad \lambda>0. \tag{3.4}
$$

*It is obvious that $C(\tilde{\mathrm{I}}, \mathbb{R})$ is a Banach space with this norm $\|\cdot\|_{\mathfrak{B}}$ since it is equivalent to the infinity norm $\|\cdot\|_\infty$. Consequently, the product space $\mathbb{X} := C(\tilde{\mathrm{I}}, \mathbb{R}) \times C(\tilde{\mathrm{I}}, \mathbb{R})$ is a generalized Banach space, endowed with the Bielecki vector-valued norm*

$$\|(u,v)\|_{\mathbb{X},\mathfrak{B}} = \begin{pmatrix} \|u\|_{\mathfrak{B}} \\ \|v\|_{\mathfrak{B}} \end{pmatrix}.$$

*Now, we apply Perov's fixed point theorem to prove that $\mathfrak{S}$ has a unique fixed point. Indeed, it is enough to show that $\mathfrak{S}$ is $\mathfrak{X}$-contraction mapping on $\mathbb{X}$ via the Bielecki's vector-valued norm. For this end, given $(u_1, v_1), (u_2, v_2) \in \mathbb{X}$ and $(x,y) \in \tilde{\mathrm{I}}$, using $(H_2')$, and Lemma 3, we can get*

$$\begin{aligned}
&|(\mathfrak{S}_1(u_1,v_1))(x,y) - (\mathfrak{S}_1(u_2,v_2))(x,y)| \\
&\le \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left(\mathring{\mathrm{p}}_1 |u_1(s,t) - u_2(s,t)| + \mathring{\mathrm{q}}_1 |v_1(s,t) - v_2(s,t)|\right) \mathrm{dtds} \\
&+ \int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} \left(\mathring{\mathrm{p}}_2 |u_1(s,t) - u_2(s,t)| + \mathring{\mathrm{q}}_2 |v_1(s,t) - v_2(s,t)|\right) \mathrm{dtds} \\
&\le \left(\mathring{\mathrm{p}}_1 \|u_1 - u_2\|_\infty + \mathring{\mathrm{q}}_1 \|v_1 - v_2\|_\infty\right) \left(\int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\lambda(s+t)} \mathrm{dtds}\right) \\
&+ \left(\mathring{\mathrm{p}}_2 \|u_1 - u_2\|_\infty + \mathring{\mathrm{q}}_2 \|v_1 - v_2\|_\infty\right) \left(\int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} e^{\lambda(s+t)} \mathrm{dtds}\right) \\
&\le \frac{e^{\lambda(x+y)}}{\lambda^{\alpha_1+\alpha_2}} \left(\mathring{\mathrm{p}}_1 \|u_1 - u_2\| \|_{\mathfrak{B}} + \mathring{\mathrm{q}}_1 \| v_1 - v_2\|_{\mathfrak{B}}\right) \\
&+ \frac{e^{\lambda(x+y)}}{\lambda^{\alpha_1+\beta_1+\alpha_2+\beta_2}} \left(\mathring{\mathrm{p}}_2 \|u_1 - u_2\| \|_{\mathfrak{B}} + \mathring{\mathrm{q}}_2 \| v_1 - v_2\|_{\mathfrak{B}}\right).
\end{aligned}$$

*Hence*

$$\begin{aligned}
\|\mathfrak{S}_1(u_1,v_1) - \mathfrak{S}_1(u_1,v_2)\|_{\mathfrak{B}} \le &\left(\frac{\mathring{\mathrm{p}}_1}{\lambda^{\alpha_1+\alpha_2}} + \frac{\mathring{\mathrm{p}}_2}{\lambda^{\alpha_1+\beta_1+\alpha_2+\beta_2}}\right) \|u_1 - u_2\|_{\mathfrak{B}} \\
&+ \left(\frac{\mathring{\mathrm{q}}_1}{\lambda^{\alpha_1+\alpha_2}} + \frac{\mathring{\mathrm{q}}_2}{\lambda^{\alpha_1+\beta_1+\alpha_2+\beta_2}}\right) \|v_1 - v_2\|_{\mathfrak{B}}
\end{aligned}$$

