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Relationship between some Linear Matrix Functions

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In front of a jury composed of :

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Dedication

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ملخص

في هذا العمل نركز على بعض دوال المصفوفة الخطية ومجالاتها المحددة على مجال الأعداد المركبة $\mathbb{C}$, نناقش التقاطعات، وإدراج مجالين لبعض دوال المصفوفة الخطية والمساواة بينهما، ثم بتطبيق مصفوفة محددة من الأدوات التحليلية على الرتب والنطاقات، ننظر في أشكال معينة من معادلات المصفوفة الخطية، التي يمكن التعبير عن الحلول العامة لها عن طريق دوال مصفوفة خطية صريحة محددة لإقامة بعض العلاقات بين مجالاتها. نتيجة لذلك، نشتق وصلات بين بعض المجالات المقيدة ببعض معادلات المصفوفة. علاوة على ذلك، فإننا نوصف العلاقة بين المعكوسات المعممة لمصفوفتين.

الكلمات المفتاحية: مصفوفة الكتلة, الحل العام, معكوس معمم, معادلة المصفوفة, الصورة, الرتبة.

Abstract

In this work we focus on some linear matrix functions and their domains defined over the field of complex numbers $\mathbb{C}$, we discuss the intersections, the inclusion as well as the equality of two domains of some linear matrix functions, then by applying a specific matrix analytic tools on ranks and ranges, we consider certain forms of linear matrix equations, the general solutions to which can be expressed via specific explicit linear matrix functions to establish some relationships between its domains. As a consequence, we derive connections between some domains constrained by some matrix equations. Further, we characterize the relationship between generalized inverses of two matrices.

Keywords: block matrix; general solution; generalized inverse; matrix equation; matrix expression; range; rank.

Résumé

Dans ce travail, nous nous concentrons sur certaines fonctions matricielles linéaires et leurs domaines définis sur le corps des nombres complexes $\mathbb{C}$, nous discutons des intersections, de l'inclusion ainsi que de l'égalité de deux domaines de certaines fonctions matricielles linéaires, puis en appliquant des outils analytiques matriciels spécifiques sur les rangs et les images, nous considérons certaines formes d'équations matricielles linéaires, dont les solutions générales peuvent être exprimées via des fonctions matricielles linéaires explicites spécifiques afin d'établir certaines relations entre ses domaines. En conséquence, nous dérivons des connexions entre certains domaines contraints par certaines équations matricielles. En outre, nous caractérisons la relation entre les inverses généralisés de deux matrices.

Mots-clés : matrice par blocs ; solution générale ; inverse généralisé ; équation matricielle ; expression matricielle ; image; rang.

Contents

RATING

Rating	Definition
$\mathbb{K}$	the field of real or complex numbers.
$M_{m\times n}(\mathbb{C})$	the set of matrices of type $m \times n$ over $\mathbb{C}$.
$\mathbb{C}^{m\times n}$	the set of complex matrices of type $m \times n$.
$\mathbb{C}_r^{m\times n}$	the set of complex matrices of type $m \times n$, of rank r .
A^{-1}	the ordinary inverse of A .
$A^{(1)}$	the generalized inverse of A .
A^{+}	the Moore-Penrose inverse of A .
A^{T}	the transpose matrix of A .
A^{*}	the adjoint matrix of A .
$r(A)$	rank of matrix A .
$\mathfrak{R}(A) = \{Ax \mid x \in \mathbb{C}^n\}$	the range of the matrix A .
$N(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$	the null space of the matrix A .
$\mathrm{tr}(A)$	the trace of the matrix A .
$\Vert A\Vert_F$	the Frobinius norm
$E_A = I - AA^{+}$ and $F_A = I - A^{+}A$	the orthogonal projectors induced by A .

Introduction

The concept of a matrix's generalized inverse was first developed by E. H. Moore in 1920. In 1955, M. Penrose redefine the idea of the generalized inverse and provided an effective tool for solving a system of linear equations.

The starting point of generalized inverses theory originates mainly in linear problems represented by an equation of the form $Ax = b$, where A represents a linear transformation. It is necessary for A to have an inverse in order for the proposed equation to have a solution for x. Since this isn't always the case, it could be preferable to search for a matrix that has explicit properties to this inverse. Penrose made this point by introducing a matrix A that verified the following four equations:

$$AXA = A, \tag{1}$$
$$XAX = X, \tag{2}$$
$$(AX)^* = AX, \tag{3}$$
$$(XA)^* = XA. \tag{4}$$

Thus the systems of linear equations know the appearance of the approached solution as: "the solution with least squares", "the solution with minimum norm", "the solution with least squares and minimum norm", "the solution with least rank". generalized inverses and Moore-Penrose inverse have been the subject of many researches. see for examples: [4], [25], [5].

When A is inversible, a matrix with certain characteristics of its inverse matrix is called a generalized inverse of A. Creating a matrix that can function as the inverse for a class of matrices larger than the invertible matrices is the aim of the generalized inverse construction. Put otherwise, the generalized inverse of each given matrix exists, and if a matrix is inversible, then its inverse and its generalized inverse coincide.

The following are the matrix equations that are most well-known in matrix theory:

$$AX = C$$
$$AXB = C$$
$$AXA^* = B$$
$$AX = C, XB = D$$

Where A, B, C and D are known matrices and X is unknown, so that the third equation is a special case of the second equation. As is generally known, computational mathematics

has extensively researched the solution of linear matrix problems and has found applications in a variety of fields, including control theory, vibration theory, biologie and so on. In literature , the notion of generalized inverses of matrices was used when Penrose considered general solutions of the matrix equations $AX = B$ and $AXB = C$ see [25], Mitra in [21], [22] gave the conditions to the paire of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ to have a common solution and a representation of this common solution is given.
The majority of problems with linear or nonlinear matrix functions should be understood in terms of their algebraic aspects and behaviors, as well as used to solve matrix function-related problems in both computational and pure mathematics. For instance, Tian and Yuan [42] studied and suggested connections between specific linear matrix functions, then explored some specific subjects about the algebraic relationships between the reduced equations and solutions of a certain linear matrix problem, Özgüler and Akar [24] provided equivalent conditions for the existence of a common solution to a pair of linear matrix equations over a principal ideal domain. Whereas, all matrix functions possess a class of fundamental types called linear matrix functions, or LMFs, and they can be defined consistently using matrices' additions and multiplications. For further related works about nonlinear matrix functions, one may refer to ([40], [39], [38], [45]).

To investigate the connections between several matrix equations and matrix functions, we refer to the use of some basic tools. Among them are matrix rank and the matrix range method. The rank of a matrix is one of the most basic quantities and useful methods and tools that are widely used in linear algebra specifically, in matrix theory and its applications. This finite nonnegative integer can be used to represent many properties of matrices such as singularity or nonsingularity of a matrix, identification of matrices, consistency of matrix equation, etc...., to review further relevant works, see ([34], [2], [18], [36], [33], [25]). The rank of matrices or partitioned matrices was first studied by Marsaglia and Styan [19] , where they provided various formulas that simplify complicated matrix expressions or equalities, as shown in Lemma 1 below

The thesis consists of three chapters, in the first chapter we gave some preliminary notions, and since the inverse of Moore-Penrose is the most important tool throughout this work, It was necessary to recall its algebraic properties and its role in the resolution of linear equations.

In the second chapter, we suggest to investigate the connections among specific linear matrix functions that often occur in matrix theory and its uses. With an effective application of some extremely chosen formulas and facts about ranks, ranges, and generalized inverses of block matrix operations, we shall deduce a number of relevant, necessary, and sufficient requirements for the collections of values of two given matrix functions to be equal.

In the third chapter, we are interested by the same purpose of the last chapter but here: given two linear matrix functions and thier domains as

$$\mathcal{T}_1 = \{A_1 + B_1X_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times n_1}\} \tag{5}$$
$$\mathcal{T}_2 = \{A_2 + B_2X_2C_2 + B_3X_3C_3 \mid X_2 \in \mathbb{C}^{p_2\times n_2}, X_3 \in \mathbb{C}^{p_3\times n_3}\} \tag{6}$$

where $A_1, A_2 \in \mathbb{C}^{l\times n}$, $B_i \in \mathbb{C}^{l\times p_i}$, $C_i \in \mathbb{C}^{n_i\times n}$, for $i = \overline{1,3}$, we establish the necessary and sufficient conditions for the two relations $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ to hold. Therefore, we provide a collection of results and details regarding the connections between specific linear interpretations of the basic linear matrix equation solutions. As a consequence, we characterize the relationship between the generalized inverses of two matrices.

Chapter 1

Preliminary

In this chapter, we introduced definitions, theorems and propositions that will be used in the sequal of this thesis.

1.1 Basic concepts

1.1.1 Linear transformations

In this section, $\mathbb{K}$ denotes the field of real or complex numbers.

Definition 1 [7] Let V and W be two vector spaces over the field $\mathbb{K}$, and let $f : V \longrightarrow W$ an application.
Then f is called lineary if it satisfies the two conditions:
• for all vectors $v_1, v_2 \in V; f(v_1 + v_2) = f(v_1) + f(v_2)$.
• for any vector $v \in V$ and scalar $t \in \mathbb{K}$ $f(tv) = tf(v)$.
We denote $\mathcal{L}(V, W)$, the set of linear transformations of V in W on $\mathbb{K}$.

Definition 2 [7] Let $f : V \longrightarrow W$ be a linear transformation

1. **The Kernel** (or null space) of f is the subset of V noted $N(f) \subset V$
   $N(f) = \{v \in V, f(v) = 0\}$

2. **The image** of f is the subset of W noted $\mathfrak{R}(f) \subset W$
   $\mathfrak{R}(f) = \{w \in W, \exists v \in V, f(v) = w\}$

1.1.2 Matrices associated to linear applications between finite dimensional vector spaces

Let V and W be two finite dimensional vector spaces n and m respectively over the field $\mathbb{K}$ $f : V \longrightarrow W$ a linear application, and let $\{e_1, e_2, ..., e_n\}$, $\{u_1, u_2, ..., u_m\}$ two bases of V and W respectively.

Definition 3 [7] The matrix of the application f in the bases $\{e_i\}_{i=\overline{1,n}}$ and $\{u_j\}_{j=\overline{1,m}}$ is the matrix denoted by $M(f)_{e_i,u_j}$, belonging to $\mathcal{F}^{m\times n}$, whose columns are the components of the vectors $f(e_1), f(e_2), ..., f(e_n)$ in the basis $\{u_1, u_2, ..., u_m\}$. In particular, ($\mathbb{C}^{m\times n}$ or $\mathbb{R}^{m\times n}$) denote the set of $m \times n$ (complex or real respectively) matrices. The matrix $A \in \mathbb{K}^{m\times n}$ is square if $m = n$, rectangular otherwise.

Proposition 1 *The application*

$$\begin{aligned} M : \mathcal{L}(V, W) &\longrightarrow \mathbb{K}^{m\times n} \\ f &\longmapsto M(f)_{e_i,u_j} \end{aligned}$$

is an isomorphism of vector spaces So for all linear applications f and g of V in W, and for all $\lambda \in \mathbb{K}$:
$M(f+g) = M(f) + M(g)$.
$M(\lambda f) = \lambda M(f)$.
and M is bijective.

Definition 4 Let A be a matrix in $\mathbb{C}^{m\times n}$,

- the transpose of A is the matrix $A^T \in \mathbb{C}^{n\times m}$ with $A^T[i,j] = A[j,i]$, for all i, j.
- the adjoint of A is the matrix $A^* \in \mathbb{C}^{n\times m}$ with $A^*[i,j] = \overline{A[j,i]}$, for all i, j.

Definition 5

1. The square matrix $A \in \mathbb{R}^{n\times n}$ is said to be
   - symmetric if $A^T = A$.
   - anti-symmetrical if $A^T = -A$.
   - orthogonal if $A^T = A^{-1}$.
2. A square matrix $A \in \mathbb{C}^{n\times n}$ is said to be
   - Hermitian (self-adjoint) if $A^* = A$.
   - anti-hermitian if $A^* = -A$.
   - unitary if $A^* = A^{-1}$.

Proposition 2

1. *For all matrices $A \in \mathbb{K}^{l\times m}$ and $B \in \mathbb{K}^{m\times n}$, $(AB)^T = B^T A^T$.*
2. *For all matrices $P \in \mathbb{C}^{l\times m}$ and $Q \in \mathbb{C}^{m\times n}$, $(PQ)^* = Q^* P^*$.*

1.1.3 Partitioned matrices

A block matrix or a partitioned matrix is a matrix that is interpreted as having been broken into sections called blocks or sub-matrices. It is of course necessary that the dimensions of the blocks are compatible.

Example 1

The matrix $P = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 6 & 6 & 4 & 3 \\ 5 & 3 & 4 & 0 \end{bmatrix}$*, can be divided into :*

$P_{11} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 6 & 6 & 4 \end{bmatrix}$, $P_{12} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, $P_{21} = \begin{bmatrix} 5 & 3 & 4 \end{bmatrix}$, $P_{22} = \begin{bmatrix} 0 \end{bmatrix}$

the partitioned matrix can be written as

$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$

1.2 Complex matrices

1.2.1 Conjugate matrix

Definition 6 Let A be a complex matrix, the conjugate matrix of a matrix A with complex coefficients is the matrix $\bar{A}$ made up of the conjugate elements of A. More precisely, if we note a_{ij} and b_{ij} the respective coefficients of A and $\bar{A}$ then $b_{ij} = \bar{a_{ij}}$
for example:
if $A = \begin{bmatrix} 2+i & 1-i \\ 4 & 4+3i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 2-i & 1+i \\ 4 & 4-3i \end{bmatrix}$

Proposition 3 *Let A and B be any two matrices of $M_{m\times n}(\mathbb{C})$ and α a scalar in $\mathbb{C}$*

1. $\overline{A+B} = \overline{A} + \overline{B}$.
2. $\overline{AB} = \overline{A} \cdot \overline{B}$.
3. $\overline{\alpha A} = \overline{\alpha}\overline{A}$.
4. $\overline{\overline{A}} = A$.
5. *If A is an invertible square matrix* $\overline{(A^{-1})} = (\overline{A})^{-1}$.

1.2.2 adjoint matrix

Definition 7 An adjoint matrix (also called a transconjugate matrix) of a matrix $A \in M_{m\times n}(\mathbb{C})$ is the transpose matrix of the conjugate matrix of A, and is denoted by A^*
In the special case where $A \in M_{m\times n}(\mathbb{R})$, its adjoint matrix is therefore simply its transpose

matrix. Thus we have :
$A^* = (\overline{A})^T = \overline{(A)^T}$

Example 2

$$A = \begin{bmatrix} 1-i & 2 \\ 5+i & 4+i \\ i & 3+i \end{bmatrix} \text{ So } A^* = \begin{bmatrix} 1-i & 2 \\ 5+i & 4+i \\ i & 3+i \end{bmatrix}^* = \begin{bmatrix} 1+i & 5-i & -i \\ 2 & 4-i & 3-i \end{bmatrix}$$

Proposition 4 *Let $A, B \in M_{m\times n}(\mathbb{C})$, and $C \in M_{m\times n}(\mathbb{R})$.*
Then :

1. $(A^*)^* = A$.
2. $(AB)^* = B^*A^*$.
3. $det A^* = \overline{det A}$.
4. *if $A = A^*$, the matrix A is said to be Hermitian or self-adjoint.*
5. *if $C = C^T$, the matrix C is said to be symmetric .*
6. *if $A = -A^*$, the matrix A is said to be anti-Hermitian.*
7. *if $C = -C^T$, the matrix C is said to be anti-symmetric.*
8. *if $AA^* = A^*A$, the matrix A is said to be normal.*
9. *if $AA^* = A^*A = I$, the matrix A is said to be unitary.*
10. *if $CC^T = C^TC = I$, the matrix C is said to be orthogonal.*

1.2.3 Frobenius Norm

Definition 8 [4] (Trace of a square matrix)
The trace of a square matrix A, denoted $tr(A)$, is defined to be the sum of elements on the main diagonal (from the upper left to the lower right) of A. The trace is only defined for a square matrix $(n \times n)$.

Let $A, B \in \mathbb{C}^{n\times n}$ and c a scalar in $\mathbb{K}$. Then

1. $tr(A) = \sum_{i=1}^{n} a_{ii}$.
2. $tr(A+B) = tr(A) + tr(B)$.
3. $tr(cA) = c.tr(A)$.

Definition 9 [7] Let $A \in \mathbb{C}^{n\times m}$, we define the Frobenius matrix norm denoted by $\|A\|_F$ such as :
$\|A\|_F = \sqrt{tr(A^*A)} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$.

1.2.4 Rank of a matrix

Definition 10

1. Let $\{v_i\}_{i\in I}$ be a family of vectors, We call the rank of the family $\{v_i\}$ the dimension of the space generated by this family.
2. Let $A \in \mathbb{K}^{m\times n}$, We call the rank of A the rank of the family formed by the column vectors of A, it is also the rank of the family formed by the vector rows.

Example 3 *The rank of the matrix* $A = \begin{pmatrix} 1 & 2 & \frac{-1}{2} & 0 \\ 2 & 4 & -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 4}$

is by definition the rank of the family of vectors of

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} \frac{-1}{2} \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

All these vectors are collinear to v_1*, so the rank of the family* $\{v_1, v_2, v_3, v_4\}$ *is 1 and so* $r(A) = 1$.

1.2.5 Full rank factorization

Proposition 5
Let $A \in \mathbb{C}^{m\times n}$ *of rank* r, $r \neq 0$.
Then there exist matrices $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$ *such that* $r(B) = r(C) = r$ *and* $A = BC$.
This decomposition is called a full rank factorization of the matrix A.

Proof *Let* $A \in \mathbb{C}^{m\times n}$ *of rank* r*, let* $\{b_1, b_2, ..., b_r\}$ *be a basis of* $\Re(A)$*, let* $B \in \mathbb{C}^{m\times r}$ *whose column vectors are* $b_1, b_2, ..., b_r$ *so* $r(B) = r$.
For a matrix $C \in \mathbb{C}^{r\times n}$*, each row vector of* A *is a linear combination of the row vectors of* C.
of the row vectors of C *from which we can write :* $A = BC$ *for a matrix* $C \in \mathbb{C}^{r\times n}$*, so* $r(C) \leq r$*, from the property* $r(BC) \leq \min(r(B), r(C))$*, we have* $r = r(A) \leq r(C)$*, consequently* $r(C) = r$.

Proposition 6 *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ *. Then:*
i) $r(AB) \leq \min(r(A), r(B))$.
ii) For $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times m}$ *, then* $r(A+B) \leq r(A) + r(B)$.

Proof

1. *the vector of* $\Re(AB)$ *is of the form* ABx *for a certain vector* x*,*
   and therefore it belongs to $\Re(A)$*, then* $\Re(AB) \subset \Re(A)$.
   accordingly $r(AB) = \dim \Re(AB) \leq \dim \Re(A) = r(A)$.
   Now, using this fact we have : $r(AB) = r(B^*A^*) \leq r(B^*) = r(B)$.
2. *Let* $A = XY$*,* $B = UV$ *full rank factorizations of* A *and* B *respectively.*
   So, $A + B = XY + UV = [X, U] \begin{bmatrix} Y \\ V \end{bmatrix}$ *therefore and according to 1.* $r(A+B) \leq r[X, U]$
   Let $\{x_1, ..., x_p\}$ *and* $\{u_1, ..., u_q\}$ *be bases for* $\Re(X)$ *and* $\Re(U)$ *respectively.*
   Any vector in the image space of $[X, U]$ *can be written as a linear combination of these* $p + q$ *vectors. so* $r[X, U] \leq r(X) + r(U) = r(A) + r(B)$.

1.2.6 Elementary block matrix operations (EBMO)

In order to conduct explicit formulas for the rank of block matrices, we use the three types of elementary operations on block matrices (abbreviated as EBMO):

1. interchange two row (column) blocks in a partitioned matrix.
2. multiply a row (column) block by a non- singular matrix on the left (on the right) in a partitioned matrix.
3. Add to a row (column) block multiplied by an appropriate matrix on the left (right) to another row (column) block.

Example 4 *Let*

$$M = \begin{bmatrix} -X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix}$$

$$\begin{aligned} \textit{with (EBMO)} \quad M &= \begin{bmatrix} -X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix} \{C_3 \longmapsto C_3 + C_1 \\ &= \begin{bmatrix} -X_1 & 0 & 0 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & X_1 \end{bmatrix} \{C_3 \longmapsto C_3 - C_2 \\ &= \begin{bmatrix} -X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ X_1 & X_2 & X_1 - X_2 \end{bmatrix} \{L_3 \longmapsto L_3 + L_1 \\ &= \begin{bmatrix} -X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & X_2 & X_1 - X_2 \end{bmatrix} \{L_3 \longmapsto L_3 - L_2 \\ &= \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_1 - X_2 \end{bmatrix}. \end{aligned}$$

So

$$r(M) = r(X_1) + r(X_2) + r(X_1 - X_2).$$

1.3 The generalized inverse of matrices

Definition 11 [4] If A is a non-singular matrix, then there is a unique inverse noted A^{-1} as

$$AA^{-1} = A^{-1}A = I$$

This inverse has the following properties :

1. $(A^{-1})^{-1} = A$.
2. $(A^T)^{-1} = (A^{-1})^T$.
3. $(A^*)^{-1} = (A^{-1})^*$.
4. $(AB)^{-1} = B^{-1}A^{-1}$.

A^T and A^* denote the transpose and conjugate transpose of A, respectively. We recall that a real or complex number λ is called an eigenvalue of a square matrix A, and a non-zero vector X is called an eigenvector of A corresponding to λ, if $AX = \lambda X$
Consider the linear system

$$Ax = b \tag{1.1}$$

Let A be a square matrix of order n, If A is nonsingular, so $ker(A) = \{0\}$, then the solution vector x of the linear equation $Ax = b$ is only determined by $x = A^{-1}b$.
Here, A^{-1} is the inverse matrix of A.

Definition 12 [4] Let $A \in \mathbb{C}^{m\times n}$, the matrix $X \in \mathbb{C}^{n\times m}$ if said to be the generalized inverse or(g-inverse) of the matrix A if $AXA = A$.

If A is square and non singular, then A^{-1} is the unique generalized inverse of A, otherwise A has several generalized inverses, we denote by $A^{(1)}$ for a generalized inverse of A.

Theorem 1 *[4] Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$, then the following two conditions are equivalent*
i) X is a g-inverse of A.
ii) for any $b \in \Re(A)$, $x = Xb$ is a solution of $Ax = b$.

Proof
i)$\Longrightarrow$ii) $\forall b \in \Re(A)$, b is of the form $b = Ay$, such as y, then $A(Xb) = AXAy = Ay = b$.
ii)$\Longrightarrow$i) when $AXb = b$ for all $b \in \Re(A)$, we have $AXAy = Ay$ for all y, than $AXA = A$.

Definition 13 Let $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is said to be the reflexive generalized inverse of the matrix A, if it satisfies the two conditions: $\begin{cases} AXA = A \\ XAX = X \end{cases}$

1.3.1 Properties of the generalized inverse

Let $A \in \mathbb{C}^{m\times n}$ matrix we give some properties of $A^{(1)}$

Theorem 2 *[4]*

1. $r(A^{(1)}) \geq r(A) = r(A^{(1)}A) = r(AA^{(1)})$.
2. *if* A *is square and nonsingular, then* $A^{(1)} = A^{-1}$ *is unique.*
3. $AA^{(1)}$ *and* $A^{(1)}A$ *are idempotent (i.e* $(AA^{(1)})^2 = AA^{(1)}$ *and* $(A^{(1)}A)^2$=$A^{(1)}A$*).*
4. $r\left(A^{(1)}AA^{(1)}\right) = r\left(A\right)$.

Proof

1. *For two matrices B and C, we have* $r(BC) \leq \min\left(r(A), r(B)\right)$*, then*

$$r(A) \geq r(AA^{(1)}) \geq r(AA^{(1)}A) = r(A) \Longrightarrow r(A) = r(AA^{(1)})$$

$$r(A) \geq r(A^{(1)}A) \geq r(AA^{(1)}A) = r(A) \Longrightarrow r(A) = r(A^{(1)}A)$$

Hence

$$r(A) = r(A^{(1)}A) = r(AA^{(1)})$$

On the other hand

$$r(A^{(1)}) \geq r(AA^{(1)}) \geq r(AA^{(1)}A) = r(A).$$

2. *We have* $AA^{(1)}A = A$
If A *is non-singular, then multiplying by* A^{-1} *on both the left and the right would give*

$$A^{(1)} = A^{-1}$$

3.

$$(AA^{(1)})^2 = (AA^{(1)}A)A^{(1)} = AA^{(1)}$$
$$(A^{(1)}A)^2 = (A^{(1)}AA^{(1)})A = A^{(1)}A$$

4. $r\left(A^{(1)}AA\right) = r\left(A^{(1)}A\right)$*, then* $r\left(A^{(1)}AA^{(1)}\right) = r\left(A^{(1)}A\right) = r\left(A\right)$.

Theorem 3 *[4]*
Let $A \in \mathbb{C}_r^{m\times n}$ *So we have :*

1. $A^{(1)}A = I_n$ *if and only if* $r = n$.
2. $AA^{(1)} = I_m$ *if and only if* $r = m$.

Example 5 *Determine a generalized inverse of* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Let $A^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

for $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

you must have $a + 2b + 2c + 4d = 1$ *then*

$A^{(1)} = \begin{bmatrix} 1 - 2b - 2c - 4d & b \\ c & d \end{bmatrix}$ *where* a, b *and* c *are arbitrary*

for example if we choose $d = -2$, $b = 2$, $c = 0$, *we will find*

$A^{(1)} = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix}$

Example 6 *Determine a generalized inverse of* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Let $A^{(1)}$ *the g-inverse of* A *such that*

$$A^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

from $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

you must have $a + b + c + d = 1$, *then* $A^{(1)} = \begin{bmatrix} a & b \\ c & 1 - a - b - c \end{bmatrix}$,

for a, b and c are arbitrary.