*As previously, we can derive*

$$\begin{aligned}
\|\mathfrak{S}_2(u_1,v_1) - \mathfrak{S}_2(u_1,v_2)\|_{\mathfrak{B}} \le &\left(\frac{\overline{\mathrm{p}}_1}{\lambda^{\gamma_1+\gamma_2}} + \frac{\overline{\mathrm{p}}_2}{\lambda^{\gamma_1+\delta_1+\gamma_2+\delta_2}}\right) \|u_1 - u_2\|_{\mathfrak{B}} \\
&+ \left(\frac{\overline{\mathrm{q}}_1}{\lambda^{\gamma_1+\gamma_2}} + \frac{\overline{\mathrm{q}}_2}{\lambda^{\gamma_1+\delta_1+\gamma_2+\delta_2}}\right) \|v_1 - v_2\|_{\mathfrak{B}}
\end{aligned}$$

*This implies that*

$$\|\mathfrak{S}(u_1,v_1) - \mathfrak{S}(u_2,v_2)\|_{\mathbb{X},\mathfrak{B}} \preceq \mathfrak{X} \|(u_1,v_1) - (u_2,v_2)\|_{\mathbb{X},\mathfrak{B}}$$

*where*

$$\mathfrak{X} = \begin{pmatrix} \frac{\mathring{\mathrm{p}}_1}{\lambda^{\alpha_1+\alpha_2}} + \frac{\mathring{\mathrm{p}}_2}{\lambda^{\alpha_1+\beta_1+\alpha_2+\beta_2}} & \frac{\mathring{\mathrm{q}}_1}{\lambda^{\alpha_1+\alpha_2}} + \frac{\mathring{\mathrm{q}}_2}{\lambda^{\alpha_1+\beta_1+\alpha_2+\beta_2}} \\ \frac{\overline{\mathrm{p}}_1}{\lambda^{\gamma_1+\gamma_2}} + \frac{\overline{\mathrm{p}}_2}{\lambda^{\gamma_1+\delta_1+\gamma_2+\delta_2}} & \frac{\overline{\mathrm{q}}_1}{\lambda^{\gamma_1+\gamma_2}} + \frac{\overline{\mathrm{q}}_2}{\lambda^{\gamma_1+\delta_1+\gamma_2+\delta_2}} \end{pmatrix}. \tag{3.5}$$

*Taking $\lambda$ large enough it follows that the matrix $\mathfrak{X}$ is convergent to zero and thus, an application of Perov's theorem shows that $\mathfrak{S}$ has a unique fixed point. So the coupled system (2) has a unique solution in $\mathbb{X}$.*

Now we give our existence result for the problem (2). The arguments are based on the Schauder's fixed point theorem in generalized Banach spaces.

**Theorem 14** *Assume that $(H_1')$ holds. In addition assume that:*
*( $H_3'$ ) There exist positive real constants* $\mathrm{a}_i, \mathrm{b}_i, \mathrm{c}_i, \mathrm{a}_i^*, \ \mathrm{b}_i^*, \mathrm{c}_i^*, i = 1, 2,$ *such that*

$$|f_i(x, u, v)| \leq \mathrm{a}_i + \mathrm{b}_i|u| + \mathrm{c}_i|v|,$$
$$|g_i(x, u, v)| \leq \mathrm{a}_i^* + \mathrm{b}_i^*|u| + \mathrm{c}_i^*|v|,$$

*for all $(x, y) \in \tilde{\mathrm{I}}$ and each $(u, v) \in \mathbb{R}^2$.*
*Also, if $\rho(\mathbb{D}) < 1$, such that*

$$\mathbb{D} = \begin{pmatrix} \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{b}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{b}_2 & \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{c}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{c}_2 \\ \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{b}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{b}_2^* & \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{c}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{c}_2^* \end{pmatrix}. \tag{3.6}$$

*Then the coupled system mentioned in (2) has at least one solution.*

**Proof 6** *We shall show that the operator $\mathfrak{S}$ defined in (3.3) satisfies the hypotheses of the Schauder's fixed point theorem generalized Banach space ( Theorem 7). Define a subset $\mathbb{B}_\varrho$ of $\mathbb{X}$ by*

$$\mathbb{B}_\varrho = \{(u, v) \in \mathbb{X} : \|(u, v)\|_{\mathbb{X},\infty} \leq \varrho\} \tag{3.7}$$

*with $\varrho := (\varrho_1, \varrho_2) \in \mathbb{R}_+^2$ such that*

$$\begin{cases} \varrho_1 \geq \sigma_1^*\mathbb{N}_1^* + \sigma_2^*\mathbb{N}_2^*, \\ \varrho_2 \geq \sigma_3^*\mathbb{N}_1^* + \sigma_4^*\mathbb{N}_2^*. \end{cases}$$