In particular the second Method for computing the generalized inverse, construction of {1}-inverse for any matrix $A \in \mathbb{C}^{m \times n}$ is simplified by transforming A to the Hermitian normal form as shown in the following theorem :

Theorem 4 *[4] Let* $A \in \mathbb{C}_r^{m \times n}$ *and let* $E \in \mathbb{C}_m^{m \times m}$ *and* $P \in \mathbb{C}_n^{n \times n}$ *as*

$$EAP = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix}, \tag{1.2}$$

for all $L \in \mathbb{C}^{(n-r) \times (m-r)}$ *matrices* $n \times m$

$$X = P \begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix} E \tag{1.3}$$

is a 1-inverse of A.

Example 7 *Let* $A \in \mathbb{C}^{m \times n}$, *and let* $T_0 = [A \quad I_m]$. *E tronsform* A *to the Hermitian normal form note* EA, *we use Elimination of Gausses, which* $ET_0 = [EA \quad A]$ *let*

$$A = \begin{bmatrix} 0 & 2i & i & 0 & 4+2i & 1 \\ 0 & 0 & 0 & -3 & -6 & -3-3i \\ 0 & 2 & 1 & 1 & 4-4i & 1 \end{bmatrix} \text{ with } \quad r(A) = 2 \tag{1.4}$$

$$[EA \quad E] = \begin{bmatrix} 0 & 2i & i & 0 & 4+2i & 1 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -3-3i & \vdots & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 4-4i & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} : \begin{cases} L_1 \mapsto (\frac{1}{2i})L_1 \\ L_3 \mapsto 2L_1 - L_3 \end{cases}$$

then

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i & \vdots & -\frac{1}{2}i & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -3-3i & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1+i & \vdots & i & 0 & 1 \end{bmatrix} : L_2 \mapsto (-\frac{1}{3})L_2$$

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i & \vdots & -\frac{1}{2}i & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1+i & \vdots & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & i & \frac{1}{3} & 1 \end{bmatrix} : L_3 \mapsto L_3 - L_2$$

then

$$EA = \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i \\ 0 & 0 & 0 & 1 & 2 & 1+i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

and

$$E = \begin{bmatrix} -\frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ i & \frac{1}{3} & 1 \end{bmatrix}$$

for all $L \in \mathbb{C}^{(n-r)\times(m-r)}$ we choose the permutation matrix P as

$$P = \begin{bmatrix} 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$EAP = \begin{bmatrix} 1 & 0 & \vdots & 0 & \frac{1}{2} & 1-2i & -\frac{1}{2}i \\ 0 & 1 & \vdots & 0 & 0 & 2 & 1+i \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

We choose $L = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \in \mathbb{C}^{4\times 1}$

Then

$$X = P\begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix} E \ = \begin{bmatrix} 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \alpha \\ 0 & 0 & \vdots & \beta \\ 0 & 0 & \vdots & \gamma \\ 0 & 0 & \vdots & \delta \end{bmatrix} \begin{bmatrix} -\frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ i & \frac{1}{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} i\alpha & \frac{1}{3}\alpha & \alpha \\ -\frac{1}{2} & 0 & 0 \\ i\beta & \frac{1}{3}\beta & \beta \\ 0 & -\frac{1}{3} & 0 \\ i\gamma & \frac{1}{3}\gamma & \gamma \\ i\delta & \frac{1}{3}\delta & \delta \end{bmatrix}$$

For example if we choose : $\alpha = -1$, $\beta = i$, $\gamma = 3i$, $\delta = -6$. *we will find a generalized invese verifying*

$$X = \begin{bmatrix} -i & -\frac{1}{3} & -1 \\ -\frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{3}i & i \\ 0 & -\frac{1}{3} & 0 \\ -3 & 3 & 3i \\ -6i & -2 & -6 \end{bmatrix}.$$

1.3.2 The generalized inverse and linear equations:

In this section we represent solutions to linear equations involving the generalized inverses of matrices.

Definition 1 *Let* $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, C \in \mathbb{C}^{m\times q}$.
Then the matrix equation

$$AXB = C \tag{1.6}$$

the equation (1.6) is consistent (solvable) if and only if it has solution.

Theorem 5 *[4] Let* $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, D \in \mathbb{C}^{m\times q}$.
Then the matrix equation

$$AXB = D \tag{1.7}$$

is consistent if and only if, for some $A^{(1)}, B^{(1)}$,

$$AA^{(1)}DB^{(1)}B = D, \tag{1.8}$$

in which case the general solution is

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \tag{1.9}$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

Proof *If (1.8) holds, then* $X = A^{(1)}DB^{(1)}$ *is a solution of (1.7).*

Conversely, if X is any solution of (1.7), then
$D = AXB = AA^{(1)}AXBB^{(1)}B = AA^{(1)}DB^{(1)}B$.
Moreover, it follows from (1.8) and the definition of $A^{(1)}$ *and* $B^{(1)}$ *that every matrix X of the form (1.9) satisfies (1.7). On the other hand, let X be any solution of (1.7). Then, clearly*
$X = A^{(1)}DB^{(1)} + X - A^{(1)}AXBB^{(1)}$,
which is of the form (1.9).

We denote by $A\{1\}$ the set of all the generalized inverses of A. The following characterization of the set $A\{1\}$, in terms of an arbitrary element $A^{(1)}$ of this set.[27]

Corollary 1 *[4] Let* $A \in \mathbb{C}^{m\times n}$, $A^{(1)} \in A\{1\}$. *then*

$$A\{1\} = \left\{A^{(1)} + Z - A^{(1)}AZAA^{(1)}, Z \in \mathbb{C}^{n\times m}\right\} \tag{1.10}$$

Proof *The set* $A\{1\}$ *is obtained by writing* $Y = A^{(1)} + Z$ *in the set of solutions of* $AXA = A$ *as given by Theorem 5.*

Applying Theorem 5 to ordinary systems of linear equations gives:

Corollary 2 *let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m}$ Then the equation $Ax = b$ is consistent if and only if $AA^{(1)}b = b$ in this case the general solution is given by*

$$x = A^{(1)}b + (I - A^{(1)}A)y \tag{1.11}$$

or $y \in \mathbb{C}^{n}$ is arbitrary

Proof *The sufficient condition, it is obvious*
The necessary condition can be demonstrated by the substitution of $Ax = b$ in $AA^{(1)}Ax = Ax$

1.4 The generalized Moore-Penrose inverse:

In this section we will introduce the well-known generalized inverse of matrices, the Moore-Penrose inverse, with its properties and applications.

Definition 14 [4] Eliakim Hastings Moore introduced the notion of a generalized inverse of a matrix in 1920, the application of this notion to the solution of systems of linear equations led to a great interest in this subject. In 1955 Penrose demonstrated that, for any matrix A (square or rectangular) with real or complex elements, there exists a unique matrix X satisfying the four equations (that we call the Penrose equations):

$$AXA = A. \tag{1.12}$$
$$XAX = X. \tag{1.13}$$
$$(AX)^{*} = AX. \tag{1.14}$$
$$(XA)^{*} = XA. \tag{1.15}$$

This unique generalized inverse is commonly known as the Moore-Penrose generalized inverse and is often referred to as A^{+}.
If A is nonsingular, the matrix $X = A^{-1}$ satisfied the four equations trivially, then the Moore-Penrose inverse of a nonsingular matrix is the same as the ordinary inverse.
From equations (1.14) and (1.15) above, we have $(AA^{+})^{*}(I_n - AA^{+}) = 0$
and $(A^{+}A)^{*}(I_m - A^{+}A) = 0$ so $P_{\mathfrak{R}(A)} = AA^{+}$ and $P_{\mathfrak{R}(A^{*})} = A^{+}A$ are orthogonal projectors on $\mathfrak{R}(A)$ and $\mathfrak{R}(A^{*})$ respectively.

1.4.1 Uniqueness and existence

Existence:

Theorem 6 *[4] if $A = BC$ with $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$, and $r = r(A) = r(B) = r(C)$, then :*

$$A^{+} = C^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*}.$$

Proof *We conclude that $B^{*}B$ and CC^{*} are matrices of rank r, because according to the properties of rank we have*

$$r(B) = r(B^{*}B) = r(CC^{*}) = r(C) = r$$

We take

$$X = C^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*}$$

Then we have :

$$AX = BCC^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*} = B(B^{*}B)^{-1}B^{*} \text{ then } (AX)^{*} = AX$$

also $XA = C^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*}BC = C^{*}(CC^{*})^{-1}C$ then
$(XA)^{*} = XA$ To verify (1.12) an (1.13) We use $XA = C^{*}(CC^{*})^{-1}C$ we obtain
$A(XA) = BC(C^{*}(CC^{*})^{-1}C) = BC = A$ and $(XA)X = C^{*}(CC^{*})^{-1}CC^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*} = (CC^{*})^{-1}(B^{*}B)^{-1}B^{*} = X$.
We notice that the matrix X satisfies the four Moore-Penrose equations . Thus $X = A^{+}$ by defnition.

Uniqueness :

We assume that X_1 and X_2 are two Moore-Penrose inverses of A, then according to the definition of A^{+} we have

$$\begin{aligned} X_1 &= X_1\,(AX_1) = X_1X_1^{*}\,(A^{*}) = X_1X_1^{*}A^{*}\,(X_2^{*}A^{*}) \\ &= X_1\,(X_1^{*}A^{*})\,AX_2 = X_1\,(AX_1A)\,X_2 = X_1\,(A)\,X_2 \\ &= X_1\,(AX_2A)\,X_2 = (X_1A)\,A^{*}X_2^{*}X_2 = (A^{*}X_1^{*}A^{*})\,X_2^{*}X_2 \\ &= (A^{*}X_2^{*})\,X_2 = X_2AX_2 \\ &= X_2. \end{aligned}$$

Example 8
Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, $r(A) = 1$ and $A = BC$ with $B \in \mathbb{C}^{2\times 1}$ and $C \in \mathbb{C}^{1\times 3}$.

We can take : $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$
then,

$B^{*}B = [5], C^{*}C = [6]$ Thus, $A^{+} = \frac{1}{30}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ Special case,
if $A \in \mathbb{C}^{m\times n}$ and $r(A) = 1$.
then $A^{+} = (\frac{1}{\alpha})A^{*}$ with $\alpha = trace(A^{*}A) = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2$

1.4.2 Main properties of the generalized Moore-Penrose inverse

Theorem 7 *[4] Let $A \in \mathbb{C}^{m\times n}$. then,*

- $(A^+)^+ = A$.
- $(A^+)^* = (A^*)^+$.
- *if* $\lambda \in \mathbb{C}$, $(\lambda A)^+ = \lambda^+ A^+$ *with* $\lambda^+ = \dfrac{1}{\lambda}$ *if* $\lambda \neq 0$, *and* $\lambda^+ = 0$ *if* $\lambda = 0$.
- $A^* = A^* A A^+ = A^+ A A^*$.
- $(A^* A)^+ = A^+ (A^*)^+$.
- $A^+ = (A^* A)^+ A^* = A^* (AA^*)^+$.
- $(UAV)^+ = V^* A^+ U^*$, *où* U, V *are arbitrary matrices.*

Theorem 8 *[4] If $A \in \mathbb{C}^{m\times n}$, then*

- $\mathfrak{R}(A) = \mathfrak{R}(AA^+) = \mathfrak{R}(AA^*)$.
- $\mathfrak{R}(A^+) = \mathfrak{R}(A^*) = \mathfrak{R}(A^+ A) = \mathfrak{R}(A^* A)$.
- $\mathfrak{R}(I - AA^+) = N(AA^+) = N(A^*) = N(A^+) = \mathfrak{R}(A)^{\perp}$.
- $\mathfrak{R}(I - A^+ A) = N(A^+ A) = N(A) = \mathfrak{R}(A^*)^{\perp}$.

We need a convenient notation for a generalized inverse satisfying certain specified equations .

Definition 15 [4] For any $A \in \mathbb{C}^{m\times n}$, $A\{i,j,...,k\}$ denotes the set of matrices $X \in \mathbb{C}^{n\times m}$, which satisfy equations (i), (j),..., (k) among equations (1.12)-(1.15).
The matrix $X \in A\{i, j,...,k\}$ is called $\{i, j,...,k\}$ -inverse of A.
In particular, a matrix $X \in \mathbb{C}^{n\times m}$ of the set $A\{1\}$ is called a g-inverse of A and denoted by $A^{(1)}$.