*Where $\mathbb{N}_1, \mathbb{N}_2$ and $\sigma_i^*; i = 1, 2$ are non-negative real numbers that will be specified later. Moreover, notice that $\mathbb{B}_\varrho$ is closed, convex and bounded subset of the generalized Banach space $\mathbb{X}$. For clarity, we will divide the remain of the proof into several steps.*

***Step*** 1*: $\mathfrak{S}(u, v) \in \mathbb{B}_\varrho$, for any $(u, v)$. Indeed, for $(u, v), (\bar{x}, \bar{y}) \in \mathbb{X}$ and for each $(x, y) \in \tilde{I}$, from the definition of the operator $\mathfrak{S}_1$ and assumption $(H_3')$, we can get*

$$\begin{aligned} |\mathfrak{S}_1(u, v)(x, y)| \leq{}& |a_1(x, y)| + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (\mathrm{a}_1 + \mathrm{b}_1|u(s,t)| + \mathrm{c}_1|v(s,t)|)\,\mathrm{dtds} \\ &+ \int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} (\mathrm{a}_2 + \mathrm{b}_2|u(s,t)| + \mathrm{c}_2|v(s,t)|)\,\mathrm{dtds} \\ \leq{}& \|a_1\|_\infty + (\mathrm{a}_1 + \mathrm{b}_1\|u\|_\infty + \mathrm{c}_1\|v\|_\infty) \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\mathrm{dtds} \\ &+ (\mathrm{a}_2 + \mathrm{b}_2\|u\|_\infty + \mathrm{c}_2\|v\|_\infty) \int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}\mathrm{dtds} \\ \leq{}& \|a_1\|_\infty + \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} (\mathrm{a}_1 + \mathrm{b}_1\|u\|_\infty + \mathrm{c}_1\|v\|_\infty) \\ &+ \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)} (\mathrm{a}_2 + \mathrm{b}_2\|u\|_\infty + \mathrm{c}_2\|v\|_\infty). \end{aligned}$$

*Hence*

$$\begin{aligned} \|\mathfrak{S}_1(u, v)\|_\infty \leq{}& \|a_1\|_\infty + \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{a}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{a}_2 \\ &+ \left(\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{b}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{b}_2\right)\|u\|_\infty \\ &+ \left(\frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{c}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{c}_2\right)\|v\|_\infty \end{aligned}$$

*By similar procedure, we can obtain*

$$\begin{aligned}\|\mathfrak{S}_1(u,v)\|_\infty \le \|a_1^*\|_\infty &+ \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{a}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{a}_2^* \\ &+ \left(\frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{b}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{b}_2^*\right)\|u\|_\infty \\ &+ \left(\frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{c}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{c}_2^*\right)\|v\|_\infty\end{aligned}$$

*Thus the above inequalities can be written in the vectorial form as follows*

$$\|\mathfrak{S}(u,v)\|_{\mathbb{X},\infty} := \begin{pmatrix} \|\mathfrak{S}_1(u,v)\|_\infty \\ \|\mathfrak{S}_2(u,v)\|_\infty \end{pmatrix}$$

$$\mathfrak{D}\begin{pmatrix}\|u\|_\infty \\ \|v\|_\infty\end{pmatrix} + \begin{pmatrix} \|a_1\|_\infty + \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{a}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{a}_2 \\ \|a_2\|_\infty + \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{a}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{a}_2^*\end{pmatrix},$$

*where* $\mathbb{D}$ *is the matrix given by (3.6)*

*Now we look for* $\varrho = (\varrho_1, \varrho_2) \in \mathbb{R}_+^2$ *such that* $\mathfrak{S}(u,v) \in \mathbb{B}_\varrho$ *for any* $(u,v) \in \mathbb{B}_\varrho$*. To this end, according to (3.7), it is sufficient to show*

$$\mathbb{D}\begin{pmatrix}\varrho_1 \\ \varrho_2\end{pmatrix} + \begin{pmatrix} \|a_1\|_\infty + \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{a}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{a}_2 \\ \|a_2\|_\infty + \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{a}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{a}_2^*\end{pmatrix} \preceq \begin{pmatrix}\varrho_1 \\ \varrho_2\end{pmatrix},$$