From [25] we have the following properties of the conjugate transpose will be used :

$$\begin{array}{rcl} A^{**} & = & A, \\ (A+B)^* & = & A^* + B^*, \\ (\lambda A)^* & = & \lambda A^*, \\ (BA)^* & = & A^* B^*, \\ AA^* = 0 & \textit{implies} & A = 0, \end{array}$$

The last of these follows from the fact that the trace of AA^* is the sum of the squares of the moduli of the elements of A. From the last two we obtain the rule

$$BAA^* = CAA^* \quad \textit{implies} \quad BA = CA. \tag{1.16}$$

since

$$(BAA^* - CAA^*)(B - C)^* = (BA - CA)(BA - CA)^*.$$

Similarly

$$BA^* A = CA^* A \quad \textit{implies} \quad BA^* = CA^*. \tag{1.17}$$

Theorem 9 *[25] The four equations*

$$AXA = A.$$

$$XAX = X.$$

$$(AX)^* = AX.$$

$$(XA)^* = XA.$$

have a unique solution for any A

Proof *We first show that equations (1.13) and (1.14) are equivalent to the single equation*

$$XX^*A^* = X. \tag{1.18}$$

Equation (1.18) follows from (1.13) and (1.14), since it is merely (1.14) substituted in (1.13). Conversely, (1.18) implies $AXX^*A^* = AX$, the left-hand side of which is hermitian. Thus (1.14) follows, and substituting (1.14) in (1.18) we get (1.13).
Similarly, (1.12) and (1.15) can be replaced by the equation

$$XAA^* = A^*. \tag{1.19}$$

Thus it is sufficient to find an X satisfying (1.18) and (1.19). Such an X will exist if a B can be found satisfying

$$BA^*AA^* = A^*$$

For then $X = BA^*$ satisfies (1.19). Also, we have seen that (1.19) implies $A^*X^*A^* = A^*$ and therefore $BA^*X^*A^* = BA^*$. Thus X also satisfies (1.18).
Now the expressions $A^*A, (A^*A)^2, (A^*A)^3, ...$ cannot all be linearly independent, i.e. there exists a relation

$$\lambda_1 A^*A + \lambda_2 (A^*A)^2 + ... + \lambda_k (A^*A)^k = 0. \tag{1.20}$$

where $\lambda_1, ..., \lambda_k$ are not all zero.
Let λ_r, be the first non-zero λ and put

$$B = -\lambda_r^{-1} \left\{ \lambda_{r+1} I + \lambda_{r+2} A^*A + ... + \lambda_k (A^*A)^{k-r-1} \right\}$$

Thus (1.20) gives $B(A^*A)^{r+1} = (A^*A)^r$, and applying (1.16) and (1.17) repeatedly we obtain $BA^*AA^* = A^*$, as required
To show that X is unique, we suppose that X satisfies (1.18) and (1.19) and that Y satisfies $Y = A^*Y^*Y$ and $A^* = A^*AY$. These last relations are obtained by respectively substituting (1.15) in (1.13) and (1.14) in (1.12). (They are (1.18) and (1.19) with Y in place of X and the reverse order of multiplication and must, by symmetry, also be equivalent to (1.12), (1.13), (1.14) and (1.15.) Now

$$X = XX^*A^* = XX^*A^*AY = XAY = XAA^*Y^*Y = A^*Y^*Y = Y$$

The unique solution of (1.12), (1.13), (1.14) and (1.15) will be called the generalized inverse of A (abbreviated g.i.) and written $X = A^+$. (Note that A need not be a square matrix and may even be zero). We shall also use the notation λ^+ for scalars, where λ^+ means λ^{-1} if $\lambda \neq 0$ and 0 if $\lambda = 0$.
In the calculation of A^+ it is only necessary to solve the two unilateral linear equations $XAA^* = A^*$ and $A^*AY = A^*$. By putting $A^* = XAY$ and using the fact that XA and AY are hermitian

and satisfy $AXA = A = AYA$ we observe that the four relations $AA^+A = A, A^+AA^+ = A^+, (AA^+)^* = AA^+$ and $(A^+A)^* = A^+A$ are satisfied.
Relations satisfied by A^+ include

$$\text{and} \qquad \begin{cases} A^+A^{+*}A^* = A^+ = A^*A^{+*}A^+ \\ A^+AA^* = A^* = A^*AA^+ \end{cases} \tag{1.21}$$

these being (1.18), (1.19) and their reverses.

Chapter 2

characterizations of relationships between linear matrix functions

2.1 Introduction

Let $f(X)$ be a matrix function over the field of complex numbers , where $X = (X_1, X_2, ..., X_k)$ are a family of matrices with variable entries. In this chapter we explore and suggest connections between specific linear matrix functions (LMFs) commonly seen in matrix theory and its applications. We will identify essential properties and conditions that ensure the value sets of two provided matrix functions are equivalent, while also detailing facts about the ranks, ranges, and generalized inverses associated with block matrix operations.
As we know in linear algebra and matrix theory, a matrix A is null if and only if $r(A) = 0$. Therefore, two matrices A and B of the same size are equal if and only if $r(A - B) = 0$. Based on this fact, we may discover that if some non-trivial algebraic formulas are obtained to calculate the rank difference A - B, we can reasonably use them to describe the fundamental connections between the two matrices, and especially to characterize the equality matrix A = B in an easy way. The strong foundation of this proposed method is that we are truly able to determine or calculate the rank of a matrix through various matrix elementary operations and obtain analytic formulas to express the ranks of matrices in many matrix cases. On the other hand, it has been realized since the 1960s that generalized matrix inverses can be adopted to derive many exact and analytical expansion formulas for calculating the ranks of some matrices.
Below, we will present a series of basic facts and equality about rows of matrices and matrix equations.
Notations:
$[A, \ \ B]$ is a block row matrix consisting of the matrices A and B.
$\begin{bmatrix} A \\ B \end{bmatrix}$ is a block column matrix consisting of the two matrices A and B.
$E_A = I - AA^+$ and $F_A = I - A^+A$ the two orthogonal projectors (Hermitian idempotent matrices) induced from A.

Lemma 1 *[19] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$,and $C \in \mathbb{C}^{l\times n}$. Then,*

$$r[A,\ \ B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2.1}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{2.2}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{2.3}$$

Based on equation. (2.3),there exist the following two formulas as well.

$$r\begin{bmatrix} A & BF_P \\ E_Q C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q). \tag{2.4}$$

$$r\begin{bmatrix} E_{B_1} A F_{C_1} & E_{B_1} B \\ CF_{C_1} & 0 \end{bmatrix} = r\begin{bmatrix} A & B & B_1 \\ C & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} - r(B_1) - r(C_1). \tag{2.5}$$

In particular, the following results hold.

a) $r[A, B] = r(A) \Leftrightarrow \mathfrak{R}(A) \supseteq \mathfrak{R}(B) \Leftrightarrow AA^+B = B \Leftrightarrow E_A B = 0.$

b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathfrak{R}(C^*) \subseteq \mathfrak{R}(A^*) \Leftrightarrow CA^+A = C \Leftrightarrow CF_A = 0.$

c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) \Leftrightarrow E_B A F_C = 0.$

Lemma 2 *[25] Let*

$$AX = B \tag{2.6}$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$ are known matrices, and $X \in \mathbb{C}^{n\times p}$is an unknown matrix. Then, the following four statements are equivalent:

a) *Equation (2.6) is solvable for X.*

b) $\mathfrak{R}(A) \supseteq \mathfrak{R}(B)$.

c) $r[A, B] = r(A)$.

d) $AA^+B = B$.

In this case, the general solution of the equation (2.6) can be written in the parametric form

$$X = A^+B + F_A U, \tag{2.7}$$

where $U \in \mathbb{C}^{n\times p}$ is an arbitrary matrix. In particular, (2.6) holds for all matrices $X \in \mathbb{C}^{n\times p}$ only if both $A = 0$ and $B = 0$, or equivalently, $[A, B] = 0$.

Lemma 3 *[1] The linear matrix equation*

$$A_1X_1 + X_2B_2 = C \tag{2.8}$$

is solvable for the two unknown matrices X_1 and X_2 of appropriate sizes if and only if

$$r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2) \tag{2.9}$$

hold, or equivalently,

$$E_{A_1}CF_{B_2} = 0 \tag{2.10}$$

holds. In particular, (2.8) holds for all matrices X_1 and X_2 if and only if $\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = 0.$

Lemma 4 *[23] The linear matrix equation*

$$A_1X_1B_1 + A_2X_2B_2 = C \tag{2.11}$$

is solvable the two unknown matrices X_1 and X_2 of appropriate sizes if and only if the following four matrix rank equalities

$$r[C, A_1, A_2] = r[A_1, A_2], \ \ r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \tag{2.12}$$

$$r\begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1), \quad r\begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.13}$$

hold, or equivalently, the following four matrix equalities

$$E_AC = 0, \ \ E_{A_1}CF_{B_2} = 0, \ \ E_{A_2}CF_{B_1} = 0, \ \ CF_B = 0 \tag{2.14}$$

hold, where $A = [A_1, A_2]$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

Lemma 5 *([13],[33]) Equation (2.11) holds for all matrices X_1 and X_2 of appropriate sizes if and only if any one of the following four block matrix equalities*

$$[C, A_1, A_2] = 0, \quad \begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = 0, \quad \begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = 0, \quad \begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = 0 \tag{2.15}$$

holds.

Lemma 6 *[30] The linear matrix equation*

$$A_1X_1 + X_2B_2 + A_3X_3B_3 + A_4X_4B_4 = C \tag{2.16}$$

is solvable for the four unknown matrices X_1, X_2, X_3, and X_4 of appropriate sizes if and only if the following four matrix rank equalities hold:

$$r\begin{bmatrix} C & A_1 & A_3 & A_4 \\ B_2 & 0 & 0 & 0 \end{bmatrix} = r[A_1, A_3, A_4] + r(B_2), \tag{2.17}$$

$$r\begin{bmatrix} C & A_1 & A_3 \\ B_2 & 0 & 0 \\ B_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_4 \end{bmatrix} + r[A_1, A_3], \tag{2.18}$$

$$r\begin{bmatrix} C & A_1 & A_4 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r[A_1, A_4], \tag{2.19}$$

$$r\begin{bmatrix} C & A_1 \\ B_2 & 0 \\ B_3 & 0 \\ B_4 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \\ B_4 \end{bmatrix} + r(A_1). \tag{2.20}$$

2.2 characterizations of some algebraic relationships of LMFs

In this section, we provide some algebraic relationships between specific linear matrix functions, where we establish the intersection, the inclusion, as well as the equality of two domains of two LMFs.
First, we need these important lemmas

Lemma 7 *([6],[31],[32],[33],[41])Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$,and $C \in \mathbb{C}^{l\times n}$.be given.Then the maximal and minimal ranks of $A - BX$, $A - BXC$ and $A - BX - YC$ with respect to X and Y are given by the following closed-form formulas:*

$$\max_{X\in\mathbb{C}^{k\times n}} r(A - BX) = \min\{r[A, B], n\} \tag{2.21}$$

$$\min_{X\in\mathbb{C}^{k\times n}} r(A - BX) = r[A, B] - r(B) \tag{2.22}$$

$$\max_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = \min\left\{r[A, B], r\begin{bmatrix} A \\ C \end{bmatrix}\right\}, \tag{2.23}$$

$$\min_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = r[A, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{2.24}$$

$$\min_{X\in\mathbb{C}^{k\times l}, Y\in\mathbb{C}^{m\times l}} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C). \tag{2.25}$$

Lemma 8 *[37] Assume that two LMFs and their domains are given by*

$$\mathcal{D}_1 = \{A_1 + B_1X_1 | X_1 \in \mathbb{C}^{p_1\times n}\} \quad \textit{and} \quad \mathcal{D}_2 = \{A_2 + B_2X_2 | X_2 \in \mathbb{C}^{p_2\times n}\}, \tag{2.26}$$

where $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, and $B_2 \in \mathbb{C}^{m\times p_2}$ are known matrices, and $X_1 \in \mathbb{C}^{p_1\times n}$ and $X_2 \in \mathbb{C}^{p_2\times n}$ are variable matrices. Then, we have the following results.

(a) $\mathcal{D}_1 \bigcap \mathcal{D}_2 \neq \phi$, i.e there exist X_1 and X_2 such that $A_1 + B_1X_1 = A_2 + B_2X_2$ if and only if $\mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}[B_1, B_2]$.

(b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *if and only if* $\mathfrak{R}[A_1 - A_2, B_1] \subseteq \mathfrak{R}(B_2)$.

(c) $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if* $\mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}(B_1) = \mathfrak{R}(B_2)$.

Proof *Applying (2.22) gives*

$$\begin{aligned} \min_{M_1 \in \mathcal{D}_1, M_2 \in \mathcal{D}_2} r(M_1 - M_2) &= \min_{X_1, X_2} r(A_1 - A_2 + B_1X_1 - B_2X_2) \\ &= \min_{X_1, X_2} r\left(A_1 - A_2, [B_1, -B_2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) \\ &= r[A_1 - A_2, B_1, B_2] - r[B_1, B_2]. \end{aligned}$$

Setting both sides of the equality to zero leads to (a). From Eq (2.2) in [37] , $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *holds if and only if*

$$\max_{X_1} \min_{X_2} r[(A_1 + B_1X_1) - (A_2 + B_2X_2)] = 0. \tag{2.27}$$

It follows from (2.22) that

$$\min_{X_2} r[(A_1 + B_1X_1) - (A_2 + B_2X_2)] = r[A_1 - A_2 + B_1X_1, B_2] - r(B_2) \tag{2.28}$$

Also by (2.23),

$$\begin{aligned} \max_{X_1} r[A_1 - A_2 + B_1X_1, B_2] &= \max_{X_1} r\left([A_1 - A_2, B_2] - B_1X_1[I_n, 0]\right) \\ &= \min\left\{ r[A_1 - A_2, B_1, B_2], r\begin{bmatrix} A_1 - A_2 & B_2 \\ I_n & 0 \end{bmatrix} \right\} \\ &= \min\left\{ r[A_1 - A_2, B_1, B_2], r(B_2) + n \right\}. \end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29) yields

$$\max_{X_1} \min_{X_2} r[(A_1 + B_1X_1) - (A_2 + B_2X_2)] = \min\left\{ r[A_1 - A_2, B_1, B_2] - r(B_2), n \right\} \tag{2.30}$$

Setting the right-hand side of this equality to zero and observing that $n \neq 0$, *we see that (2.27) is equivalent to* $r[A_1 - A_2, B_1, B_2] = r(B_2)$ *Thus, we have (b).*
By symmetry, $\mathcal{D}_2 \subseteq \mathcal{D}_1$ *if and only if* $\mathfrak{R}[A_1 - A_2, B_2] \subseteq \mathfrak{R}(B_1)$. *Integrating this assertion with (b) results in (c).*

Theorem 10 *[42] Assume that two LMFs and their domains are given by*

$$\mathcal{D}_1 = \left\{ A_1 + B_1X_1 + Y_1C_1 | X_1 \in \mathbb{C}^{p_1 \times n_1},\ \ Y_1 \in \mathbb{C}^{m_1 \times q_1} \right\} \tag{2.31}$$

$$\mathcal{D}_2 = \left\{ A_2 + B_2X_2C_2 + D_2Y_2E_2 | X_2 \in \mathbb{C}^{s_2 \times t_2},\ \ Y_2 \in \mathbb{C}^{u_2 \times v_2} \right\} \tag{2.32}$$

where $A_1 \in \mathbb{C}^{m \times n}$, $B_1 \in \mathbb{C}^{m \times p_1}$, $C_1 \in \mathbb{C}^{q_1 \times n}$, $A_2 \in \mathbb{C}^{m \times n}$, $B_2 \in \mathbb{C}^{m \times s_2}$, $C_2 \in \mathbb{C}^{t_2 \times n}$, $D_2 \in \mathbb{C}^{m \times u_2}$, *and* $E_2 \in \mathbb{C}^{v_2 \times n}$ *are known matrices. Then, we have the following results.*

a) $\mathcal{D}_1 \bigcap \mathcal{D}_2 \neq \phi$ *if and only if the following four conditions hold:*

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r\,[B_1, B_2, D_2] + r(C_1), \tag{2.33}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ E_2 \end{bmatrix} + r\,[B_1, B_2]\,, \tag{2.34}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + r\,[B_1, D_2] \tag{2.35}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \\ E_2 \end{bmatrix} + r(B_1). \tag{2.36}$$

b) $\mathcal{D}_1 \supseteq \mathcal{D}_2$ *if and only if any one of the following four conditions holds:*

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \tag{2.37}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \tag{2.38}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(B_1) + r(C_1), \tag{2.39}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r(B_1) + r(C_1). \tag{2.40}$$

c) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *if and only if any one of the following four groups of conditions holds:*

$$r\,[B_2, D_2] = m \ \text{ or } \ r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r\,[B_2, D_2]\,, \tag{2.41}$$

$$r(B_2) = m \ \text{ or } \ r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r(B_2) + r(E_2), \tag{2.42}$$

$$r(C_2) = n \text{ or } r(D_2) = m \text{ or } r\begin{bmatrix} A_1 - A_2 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(C_2) + r(D_2), \tag{2.43}$$

$$r\begin{bmatrix} C_2 \\ E_2 \end{bmatrix} = n \text{ or } r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ E_2 \end{bmatrix}. \tag{2.44}$$

d) $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if both (b) and (c) hold.*

Proof 1. *The condition* $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ *is obviously equivalent to:*

$$A_1 + B_1X_1 + Y_1C_1 = A_2 + B_2X_2C_2 + D_2Y_2E_2$$

For some variable matrices X_1*,* Y_1*,* X_2*, and* Y_2*. We rewrite this equation as:*

$$B_1X_1 + Y_1C_1 - B_2X_2C_2 - D_2Y_2E_2 = A_2 - A_1. \tag{2.45}$$

In this case, applying Lemma 6 to (2.45) such that

$$\begin{cases} C = A_2 - A_1 & ; A_1 = B_1 \quad ; A_3 = B_2 \quad ; A_4 = D_2. \\ B_2 = C_1 & ; B_3 = C_2 \quad ; B_4 = E_2. \end{cases}$$

$$\begin{aligned} r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} &= r(B_1) + r(C_1), \\ r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} &= r(B_1) + r(C_1), \\ r\begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} &= r(B_1) + r(C_1), \\ r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} &= r(B_1) + r(C_1). \end{aligned}$$

This really represents a*).*

2. *The condition* $\mathcal{D}_2 \subseteq \mathcal{D}_1$ *is equivalent to the fact that:*

   - *The equation (2.45) equivalent to*

     $$B_1X_1 + Y_1C_1 = A_2 - A_1 + B_2X_2C_2 + D_2Y_2E_2$$

     By lemma (3) such that

     $$\begin{cases} C = A_2 - A_1 + B_2X_2C_2 + D_2Y_2E_2, \\ A_1 = B_1, \\ B_2 = C_1. \end{cases}$$

     is solvable for the two unknown matrices X_1 *and* Y_1 *of appropriate sizes if and only if*

     $$\begin{aligned} &E_{B_1}(A_2 - A_1 + B_2X_2C_2 + D_2Y_2E_2)F_{C_1} = 0 && \Leftrightarrow \\ &E_{B_1}(A_2 - A_1)F_{C_1} + E_{B_1}(B_2X_2C_2)F_{C_1} + E_{B_1}(D_2Y_2E_2)F_{C_1} = 0 && \Leftrightarrow \\ &E_{B_1}(B_2X_2C_2)F_{C_1} + E_{B_1}(D_2Y_2E_2)F_{C_1} = -E_{B_1}(A_2 - A_1)F_{C_1}. \end{aligned}$$

- *By applying lemma (5) to the last equation and we find*

$$\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}B_2 & E_{B_1}D_2 \end{bmatrix} = 0, \tag{2.46}$$

$$\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}B_2 \\ E_2F_{C_1} & 0 \end{bmatrix} = 0, \tag{2.47}$$

$$\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}D_2 \\ C_2F_{C_1} & 0 \end{bmatrix} = 0, \tag{2.48}$$

$$\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} \\ C_2F_{C_1} \\ E_2F_{C_1} \end{bmatrix} = 0. \tag{2.49}$$

Reminder that a matrix A is null if and only if $r(A) = 0$.Hold, where

$$r\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}B_2 & E_{B_1}D_2 \end{bmatrix} = r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} - r(B_1) - r(C_1),$$

$$r\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}B_2 \\ E_2F_{C_1} & 0 \end{bmatrix} = r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(C_1),$$

$$r\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} & E_{B_1}D_2 \\ C_2F_{C_1} & 0 \end{bmatrix} = r\begin{bmatrix} A_2 - A_1 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(C_1),$$

$$r\begin{bmatrix} E_{B_1}(A_2 - A_1)F_{C_1} \\ C_2F_{C_1} \\ E_2F_{C_1} \end{bmatrix} = r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} - r(B_1) - r(C_1).$$

hold by lemma 1 (c). Substituting these four matrix rank equalities into (2.46)–(2.49) leads to the equivalences of (2.37)–(2.40) and (2.46)–(2.49), respectively.

3. *The condition $\mathcal{D}_1 \subseteq \mathcal{D}_2$ is equivalent to the fact that: The equation (2.45) equivalent to*

$$B_2X_2C_2 + D_2Y_2E_2 = B_1X_1 + Y_1C_1 - A_2 + A_1$$

By lemma (4)

$$\begin{cases} C = B_1X_1 + Y_1C_1 - A_2 + A_1, \\ A_1 = B_2, \\ B_1 = C_2, \\ A_2 = D_2, \\ B_2 = E_2. \end{cases}$$

is solvable if the following four matrix equalities

$$E_G(B_1X_1 + Y_1C_1 + A_1 - A_2) = 0$$
$$E_{B_2}(B_1X_1 + Y_1C_1 + A_1 - A_2)F_{E_2} = 0$$
$$E_{D_2}(B_1X_1 + Y_1C_1 + A_1 - A_2)F_{C_2} = 0$$
$$(B_1X_1 + Y_1C_1 + A_1 - A_2)F_H = 0$$

Which are further equivalent to

$$E_G B_1 X_1 + E_G Y_1 C_1 + E_G(A_1 - A_2) = 0 \tag{2.50}$$
$$E_{B_2} B_1 X_1 F_{E_2} + E_{B_2} Y_1 C_1 F_{E_2} + E_{B_2}(A_1 - A_2) F_{E_2} = 0 \tag{2.51}$$
$$E_{D_2} B_1 X_1 F_{C_2} + E_{D_2} Y_1 C_1 F_{C_2} + E_{D_2}(A_1 - A_2) F_{C_2} = 0 \tag{2.52}$$
$$B_1 X_1 F_H + Y_1 C_1 F_H + (A_1 - A_2) F_H = 0 \tag{2.53}$$

holds for all matrices X_1 and Y_1, where $G = [B_2, D_2]$ and $\begin{bmatrix} C_2 \\ E_2 \end{bmatrix}$.
By lemma (5),the matrix equality in (2.50) holds for all matrices X_1 and Y_1 if and only if any one of the following two conditions holds:

$$E_G = 0 \Leftrightarrow r[B_2, D_2] = m,$$
$$\text{or} \quad r\begin{bmatrix} E_G(A_1 - A_2) & E_G B_1 \\ C_1 & 0 \end{bmatrix} = 0 \;\Leftrightarrow r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 & D_2 \\ C_1 & 0 & 0 & 0 \end{bmatrix} = r[B_2, D_2].$$

by (2.1), (2.3) , and Lemma 1(a) and (c), as stipulated by the requirements of (2.41); the matrix equality in (2.51) holds for all matrices X_1 and Y_1 if and only if any one of the following three conditions holds:

$$E_{B_2} = 0 \Leftrightarrow r(B_2) = m,$$
$$\text{or} \quad r\begin{bmatrix} E_{B_2}(A_1 - A_2)F_{E_2} & E_{B_2} B_1 \\ C_1 F_{E_2} & 0 \end{bmatrix} = 0 \;\Leftrightarrow r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = r(B_2) + r(E_2).$$
$$\text{or} \quad F_{E_2} = 0 \Leftrightarrow r(E_2) = n$$

by (2.1), (2.3), and Lemma 1(a) and (c), thus establishing (2.42), the matrix equality in (2.52) holds for all matrices X_1 and Y_1 if and only if any one of the following three conditions are true:

$$E_{D_2} = 0 \Leftrightarrow r(D_2) = m,$$
$$\text{or} \quad r\begin{bmatrix} E_{D_2}(A_1 - A_2)F_{C_2} & E_{D_2} F_1 \\ C_1 F_{C_2} & 0 \end{bmatrix} = 0 \;\Leftrightarrow r\begin{bmatrix} A_1 - A_2 & B_1 & D_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = r(C_2) + r(D_2).$$
$$\text{or} \quad F_{C_2} = 0 \Leftrightarrow r(C_2) = n$$

by(2.1), (2.3), and lemma1 (a) and (c), as defined in (2.43); The matrix equality in (2.53) holds for all matrices X_1 and Y_1 if and only if any of the following four conditions are true:

$$F_H = 0 \Leftrightarrow r\begin{bmatrix} C_2 \\ E_2 \end{bmatrix} = n,$$
$$\text{or} \quad r\begin{bmatrix} (A_1 - A_2)F_H & B_1 \\ C_1 F_H & 0 \end{bmatrix} = 0 \;\Leftrightarrow r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ E_2 \end{bmatrix}.$$

by (2.2), (2.3), and Lemma 1(b) and (c), thus establishing (2.44).

2.3 Relationships Between Linear Transformations for Solutions of Basic Linear Matrix Equations

Penrose [25] discovered that general solutions of linear matrix equations can be derived and represented by some linear algebraic matrix expressions consisting of the matrices given in the matrix equations and their generalized inverses. Based on this important fact, we can leverage previous formulas, results, and facts to describe and characterize possible relationships. differences between solutions of different linear matrix equations.
By observation, we can find that there are many types of linear matrix equations whose general solutions can be represented in some explicit linear matrix functions, as given in (2.54). In this part, we will discuss a selection of results and facts about the relationships between some linear transformations for solutions of some basic linear matrix equations.