*Equivalently*

$$\begin{pmatrix}\mathbb{N}_1^* \\ \mathbb{N}_2^*\end{pmatrix} \preceq (\mathbb{I} - \mathbb{D})\begin{pmatrix}\varrho_1 \\ \varrho_2\end{pmatrix}. \tag{3.8}$$

*Where*

$$\begin{pmatrix}\mathbb{N}_1^* \\ \mathbb{N}_2^*\end{pmatrix} = \begin{pmatrix} \|a_1\|_\infty + \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{a}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{a}_2 \\ \|a_2\|_\infty + \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{a}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{a}_2^*\end{pmatrix}.$$

*Since the matrix* $\mathbb{D}$ *is convergent to zero. It yields, from Definition 11 that the matrix* $(\mathbb{I} - \mathbb{D})$ *is nonsingular and* $(\mathbb{I} - \mathbb{D})^{-1}$ *has nonnegative elements. So, (3.8) is equivalent to*

$$\begin{pmatrix}\varrho_1 \\ \varrho_2\end{pmatrix} \succeq (\mathbb{I} - \mathbb{D})^{-1}\begin{pmatrix} \frac{\mathrm{T}_1^{\alpha_1}\ \mathrm{T}_2^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\mathrm{a}_1 + \frac{\mathrm{T}_1^{\alpha_1+\beta_1}\ \mathrm{T}_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}\mathrm{a}_2 \\ \frac{\mathrm{T}_1^{\gamma_1}\ \mathrm{T}_2^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\mathrm{a}_1^* + \frac{\mathrm{T}_1^{\gamma_1+\delta_1}\ \mathrm{T}_2^{\gamma_2+\delta_2}}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)}\mathrm{a}_2^*\end{pmatrix}.$$

*In addition, if we take*

$$(\mathbb{I} - \mathbb{D})^{-1} = \begin{pmatrix}\sigma_1^* & \sigma_2^* \\ \sigma_3^* & \sigma_4^*\end{pmatrix},$$

*we can arrive at*

$$\begin{cases}\varrho_1 \ge \sigma_1^*\mathbb{N}_1^* + \sigma_2^*\mathbb{N}_2^*, \\ \varrho_2 \ge \sigma_3^*\mathbb{N}_1^* + \sigma_4^*\mathbb{N}_2^*\end{cases}$$

*Which means that* $\mathfrak{S}(u,v) \in \mathbb{B}_\varrho$*.*

***Step 2***: $\mathfrak{S}$ *is compact and continuous. Firstly, observe that the operator* $\mathfrak{S}$ *is continuous, owing* to the continuity of functions of $f_1, f_2, g_1$ and $g_2$. The next task is to show that $\mathfrak{S}$ is uniformly bounded on $\mathbb{B}_\varrho$. From (3.7), and for each $(u,v) \in \mathbb{B}_\varrho$ we can get

$$\|\mathfrak{S}(u,v)\|_{\mathbb{X},\infty} := \left( \begin{array}{c} \|\mathfrak{S}_1(u,v)\|_\infty \\ \|\mathfrak{S}_2(u,v)\|_\infty \end{array} \right) \preceq \mathbb{D}\begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix} + \begin{pmatrix} \mathbb{N}_1^* \\ \mathbb{N}_2^* \end{pmatrix} < \infty.$$

*This proves that $\mathfrak{S}$ is uniformly bounded.*

*Finally, we show that $\mathfrak{S}(\mathbb{B}_\varrho)$ is equicontinuous. Let $(u,v) \in \mathbb{B}_\varrho$ and any $(x_1,y_1), (x_2,y_2) \in \tilde{I}$, with $x_1 < x_2$ and $y_1 < y_2$. Taking $(H_3')$ into consideration we can find*