Theorem 11 *[42] Assume that the two linear matrix equations*

$$A_1X_1 = B_1 \text{ and } A_2X_2 = B_2 \tag{2.54}$$

are solvable for X_1 and X_2, respectively, where $A_i \in \mathbb{C}^{m_i\times n_i}$ and $B_i \in \mathbb{C}^{m_i\times p}$ are given, i=1,2. We also denote

$$\mathcal{D}_1 = \{S_1X_1 + T_1 | A_1X_1 = B_1\} \text{ and } \mathcal{D}_2 = \{S_2X_2 + T_2 | A_2X_2 = B_2\}, \tag{2.55}$$

where $S_i \in \mathbb{C}^{s\times n_i}$ and $T_i \in \mathbb{C}^{s\times p}$ are given; $i = 1,2$. Then, we have the following results

a) $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ if and only if

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}. \tag{2.56}$$

b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_2 \\ A_2 \end{bmatrix} + r(A_1). \tag{2.57}$$

c) $\mathcal{D}_1 = \mathcal{D}_2$ if and only if

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_1 \\ A_1 \end{bmatrix} + r(A_2) = r\begin{bmatrix} S_2 \\ A_2 \end{bmatrix} + r(A_1). \tag{2.58}$$

Proof *By lemma 2, the general solutions of the two linear matrix equations in (2.54) can be expressed as $X_1 = A_1B_1 + F_{A_1}U$,and $X_2 = A_2B_2 + F_{A_2}V$ Where $U \in \mathbb{C}^{n_1\times p}$ and $V \in \mathbb{C}^{n_2\times p}$ are arbitrary matrices. Then, the equation (2.55) can be rewritten as*

$$\mathcal{D}_1 = \left\{S_1A_1^+B_1 + S_1F_{A_1}U + T_1\right\} \text{ and } \mathcal{D}_2 = \left\{S_2A_2^+B_2 + S_2F_{A_2}V + T_2\right\}. \tag{2.59}$$

By applying Lemma 8a) to (2.59), we obtain that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ if and only if

$$r\begin{bmatrix} S_1F_{A_1} & S_2F_{A_2} & S_1A_1^+B_1 - S_2A_2^+B_2 + T_1 - T_2 \end{bmatrix} = r\begin{bmatrix} S_1F_{A_1} & S_2F_{A_2} \end{bmatrix}, \tag{2.60}$$

We need to calculate these two rank equality

$$\begin{aligned} & r\begin{bmatrix} S_1F_{A_1} & S_2F_{A_2} & S_1A_1^+B_1 - S_2A_2^+B_2 + T_1 - T_2 \end{bmatrix} \\ =& r\begin{bmatrix} S_1 & S_2 & S_1A_1^+B_1 - S_2A_2^+B_2 + T_1 - T_2 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \end{bmatrix} - r(A_1) - r(A_2), \\ =& r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} - r(A_1) - r(A_2). \end{aligned}$$

and

$$r\begin{bmatrix} S_1F_{A_1} & S_2F_{A_2} \end{bmatrix} = r\begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} - r(A_1) - r(A_2).$$

So Eq (2.60) is equivalent to

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_1 & S_2 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{2.61}$$

thus establishing a) By applying to the lemma 8 b) to Eq (2.60), we obtain that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ if and only if

$$r\begin{bmatrix} S_1F_{A_1} & S_2F_{A_2} & S_1A_1^+B_1 - S_2A_2^+B_2 + T_1 - T_2 \end{bmatrix} = r(S_2F_{A_2}), \tag{2.62}$$

where by (2.2), the following rank equality

$$r(S_2F_{A_2}) = r\begin{bmatrix} S_2 \\ A_2 \end{bmatrix} - r(A_2) \tag{2.63}$$

So Eq (2.62) is equivalent to

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_2 \\ A_2 \end{bmatrix} + r(A_1) \tag{2.64}$$

yields Result b). Similarly , we obtain that $\mathcal{D}_1 \supseteq \mathcal{D}_2$ if and only if

$$r\begin{bmatrix} S_1 & S_2 & T_1 - T_2 \\ A_1 & 0 & -B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} S_1 \\ A_1 \end{bmatrix} + r(A_2) \tag{2.65}$$

So the equations (2.64) and (2.65) leads to c).

Corollary 3 *Assume that $A_1X_1 = B_1$ and $A_2X_2 = B_2$ in (2.54) are solvable for X_1 and X_2, respectively, and denote*

$$\mathcal{D}_1 = \{X_1 | A_1X_1 = B_1\} \ \ and \ \ \mathcal{D}_2 \{X_2 | A_2X_2 = B_2\}. \tag{2.66}$$

Then,we have the following results.

1. $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ *if and only if*

$$r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} ,i.e, \ \mathfrak{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

2. $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *if and only if*

$$r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1),i.e, \ \mathfrak{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \ \ and \ \ \mathfrak{R}(A_2^*) \subseteq \mathfrak{R}(A_1^*).$$

3. $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if*

$$r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1) = r(A_2),i.e, \ \mathfrak{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \ \ and \ \ \mathfrak{R}(A_2^*) = \mathfrak{R}(A_1^*).$$

Corollary 4 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$ be given, and suppose that $AX = B$ is solvable for $X \in \mathbb{C}^{n\times p}$. In addition, we denote*

$$\mathcal{D}_1 = \{SX | AX = B\} \ \ and \ \ \mathcal{D}_2 = \{SX | MAX = MB\}. \tag{2.67}$$

where $M \in \mathbb{C}^{t\times m}$ and $S \in \mathbb{C}^{s\times n}$ are two given matrices. Then, the following results hold.

1. $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *always holds.*
2. $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if*

$$r\begin{bmatrix} MA \\ S \end{bmatrix} = r\begin{bmatrix} A \\ S \end{bmatrix} + r(MA) - r(A).$$

Corollary 5 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$ be given, and suppose that $AX = B$ is solvable for $X \in \mathbb{C}^{n\times p}$. In addition, we denote*

$$\mathcal{D}_1 = \{X | AX = B\} \ \ and \ \ \mathcal{D}_2 \{X | MAX = MB\}. \tag{2.68}$$

where $M \in \mathbb{C}^{s\times m}$. Then, we have the following results.

1. $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *always holds.*
2. $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if $r(MA) = r(A)$.*

Chapter 3

More on relationships between some linear matrix functions

3.1 Introduction

The notations used in chapter 2 are considered in this one.

According to the previous chapter, in this one, we will extend the study of the algebraic relationships between two domains of linear matrix functions, where we will considered new domains, further we focus on the intersection, the inclusion as well as the equality of these domains. While all matrix functions have a class of fundamental types can be defined consistently using matrices' additions and multiplications. To investigate the connections between several matrix equations and matrix functions, we refer to the use of some basic tools. Among them are matrix rank and the matrix range method. The rank of a matrix is one of the most basic quantities and useful methods and tools that are widely used in linear algebra specifically, in matrix theory and its applications. This finite nonnegative integer can be used to represent many properties of matrices such as singularity or nonsingularity of a matrix, identification of matrices, consistency of matrix equation, etc. ..to review further relevant works, see ([34], [2], [18], [36], [33], [25]). The rank of matrices or partitioned matrices was first studied by Marsaglia and Styan [19] , where they provided various formulas that simplify complicated matrix expressions or equalities, as shown in Lemma 1 below, This work aims to explore and suggest some basic aspects concerning the domains of some specific examples of LMFs using the matrix rank method. Further, It is well known that there are different classes of linear matrix equations, the general solutions to which can be explained by particular explicit linear matrix functions, for instance, Jiang et al. in [14] studied the relationships between the set of solutions to $AXB = C$ and the set of solutions of its reduced equations. In addition one of the primary topics of study in the subject of linear matrix functions as well as matrix equations is derivating some properties of generalized inverses and characterizing the connections between the generalized inverses of two matrices, because of this fact and motivated by the works mentioned above, In this chapter, will given two linear matrix functions and their domains as

$$\mathcal{T}_1 = \left\{ A_1 + B_1 X_1 C_1 \mid X_1 \in \mathbb{C}^{p_1 \times n_1} \right\}. \tag{3.1}$$

$$\mathcal{T}_2 = \left\{ A_2 + B_2 X_2 C_2 + B_3 X_3 C_3 \mid X_2 \in \mathbb{C}^{p_2 \times n_2}, X_3 \in \mathbb{C}^{p_3 \times n_3} \right\}. \tag{3.2}$$

where A_1, $A_2 \in \mathbb{C}^{l\times n}$, $B_i \in \mathbb{C}^{l\times p_i}$, $C_i \in \mathbb{C}^{n_i\times n}$, for $i = \overline{1,3}$, we establish the necessary and sufficient conditions for the two relations $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ to hold. In Section 3, we provide a collection of results and details regarding the connections between specific linear interpretations of the basic linear matrix equation solutions. As a consequence, we characterize the relationship between the generalized inverses of two matrices.

Lemma 9 *[43] Let $T \in \mathbb{C}^{l\times n}$, $N \in \mathbb{C}^{l\times p}$, $B \in \mathbb{C}^{p\times k}$ and $D \in \mathbb{C}^{n\times k}$ be given. Then the system of matrix equations $TX = N$ and $XB = D$ has a solution if and only if*

$$TT^+N = N,\ DB^+B = D \quad \text{and} \quad TD = NB.$$

In this case the general solution can be written as

$$X = T^+N + F_T DB^+ + F_T V E_B, \tag{3.3}$$

where $V \in C^{n\times p}$ is arbitrary.

Lemma 10 *[25] Let*

$$AXB = C \tag{3.4}$$

be a two-sided linear matrix equation, where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$,and $C \in \mathbb{C}^{m\times q}$ are given, and $X \in \mathbb{C}^{n\times p}$ is an unknown matrix.
Then, the following four statements are equivalent:

a) Equation (3.4) is solvable for X.

b) Both $\mathfrak{R}(A) \supseteq \mathfrak{R}(C)$ and $\mathfrak{R}(B^*) \supseteq \mathfrak{R}(C^*)$.

c) Both $r[A, C] = r(A)$ and $r\begin{bmatrix} B \\ C \end{bmatrix} = r(B)$.

d) $AA^+CB^+B = C$.

In this case, the general solution X of (3.4) can be written in the parametric form $X = A^+CB^+ + F_AU + VE_B$, where $U, V \in \mathbb{C}^{n\times p}$ are two arbitrary matrices. In particular, (3.4) holds for all matrices $X \in \mathbb{C}^{n\times p}$ if and only if

$$[A, C] = 0 \quad \text{or} \quad \begin{bmatrix} B \\ C \end{bmatrix} = 0. \tag{3.5}$$

Lemma 11 *[25]*

1) There exists an X such that $BXC = A$ if and only if $\mathfrak{R}(A) \subseteq \mathfrak{R}(B)$ and $\mathfrak{R}(A^*) \subseteq \mathfrak{R}(C^*)$, or equivalently, $BB^+A = AC^+C = A$ In this case, the general solution can be written in the parametric form $X = B^+AC^+ + F_BU_1 + U_2E_C$ where U_1 and U_2 are arbitrary matrices.

2) There exist X and Y such that $BX + YC = A$ if and only if $E_BAF_C = 0$.

Lemma 12 *[37] Assume that two LMFs and their domains are given by*

$$\mathcal{D}_1 = \left\{A_1 + B_1X_1C_1 | X_1 \in \mathbb{C}^{p_1\times q_1}\right\} \text{ and } \mathcal{D}_2 = \left\{A_2 + B_2X_2C_2 | X_2 \in \mathbb{C}^{p_2\times q_2}\right\}, \tag{3.6}$$

where $A_i \in \mathbb{C}^{m\times n}$,$B_i \in \mathbb{C}^{m\times p_i}$, and $C_i \in \mathbb{C}^{q_i\times n}$ are known matrices, and $X_i \in {}^{p_i\times q_i}$ are variablematrices; $i = 1, 2$.Then, we have the following results.

a) $\mathcal{D}_1 \bigcap \mathcal{D}_2 \neq \phi$ if and only if the following four conditions hold:
$\mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}[B_1, B_2]$, $\mathfrak{R}(A_1^* - A_2^*) \subseteq \mathfrak{R}[C_1^*, C_2^*]$
$r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} = r(B_1) + r(C_2)$, $r\begin{bmatrix} A_1 - A_2 & B_2 \\ C_1 & 0 \end{bmatrix} = r(B_2) + r(C_1)$.

b) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if any one of the following three conditions holds:

1) $\mathfrak{R}[A_1 - A_2, B_1] \subseteq \mathfrak{R}(B_2)$ *and* $\mathfrak{R}[A_1^* - A_2^*, C_1^*] \subseteq \mathfrak{R}(C_2^*)$
2) $B_1 = 0$, $\mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}(B_2)$, *and* $\mathfrak{R}(A_1^* - A_2^*) \subseteq \mathfrak{R}(C_2^*)]$
3) $C_1 = 0, \mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}(B_2)$, *and* $\mathfrak{R}(A_1^* - A_2^*) \subseteq \mathfrak{R}(C_2^*)$

c) $\mathcal{D}_1 = \mathcal{D}_2$if and only if any one of the following five conditions hold:

1) $\mathfrak{R}(A_1 - A_2) \subseteq \mathfrak{R}(B_1) = \mathfrak{R}(B_2)$ *and* $\mathfrak{R}(A_1^* - A_2^*) \subseteq \mathfrak{R}(C_1^*) = \mathfrak{R}(C_2^*)$.
2) $A_1 = A_2, \quad B_1 = 0, B_2 = 0$.
3) $A_1 = A_2, \quad B_1 = 0, C_2 = 0$.
4) $A_1 = A_2, \quad B_2 = 0, C_2 = 0$.
5) $A_1 = A_2, \quad C_1 = 0, C_2 = 0$.