$$\begin{aligned}
&|\mathfrak{S}_1(u,v)(x_2,y_2) - \mathfrak{S}_1(u,v)(x_1,y_1)| \le |a_1(x_2,y_2) - a_1(x_2,y_2)| \\
&+ \frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{y_1} \left[(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-1} - (x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}\right] \mathrm{dtds} \\
&+ \frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1} \mathrm{dtds} \\
&+ \frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1} \mathrm{dtds} \\
&+ \frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1} \mathrm{dtds} \\
&+ \frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} \\
&\int_0^{x_1}\int_0^{y_1} \left[(x_1-s)^{\alpha_1+\beta_1-1}(y_1-t)^{\alpha_2+\beta_2-1} - (x_2-s)^{\alpha_1+\beta_1-1}(y_2-t)^{\alpha_2+\beta_2-1}\right] \mathrm{dtds} \\
&+ \frac{\mathrm{a}_2 + \mathrm{b}_2\varrho_1 + \mathrm{c}_2\varrho_2}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{\alpha_1+\beta_1-1}(y_2-t)^{\alpha_2+\beta_2-1} \mathrm{dtds} \\
&+ \frac{\mathrm{a}_2 + \mathrm{b}_2\varrho_1 + \mathrm{c}_2\varrho_2}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{\alpha_1+\beta_1-1}(y_2-t)^{\alpha_2+\beta_2-1} \mathrm{dtds} \\
&+ \frac{\mathrm{a}_2 + \mathrm{b}_2\varrho_1 + \mathrm{c}_2\varrho_2}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{\alpha_1+\beta_1-1}(y_2-t)^{\alpha_2+\beta_2-1} \mathrm{dtds} \\
&\le |a_1(x_2,y_2) - a_1(x_2,y_2)| \\
&+ 2\frac{\mathrm{a}_1 + \mathrm{b}_1\varrho_1 + \mathrm{c}_1\varrho_2}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \left[x_2^{\alpha_1}(y_2-y_1)^{\alpha_2} - (x_2-x_1)^{\alpha_1}(y_2-y_1)^{\alpha_2} + y_2^{\alpha_2}(x_2-x_1)^{\alpha_1}\right] \\
&+ 2\frac{\mathrm{a}_2 + \mathrm{b}_2\varrho_1 + \mathrm{c}_2\varrho_2}{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)} \\
&\left[x_2^{\alpha_1+\beta_1}(y_2-y_1)^{\alpha_2+\beta_2} - (x_2-x_1)^{\alpha_1+\beta_1}(y_2-y_1)^{\alpha_2+\beta_2} + y_2^{\alpha_2+\beta_2}(x_2-x_1)^{\alpha_1+\beta_1}\right].
\end{aligned}$$

*Similarly, it can be shown that*

$$\begin{aligned}
&|\mathfrak{S}_2(u,v)(x_2,y_2) - \mathfrak{S}_2 u(x_1,y_1)| \le |a_2(x_1,y_1) - a_2(x_2,y_2)| \\
&+ 2\frac{\mathrm{a}_1^* + \mathrm{b}_1^*\varrho_1 + \mathrm{c}_1^*\varrho_2}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} \left[x_2^{\gamma_1}(y_2-y_1)^{\gamma_2} - (x_2-x_1)^{\gamma_1}(y_2-y_1)^{\gamma_2} + y_2^{\gamma_2}(x_2-x_1)^{\gamma_1}\right] \\
&+ 2\frac{\mathrm{a}_2^* + \mathrm{b}_2^*\varrho_1 + \mathrm{c}_2^*\varrho_2}{\Gamma(\gamma_1+\delta_1+1)\Gamma(\gamma_2+\delta_2+1)} \\
&\left[x_2^{\gamma_1+\delta_1}(y_2-y_1)^{\gamma_2+\delta_2} - (x_2-x_1)^{\gamma_1+\delta_1}(y_2-y_1)^{\gamma_2+\delta_2} + y_2^{\gamma_2+\delta_2}(x_2-x_1)^{\gamma_1+\delta_1}\right].
\end{aligned}$$

*As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the above inequalities tends to zero independently of $(u,v) \in \mathbb{B}_\varrho$. Hence, the operators $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equicontinuous and thus the operator $\mathbb{G}$ is equicontinuous. By Arzelà-Ascoli's theorem, we deduce that $\mathfrak{S}$ is a compact operator. Thus all the assumptions of* (7) *are satisfied. As a consequence of Schauder's fixed point theorem, we conclude that the operator $\mathfrak{S}$ defined by (3.3) has at least one fixed point $(u,v) \in \mathbb{B}_\varrho$, which is just the solution of system (2). This completes the proof of the Theorem(3.3).*

## 3.1 Applications

We finish this chapter by constructing examples regarding the above results.

**Exemple 2** *Consider the following system:*

$$\begin{cases} u(x,y) = (\mathrm{x}-\mathrm{y})e^{-(x^2+y^2)} + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f_1(s,t,u(s,t),v(s,t))\mathrm{dtds} \\ \qquad + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} f_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ v(x,y) = \sin(\mathrm{x}+\mathrm{y}) + \int_0^x \int_0^y \frac{(x-s)^{\gamma_1-1}(y-t)^{\gamma_2-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} g_1(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ \qquad + \int_0^x \int_0^y \frac{(x-s)^{\gamma_1+\delta_1-1}(y-t)^{\gamma_2+\delta_2-1}}{\Gamma(\gamma_1+\delta_1)\Gamma(\gamma_2+\delta_2)} g_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \end{cases} \quad (x,y)\in\tilde{\mathrm{I}} \tag{3.9}$$

*where*

$$\alpha=(\alpha_1,\alpha_2)=(0.5,0.6), \beta=(\beta_1,\beta_2)=(0.8,0.75), \gamma=(\gamma_1,\gamma_2)=(0.25,0.75),$$
$$\delta=(\delta_1,\delta_2)=(0.75,0.25), \mathrm{T}_1=\mathrm{T}_2=1, \tilde{\mathrm{I}}=[0,1]\times[0,1],$$

*and*

$$\begin{aligned} \zeta_1(\mathrm{x},\mathrm{y}) &= (\mathrm{x}-\mathrm{y})e^{-(x^2+y^2)} \\ \zeta_2(\mathrm{x},\mathrm{y}) &= \sin(\mathrm{x}+\mathrm{y}) \\ \mathrm{f}_1(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \frac{\sin(|\mathrm{u}^3(\mathrm{x},\mathrm{y})|)}{\sqrt{(\mathrm{x}+\mathrm{y})^2+16}} + \sin^{-1}\left(\frac{(\mathrm{x}+\mathrm{y})}{4}\right) cos\left(|\mathrm{v}^3(\mathrm{x},\mathrm{y})|\right) + 4(\mathrm{x}+\mathrm{y}) \\ \mathrm{f}_2(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \cos^{-1}\left(\frac{|\mathrm{u}(\mathrm{x},\mathrm{y})|}{4}\right) + \frac{\frac{1}{4}e^{-(\mathrm{x}+\mathrm{y})}}{1+|\mathrm{v}(\mathrm{x},\mathrm{y})|} \\ \mathrm{g}_1(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \frac{\mathrm{u}(\mathrm{x},\mathrm{y})}{e^{\mathrm{x}+\mathrm{y}+4}(1+|\mathrm{u}(\mathrm{x},\mathrm{y})|)} + \frac{\mathrm{v}(\mathrm{x},\mathrm{y})}{\mathrm{x}+\mathrm{y}+2} \\ \mathrm{g}_2(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \frac{1}{(\mathrm{x}+\mathrm{y}+2)^2}\left(\frac{\mathrm{u}(\mathrm{x},\mathrm{y})+\sqrt{1+\mathrm{u}^2(\mathrm{x},\mathrm{y})}}{2} + \sin|\mathrm{v}(\mathrm{x},\mathrm{y})|\right) \end{aligned}$$

*Clearly, the functions* $g_1$ *and* $g_2$ *are continuous. Moreover, for any* $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2$ *and* $(\mathrm{x},\mathrm{y})\in\tilde{\mathrm{I}}$ *we have*

$$\begin{aligned} |\mathrm{f}_1(\mathrm{x},\mathrm{y},\mathrm{u}_1,\mathrm{v}_1) - \mathrm{g}_1(\mathrm{x},\mathrm{y},\mathrm{u}_2,\mathrm{v}_2)| &\le 0.25|\mathrm{u}_1-\mathrm{u}_2| + 0.25|\mathrm{v}_1-\mathrm{v}_2| \\ |\mathrm{f}_2(\mathrm{x},\mathrm{y},\mathrm{u}_1,\mathrm{v}_1) - \mathrm{g}_2(\mathrm{x},\mathrm{y},\mathrm{u}_2,y_2)| &\le 0.25|\mathrm{u}_1-\mathrm{u}_2| + 0.25|\mathrm{v}_1-\mathrm{v}_2| \\ |\mathrm{g}_1(\mathrm{x},\mathrm{y},\mathrm{u}_1,\mathrm{v}_1) - \mathrm{g}_1(\mathrm{x},\mathrm{y},\mathrm{u}_2,\mathrm{v}_2)| &\le e^{-2}|\mathrm{u}_1-\mathrm{u}_2| + 0.25|\mathrm{v}_1-\mathrm{v}_2| \\ |\mathrm{g}_2(\mathrm{x},\mathrm{y},\mathrm{u}_1,\mathrm{v}_1) - \mathrm{g}_2(\mathrm{x},\mathrm{y},\mathrm{u}_2,y_2)| &\le 0.25|\mathrm{u}_1-\mathrm{u}_2| + 0.25|\mathrm{v}_1-\mathrm{v}_2|. \end{aligned}$$