Proof
Observe that the difference $M_1 - M_2$ for $M_1 \in \mathcal{D}_1$ and $M_2 \in \mathcal{D}_2$ can be written as
$M_1 - M_2 = A_1 - A_2 + B_1X_1C_1 - B_2X_2C_2$.
$M_1 - M_2$ gives

$$\min_{M_1\in\mathcal{D}_1, M_2\in\mathcal{D}_2} r(M_1 - M_2) = r\begin{bmatrix} A_1 - A_2 \\ C_1 \\ C_2 \end{bmatrix} + r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \end{bmatrix} + \max\{s_1, s_2\}, \tag{3.7}$$

Where

$$s_1 = r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A_1 - A_2 & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 - A_2 & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

Result (a) follows immediately from (3.7). From Eq (2.2) in [37], $\mathcal{D}_1 \subseteq \mathcal{D}_2$ holds if and only if

$$\max_{X_1} \min_{X_2} r[(A_1 + B_1X_1C_1) - (A_2 + B_2X_2C_2)] = 0.$$

Applying (2.24) gives

$$\min_{X_2} r[(A_1 + B_1X_1C_1) - (A_2 + B_2X_2C_2)] = r[A_1 - A_2 + B_1X_1C_1, B_2] + r\begin{bmatrix} A_1 - A_2 + B_1X_1C_1 \\ C_2 \end{bmatrix} - r\begin{bmatrix} A_1 - A_2 + B_1X_1C_1 & B_2 \\ C_2 & 0 \end{bmatrix}.$$

It is difficult to find the maximal rank of the expression with respect to X_1. From 11 (1), $\mathcal{D}_1 \subseteq \mathcal{D}_2$ holds if and only if

$$r[A_1 - A_2 + B_1X_1C_1, B_2] = r(B_2) \ \ and \ \ r[A_1^* - A_2^* + C_1^*X_1^*C_1^*, B_2^*] = r(C_2) \tag{3.8}$$

hold for all X_1. Applying (2.23) gives

$$\begin{aligned} \max_{X_1} r[A_1 - A_2 + B_1X_1C_1, B_2] &= \min\left\{ r[A_1 - A_2, B_1, B_2],\ r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_1 & 0 \end{bmatrix} \right\}, \\ \max_{X_1} r[A_1^* - A_2^* + C_1^*X_1^*B_1^*, C_2^*] &= \min\left\{ r\begin{bmatrix} A_1 - A_2 \\ C_1 \\ C_2 \end{bmatrix},\ r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} \right\}, \end{aligned}$$

Thus, the first equality in (3.8) is equivalent to

$$r[A_1 - A_2, B_1, B_2] = r(B_2) \ \ or \ \ r\begin{bmatrix} A_1 - A_2 & B_2 \\ C_1 & 0 \end{bmatrix} = r(B_2) \tag{3.9}$$

and the second equality in (3.8) is equivalent to

$$r\begin{bmatrix} A_1 - A_2 \\ C_1 \\ C_2 \end{bmatrix} = r(C_2) \ \ or \ \ r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} = r(C_2) \tag{3.10}$$

Note that B_1, B_2, C_1 and C_2 are nonzero. Hence, (3.9) and (3.10) are equivalent to (b).
If at least one of B_1, B_2, C_1 and C_2 is zero, the relations between the two sets $\mathcal{D}_1$ and $\mathcal{D}_2$ in (lemma 12) become trivial.

Lemma 13 *[30] The linear matrix equation*

$$A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 = C \tag{3.11}$$

is solvable for the three unknown matrices X_1, X_2 and X_3 of appropriate sizes if and only if the following nine matrix rank equalities

$$r\begin{bmatrix} A_1 & A_2 & A_3 & C \end{bmatrix} = r\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}, r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ C \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},$$

$$r\begin{bmatrix} C & A_1 & A_2 \\ B_3 & 0 & 0 \end{bmatrix} = r[A_1, A_2] + r(B_3), r\begin{bmatrix} C & A_1 & A_3 \\ B_2 & 0 & 0 \end{bmatrix} = r[A_1, A_3] + r(B_2),$$

$$r\begin{bmatrix} C & A_2 & A_3 \\ B_1 & 0 & 0 \end{bmatrix} = r[A_2, A_3] + r(B_1), r\begin{bmatrix} C & A_3 \\ B_1 & 0 \\ B_2 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + r(A_3),$$

$$r\begin{bmatrix} C & A_2 \\ B_1 & 0 \\ B_3 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} + r(A_2), r\begin{bmatrix} C & A_1 \\ B_2 & 0 \\ B_3 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r(A_1),$$

$$r\begin{bmatrix} C & 0 & A_1 & 0 & A_3 \\ 0 & -C & 0 & A_2 & A_3 \\ B_2 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 \\ B_3 & B_3 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & 0 \\ 0 & B_1 \\ B_3 & B_3 \end{bmatrix} + r\begin{bmatrix} A_1 & 0 & A_3 \\ 0 & A_2 & A_3 \end{bmatrix}.$$

hold.

3.2 Relationship between linear matrix functions

In this section, we consider the two domains given in (3.1) and (3.2), we discuss the necessary and sufficient conditions for the two relations $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ to hold, then we derive connections between domains of some well known linear matrix functions.

Theorem 12 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be as given in (3.1) and (3.2) respectively. Then,*
(a) $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$ if and only if all the next equalities hold

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}, \; r\begin{bmatrix} A_2 - A_1 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix},$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_2 \end{bmatrix} + r(C_3),$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_3 \end{bmatrix} + r(C_2),$$

$$r\begin{bmatrix} A_2 - A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix} + r(C_1),$$

$$r\begin{bmatrix} A_2 - A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + r(B_3), \; r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix} + r(B_2),$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} + r(B_1),$$

$$r\begin{bmatrix} A_2 - A_1 & 0 & B_1 & 0 & B_3 \\ 0 & A_1 - A_2 & 0 & B_2 & B_3 \\ C_2 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 \\ C_3 & C_3 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 & 0 \\ 0 & C_1 \\ C_3 & C_3 \end{bmatrix} + r\begin{bmatrix} B_1 & 0 & B_3 \\ 0 & B_2 & B_3 \end{bmatrix}.$$

(b) $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if and only if any one of the next equalities hold

$$\begin{gathered} r\begin{bmatrix} B_2 & B_3 \end{bmatrix} = l \;\; or\; r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}, \\ or\; r\begin{bmatrix} A_2 - A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}. \end{gathered} \tag{3.12}$$

$$\begin{gathered} r(B_2) = l \;\; or\; r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r(B_2) + r(C_3), \\ or\; r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r(C_3) + r(B_2) \;\; or\; r(C_3) = n. \end{gathered} \tag{3.13}$$

$$\begin{gathered} r(B_3) = l \;\; or\; r\begin{bmatrix} A_2 - A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix} = r(B_3) + r(C_2), \\ or\; r\begin{bmatrix} A_2 - A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} = r(C_2) + r(B_3) \;\; or\; r(C_2) = n. \end{gathered} \tag{3.14}$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix}, \text{ or}$$
$$r\begin{bmatrix} A_2 - A_1 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix}, \text{ or } r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = n. \tag{3.15}$$

Proof *(a) The intersection* $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$ *is obviously equivalent to*

$$A_1 + B_1X_1C_1 = A_2 + B_2X_2C_2 + B_3X_3C_3. \tag{3.16}$$

Eq (3.16) can be written as

$$B_1X_1C_1 - B_2X_2C_2 - B_3X_3C_3 = A_2 - A_1. \tag{3.17}$$

By applying Lemma 13 to the Eq (3.17) we get (a).
(b) Eq (3.16) can be written as

$$B_2X_2C_2 + B_3X_3C_3 = A_1 - A_2 + B_1X_1C_1 \tag{3.18}$$

From Lemma 4, Eq (3.18) holds for the two matrices X_2 *and* X_3 *if and only if all the next four conditions hold*

$$E_{[B_2,B_3]}(A_1 - A_2 + B_1X_1C_1) = 0, \tag{3.19}$$
$$E_{B_2}(A_1 - A_2 + B_1X_1C_1)F_{C_3} = 0, \tag{3.20}$$
$$E_{B_3}(A_1 - A_2 + B_1X_1C_1)F_{C_2} = 0, \tag{3.21}$$
$$((A_1 - A_2) + B_1X_1C_1)F_Z = 0 \tag{3.22}$$

where $Z = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$. *By Lemma 10, Eq (3.19) holds for all* X_1 *if and only if*

$$E_{[B_2,B_3]} = 0 \text{ or } \begin{bmatrix} E_{[B_2,B_3]}B_1 & E_{[B_2,B_3]}(A_2 - A_1) \end{bmatrix} = 0 \text{ or } \begin{bmatrix} C_1 \\ E_{[B_2,B_3]}(A_2 - A_1) \end{bmatrix} = 0,$$

which are equivalent respectively to

$$r\begin{bmatrix} B_2 & B_3 \end{bmatrix} = l \text{ or } r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix},$$
$$\text{or } r\begin{bmatrix} A_2 - A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}.$$

this proves (3.12). Eq (3.20) holds for all X_1 *if and only if*

$$E_{B_2} = 0 \text{ or } \begin{bmatrix} E_{B_2}B_1 & E_{B_2}(A_2 - A_1)F_{C_3} \end{bmatrix} = 0 \text{ or } \begin{bmatrix} C_1F_{C_3} \\ E_{B_2}(A_2 - A_1)F_{C_3} \end{bmatrix} \text{ or } F_{C_3} = 0,$$

which, in consequence, equal

$$r(B_2) = l \text{ or } r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r(B_2) + r(C_3),$$
$$\text{or } r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r(C_3) + r(B_2) \text{ or } r(C_3) = n.$$

which yields (3.13). Similarly, Eq (3.21) holds for all X_1 *if and anly if*

$$r\left(B_3\right)=l \text{ or } r\begin{bmatrix} A_2-A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix}=r\left(B_3\right)+r\left(C_2\right),$$

$$\text{or } r\begin{bmatrix} A_2-A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}=r\left(C_2\right)+r\left(B_3\right) \text{ or } r\left(C_2\right)=n.$$

Then we get (3.14). Eq (3.22) holds for all X_1 *if and only if*

$$\begin{bmatrix} B_1 & (A_2-A_1)F_Z \end{bmatrix}=0 \text{ or } \begin{bmatrix} C_1F_Z \\ (A_2-A_1)F_Z \end{bmatrix}=0 \text{ or } F_Z=0,$$

which, then equal

$$r\begin{bmatrix} A_2-A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix}=r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} \text{ or } r\begin{bmatrix} A_2-A_1 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}=r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} \text{ or } r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix}=n,$$

which proves (3.15). Hence we establish (b).