*So assumption (* $H_2'$ *) is satisfied with*

$$\overline{\mathrm{p}}_1 = e^{-2}, \mathring{\mathrm{q}}_1 = \overline{\mathrm{q}}_1 = \mathring{\mathrm{p}}_1 = \mathring{\mathrm{p}}_2 = \mathring{\mathrm{q}}_2 = \overline{\mathrm{p}}_2 = \overline{\mathrm{q}}_2 = 0.25.$$

*Furthermore, the matrix* $\mathbb{A}$ *given by* (3.2) *has the following form*

$$\mathbb{A} = \begin{pmatrix} \frac{1}{4\Gamma(1.5)\Gamma(1.6)} + \frac{1}{4\Gamma(2.3)\Gamma(2.35)} & \frac{1}{4\Gamma(1.5)\Gamma(1.6)} + \frac{1}{4\Gamma(2.3)\Gamma(2.35)} \\ \frac{1}{e^{-2}\Gamma(1.25)\Gamma(1.75)} + \frac{1}{4\Gamma(2)\Gamma(2)} & \frac{1}{4\Gamma(1.75)\Gamma(1.25)} + \frac{1}{4\Gamma(2)\Gamma(2)} \end{pmatrix}$$

*Using the Matlab program we can get the eigenvalues of* $\mathbb{A}$ *as follows* $\sigma_1 = 0.0698, \sigma_2 =$ *0.9742. This shows that* $\mathbb{A}$ *converges to zero. Therefore, by Theorem 12, the coupled system* (3.9) *has a unique solution.*

**Exemple 3** *Consider the following system:*

$$\begin{cases} u(x,y) = \mathrm{x}+\mathrm{y}^2 + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1+)\Gamma(\alpha_2)} f_1(s,t,u(s,t),v(s,t))\mathrm{dtds} \\ \qquad\qquad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\alpha_1+\beta_1-1}(y-t)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)} f_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ v(x,y) = \mathrm{x}e^{\mathrm{x}}+\mathrm{y} + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\gamma_1-1}(y-t)^{\gamma_2-1}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} g_1(s,t,u(s,t),v(s,t))\mathrm{dtds}, \\ \qquad\qquad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{\gamma_1+\delta_1-1}(y-t)^{\gamma_2+\delta_2-1}}{\Gamma(\gamma_1+\delta_1)\Gamma(\gamma_2+\delta_2)} g_2(s,t,u(s,t),v(s,t))\mathrm{dtds}, \end{cases} \quad (x,y)\in\tilde{\mathrm{I}} \tag{3.10}$$

*where*
$\alpha=(\alpha_1,\alpha_2)=(0.25,0.75), \beta=(\beta_1,\beta_2)=(0.75,0.25), \gamma=(\gamma_1,\gamma_2)=(0.25,0.75), \delta=(\delta_1,\delta_2)=(0.75,0.25), \mathrm{T}_1=10,\ \mathrm{T}_2=20, \tilde{\mathrm{I}}=[0,10]\times[0,20]$,
*and*

$$\begin{aligned} \zeta_1(\mathrm{x},\mathrm{y}) &= \mathrm{x}+\mathrm{y}^2 \\ \zeta_2(\mathrm{x},\mathrm{y}) &= \mathrm{x}e^{\mathrm{x}}+\mathrm{y} \\ \mathrm{f}_1(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \left(1+e^{\mathrm{x}+\mathrm{y}}\right)\ln(1+|\mathrm{u}(\mathrm{x},\mathrm{y})|)+e^{\mathrm{xy}}\arctan\mathrm{v}(\mathrm{x},\mathrm{y}) \\ \mathrm{f}_2(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \frac{\mathrm{x}+\mathrm{y}}{(1+|\mathrm{u}(\mathrm{x},\mathrm{y})|+|\mathrm{v}(\mathrm{x},\mathrm{y})|)} \\ \mathrm{g}_1(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \left(1+e^{\mathrm{x}+\mathrm{y}}\right)\ln(1+|\mathrm{u}(\mathrm{x},\mathrm{y})|)+e^{\mathrm{xy}}\arctan\mathrm{v}(\mathrm{x},\mathrm{y}) \\ \mathrm{g}_2(\mathrm{x},\mathrm{y},\mathrm{u}(\mathrm{x},\mathrm{y}),\mathrm{v}(\mathrm{x},\mathrm{y})) &= \frac{\mathrm{x}+\mathrm{y}}{(1+|\mathrm{u}(\mathrm{x},\mathrm{y})|+|\mathrm{v}(\mathrm{x},\mathrm{y})|)}. \end{aligned}$$

*Observe that* $\mathrm{f}_1,\ \mathrm{f}_2, \mathrm{g}_1,\ \mathrm{g}_2$ *are continuous and satisfy the condition* $(H_2')$ *with*

$$\mathring{\mathrm{p}}_1=\overline{\mathrm{p}}_1=1+e^2, \mathring{\mathrm{q}}_1=\overline{\mathrm{q}}_1=e, \mathring{\mathrm{p}}_2=\mathring{\mathrm{q}}_2=\overline{\mathrm{p}}_2=\overline{\mathrm{q}}_2=2.$$

*In addition, the matrix* $\mathfrak{X}$ *given by* (3.5) *has the following form*

$$\mathfrak{X}=\left(\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)\begin{pmatrix} e^2+3 & e+2 \\ e+3 & e+2 \end{pmatrix}.$$

*Taking* $\lambda$ *large enough it follows that the matrix* $\mathfrak{X}$ *is convergent to zero and thus, an application of Theorem 13 shows that the coupled system* (3.10) *has a unique solution.*

# Conclusion and Perspectives

In this memory, we have conducted a comprehensive study on the existence, uniqueness, and stability of solutions for a class of coupled systems of nonlinear integral equations involving the $\psi$-Riemann–Liouville fractional integral. Our investigation has been carried out within the framework of generalized Banach spaces endowed with vector-valued norms, commonly referred to as Perov-type spaces. This choice of analytical setting has allowed us to deal effectively with the complex structure of the systems under consideration, particularly due to the nonlocal and memory-dependent nature of the fractional integral operator. By combining the tools of fixed point theory with the structure of vector-valued norms and convergent-to-zero matrices, we have been able to obtain several important theoretical results. Using Schauder's fixed point theorem, we established the existence of at least one solution under suitable compactness and continuity assumptions. Then, by applying Perov's fixed point theorem, which generalizes Banach's contraction principle to the setting of vector-valued norms, we proved the uniqueness of the solution. Moreover, we have extended our analysis to address the concept of Ulam–Hyers stability, a crucial property that ensures the robustness of solutions under small perturbations in the data. By employing the matrix convergence technique, we derived sufficient conditions under which the system exhibits Ulam–Hyers and generalized Ulam–Hyers–Rassias stability. The theoretical findings were supported by carefully chosen illustrative examples that confirm the effectiveness and applicability of our approach.

Beyond the results obtained, this work opens up several promising directions for further research. One natural extension would be to apply similar analytical techniques to systems of nonlinear fractional differential equations, particularly those involving Caputo or Hadamard derivatives, which are also commonly used in modeling real-world phenomena with memory. Furthermore, incorporating time-delay effects or impulsive dynamics into the fractional framework would offer a more realistic representation of many physical, biological, and engineering systems. Another valuable direction would be the development of numerical methods aligned with the theoretical conditions established in this thesis, allowing for practical approximations of solutions while preserving properties such as stability and convergence. Additionally, exploring stochastic variants of fractional integral equations could be of great interest, especially in contexts where uncertainty and random fluctuations play a significant role. Finally, applying the analytical tools developed here to specific models in population dynamics, viscoelastic materials, control theory, or finance would further demonstrate the practical relevance of this work. In conclusion, this thesis contributes to the growing literature on fractional calculus and fixed point theory by bridging abstract mathematical analysis with applicable results, providing a solid foundation for both theoretical advancement and real-world modeling.

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# شهادة الترخيص بالإيداع

أنا الأستاذ:

بوطيارة عبداللطيف

بصفتي مشرف والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـــ

Some qualitative results for a new class of fractional integral Equations

من انجاز الطالب(ة):

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امضاء المسؤول عن التصحيح

A. BOUTIARA

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