Setting $B_3 = I_p$, $C_2 = I_n$ in Theorem 12 we get the following result

Corollary 6 *Consider the two domains of two linear matrix functions*

$$\mathcal{T}_1=\left\{A_1+B_1X_1C_1 \mid X_1\in\mathbb{C}^{p_1\times n_1}\right\}, \tag{3.23}$$

$$\mathcal{T}_2=\left\{A_2+B_2X_2+X_3C_3 \mid X_2\in\mathbb{C}^{p_2\times n},\ X_3\in\mathbb{C}^{l\times p_3}\right\}. \tag{3.24}$$

where $A_1, A_2 \in \mathbb{C}^{l\times n}$, $B_1 \in \mathbb{C}^{l\times p_1}$, $B_2 \in \mathbb{C}^{l\times p_2}$*and* $C_1 \in \mathbb{C}^{n_1\times n}$, $C_3 \in \mathbb{C}^{p_3\times n}$*are known matrices. Then,*
(a) $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$ *if and only if the next rank equalities hold*

$$r\begin{bmatrix} A_2-A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix}=r\begin{bmatrix} B_1 & B_2 \end{bmatrix}+r(C_3),$$

$$r\begin{bmatrix} A_2-A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix}=r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}+r(B_2),$$

$$r\begin{bmatrix} A_2-A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix}=r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}+r\begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

(b) $\mathcal{T}_1 \subseteq \mathcal{T}_2$ *if and only if*

$$r(B_2) = l \text{ or } r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r(B_2) + (C_3)$$
$$\text{or } r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r(C_3) + r(B_2) \text{ or } r(C_3) = n. \tag{3.25}$$

From Colrollary 6 we can deduce the next result:

Corollary 7 *Let* $A_4 \in \mathbb{C}^{l\times n}$, $B_4 \in \mathbb{C}^{p\times s}$, $C_4 \in \mathbb{C}^{l\times p}$, $D_4 \in \mathbb{C}^{n\times s}$, $A_5 \in \mathbb{C}^{k\times t}$, $B_5 \in \mathbb{C}^{k\times n}$, $C_5 \in \mathbb{C}^{p\times t}$*be given,* X_4, $X_5 \in \mathbb{C}^{n\times p}$ *be unknown matrices, and assume that the system* $A_4X = C_4$, $XB_4 = D_4$, *and the matrix equation* $B_5XC_5 = A_5$ *are solvable for* X_4 *and* X_5 *respectively. Denote:*

$$\mathcal{T}_1 = \left\{X_4 \in \mathbb{C}^{n\times p} \mid A_4X_4 = C_4, X_4B_4 = D_4\right\}, \tag{3.26}$$
$$\mathcal{T}_2 = \left\{X_5 \in \mathbb{C}^{n\times p} \mid B_5X_5C_5 = A_5\right\}. \tag{3.27}$$

Then,
(a) $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$, *that is the system* $A_4X_4 = C_4$, $X_4B_4 = D_4$ *and* $B_5X_5C_5 = A_5$ *have a common solution if and only if*

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix},$$
$$r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix},$$
$$r\begin{bmatrix} 0 & B_4 & C_5 \\ A_4 & -C_4B_4 & 0 \\ B_5 & 0 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix} + r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix}.$$

(b) $\mathcal{T}_1 \subseteq \mathcal{T}_2$, *that is all the solutions of* $A_4X_4 = C_4$, $X_4B_4 = D_4$ *are solutions of* $B_5X_5C_5 = A_5$ *if and only if* $r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r(A_4)$ *or* $r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r(B_4)$.

Proof *It follows from Lemmas 9 and 10, the solutions of system* $A_4X_4 = C_4$, $X_4B_4 = D_4$ *and equation* $B_5X_5C_5 = A_5$ *can be expressed respectively as*

$$X_4 = A_4^+C_4 + F_{A_4}D_4B_4^+ + F_{A_4}VE_{B_4},$$
$$X_5 = B_5^+A_5C_5^+ + F_{B_5}U + WE_{C_5},$$

where V, U *and* W *are arbitrary, So the two sets in (3.26) and (3.27) can be represented respectively as :*

$$\mathcal{T}_1 = \left\{A_4^+C_4 + F_{A_4}D_4B_4^+ + F_{A_4}VE_{B_4}\right\},$$
$$\mathcal{T}_2 = \left\{B_5^+A_5C_5^+ + F_{B_5}U + WE_{C_5}\right\}.$$

From Corollary 6, the relation $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \varnothing$ holds if and only if the next equalities hold

$$r\begin{bmatrix} B_5^+ A_5 C_5^+ - A_4^+ C_4 - F_{A_4} D_4 B_4^+ & F_{A_4} & F_{B_5} \\ E_{C_5} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} F_{A_4} & F_{B_5} \end{bmatrix} + r(E_{C_5}), \tag{3.28}$$

$$r\begin{bmatrix} B_5^+ A_5 C_5^+ - A_4^+ C_4 - F_{A_4} D_4 B_4^+ & F_{B_5} \\ E_{B_4} & 0 \\ E_{C_5} & 0 \end{bmatrix} = r\begin{bmatrix} E_{B_4} \\ E_{C_5} \end{bmatrix} + r(F_{B_5}), \tag{3.29}$$

$$r\begin{bmatrix} B_5^+ A_5 C_5^+ - A_4^+ C_4 - F_{A_4} D_4 B_4^+ & F_{A_4} & F_{B_5} \\ E_{B_4} & 0 & 0 \\ E_{C_5} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} E_{B_4} \\ E_{C_5} \end{bmatrix} + r\begin{bmatrix} F_{A_4} & F_{B_5} \end{bmatrix}. \tag{3.30}$$

By Lemma 1 and simplifying by $C_4B_4 = A_4D_4$, $A_4A_4^+C_4 = C_4$, $D_4B_4^+B_4 = D_4$, $B_5B_5^+A_5 = A_5$, $A_5C_5^+C_5 = A_5$, we find that the rank equalities in (3.28)-(3.30) are equivalent respectively to

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix},$$
$$r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix},$$
$$r\begin{bmatrix} 0 & B_4 & C_5 \\ A_4 & -C_4B_4 & 0 \\ B_5 & 0 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix} + r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix}.$$

thus (a) is proved.
(b) $\mathcal{T}_1 \subseteq \mathcal{T}_2$ holds if and only if

$$r\begin{bmatrix} B_5^+ A_5 C_5^+ - A_4^+ C_4 - F_{A_4} D_4 B_4^+ & F_{A_4} & F_{B_5} \\ E_{C_5} & 0 & 0 \end{bmatrix} = r(F_{B_5}) + r(E_{C_5}),$$
$$\text{or } r\begin{bmatrix} B_5^+ A_5 C_5^+ - A_4^+ C_4 - F_{A_4} D_4 B_4^+ & F_{B_5} \\ E_{B_4} & 0 \\ E_{C_5} & 0 \end{bmatrix} = r(E_{C_5}) + r(F_{B_5}),$$

which, then equal

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r(A_4) \ \text{ or } \ r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r(B_4).$$

Then we establish (b).

Remark 1 Result (a) of corollary 7 is the same that in [16] (Theorem 2.4).

3.3 Relations between linear matrix function with constraints

In this section, we examine some forms of LMFs constrained by linear matrix equations, we present results and information about the connections between various linear conceptions of the basic linear matrix equation solutions. Therefore, we define the connections between the generalized inverses of two matrices.

Theorem 13 *Let $A_1 \in \mathbb{C}^{l\times n}$, $B_1 \in \mathbb{C}^{l\times n_1}$, $C_1 \in \mathbb{C}^{p_1\times n}$, $A_2 \in \mathbb{C}^{l\times n}$, $B_2 \in \mathbb{C}^{l\times n_2}$, $C_2 \in \mathbb{C}^{p_2\times n}$ be given, $X_1 \in \mathbb{C}^{n_1\times p_1}$, $X_2 \in \mathbb{C}^{n_2\times p_2}$ be unknown matrices. Suppose that, the two linear matrix equations*

$$B_1X_1C_1 = A_1 \text{ and } B_2X_2C_2 = A_2 \tag{3.31}$$

are solvable for X_1 and X_2 respectively. Denote

$$\mathcal{T}_3 = \{D_1 + M_1X_1N_1 \ / \ A_1 = B_1X_1C_1\}. \tag{3.32}$$

$$\mathcal{T}_4 = \{D_2 + M_2X_2N_2 \ / \ A_2 = B_2X_2C_2\}. \tag{3.33}$$

where $D_i \in \mathbb{C}^{k\times n}$, $M_i \in \mathbb{C}^{k\times n_i}$, $N_i \in \mathbb{C}^{p_i\times n}$ are known matrices, $i = 1, 2$. Then,
(a) $\mathcal{T}_3 \cap \mathcal{T}_4 \neq \varnothing$ if and only if all the next conditions hold

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \end{bmatrix}, \tag{3.34}$$

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix} + r\begin{bmatrix} N_1 & C_1 & 0 \\ N_2 & 0 & C_2 \end{bmatrix}, \tag{3.35}$$

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 \\ N_1 & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \end{bmatrix} + r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \tag{3.36}$$

$$r\begin{bmatrix} D_2 - D_1 \\ N_1 \\ N_2 \end{bmatrix} = r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}. \tag{3.37}$$

(b) $\mathcal{T}_3 \subseteq \mathcal{T}_4$ if and only if at least one of the next conditions holds

$$r(M_2) = k \text{ or } r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 \\ N_1 & 0 & 0 & C_1 \\ 0 & -B_1 & 0 & A_1 \end{bmatrix} = r(B_1) + r(C_1) + r(M_2), \tag{3.38}$$

$$r\begin{bmatrix} M_2 \\ B_2 \end{bmatrix} = r(B_2) + k \text{ or } r\begin{bmatrix} N_2 & C_2 \end{bmatrix} = r(C_2) + n,$$

$$\text{or} \quad r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} M_2 \\ B_2 \end{bmatrix} + r\begin{bmatrix} N_2 & C_2 \end{bmatrix} + r(B_1) + r(C_1), \tag{3.39}$$

$$r(M_2) = k \text{ or } (N_2) = n \text{ or } r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 \\ N_1 & 0 & 0 & C_1 \\ N_2 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & A_1 \end{bmatrix} = r(M_2) + r(N_2) + r(C_1) + r(B_1), \tag{3.40}$$

$$r(N_2) = n \text{ or } r\begin{bmatrix} D_2 - D_1 & M_1 & 0 \\ N_1 & 0 & C_1 \\ N_2 & 0 & 0 \\ 0 & -B_1 & A_1 \end{bmatrix} = r(N_2) + r(C_1) + r(B_1), \tag{3.41}$$

(c) $\mathcal{T}_3 \supseteq \mathcal{T}_4$ *if and only if at least one of the next conditions holds*

$$r(M_1) = k \text{ or } r\begin{bmatrix} D_1 - D_2 & M_2 & M_1 & 0 \\ N_2 & 0 & 0 & C_2 \\ 0 & -B_2 & 0 & A_2 \end{bmatrix} = r(M_1) + r(B_2) + r(C_2), \tag{3.42}$$

$$\begin{aligned} & r\begin{bmatrix} M_1 \\ B_1 \end{bmatrix} = r(B_1) + k \text{ or } r\begin{bmatrix} N_1 & C_1 \end{bmatrix} = r(C_1) + n, \\ \text{or} \quad & r\begin{bmatrix} D_1 - D_2 & M_2 & M_1 & 0 & 0 \\ N_2 & 0 & 0 & C_2 & 0 \\ N_1 & 0 & 0 & 0 & C_1 \\ 0 & -B_2 & 0 & A_2 & 0 \\ 0 & 0 & B_1 & 0 & A_1 \end{bmatrix} = r\begin{bmatrix} M_1 \\ B_1 \end{bmatrix} + r\begin{bmatrix} N_1 & C_1 \end{bmatrix} + r(C_2) + r(B_2), \end{aligned} \tag{3.43}$$

$$\begin{aligned} & r(M_1) = k \text{ or } (N_1) = n, \\ \text{or} \quad & r\begin{bmatrix} D_1 - D_2 & M_2 & M_1 & 0 \\ N_2 & 0 & 0 & C_2 \\ N_1 & 0 & 0 & 0 \\ 0 & -B_2 & 0 & A_2 \end{bmatrix} = r(M_1) + r(N_1) + r(C_2) + r(B_2), \end{aligned} \tag{3.44}$$

$$r(N_1) = n \text{ or } r\begin{bmatrix} D_1 - D_2 & M_2 & 0 \\ N_2 & 0 & C_2 \\ N_1 & 0 & 0 \\ 0 & -B_2 & A_2 \end{bmatrix} = r(N_1) + r(B_2) + r(C_2). \tag{3.45}$$

(d) $\mathcal{T}_3 = \mathcal{T}_4$ *if and only if*

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} M_2 \\ B_2 \end{bmatrix} + r\begin{bmatrix} N_2 & C_2 \end{bmatrix} + r(B_1) + r(C_1)$$

$$= r\begin{bmatrix} M_1 \\ B_1 \end{bmatrix} + r\begin{bmatrix} N_1 & C_1 \end{bmatrix} + r(C_2) + r(B_2).$$

Proof *(a) By Lemma 10 the general solutions of equations in (3.31) are given respectively by:*

$$X_1 = B_1^+ A_1 C_1^+ + F_{B_1} V_1 + W_1 E_{C_1}$$

$$X_2 = B_2^+ A_2 C_2^+ + F_{B_2} V_2 + W_2 E_{C_2}$$

where V_1, V_2, W_1, *and* W_2 *are arbitrary.*
So the two sets in (3.32) and (3.33) can be represented as :

$$\mathcal{T}_3 = \left\{D_1 + M_1 B_1^+ A_1 C_1^+ N_1 + M_1 F_{B_1} V_1 N_1 + M_1 W_1 E_{C_1} N_1\right\}, \tag{3.46}$$

$$\mathcal{T}_4 = \left\{D_2 + M_2 B_2^+ A_2 C_2^+ N_2 + M_2 F_{B_2} V_2 N_2 + M_2 W_2 E_{C_2} N_2\right\}. \tag{3.47}$$

The condition $\mathcal{T}_3 \cap \mathcal{T}_4 \neq \varnothing$ *is obviously equivalent to*

$$M_1 F_{B_1} V_1 N_1 + M_1 W_1 E_{C_1} N_1 - M_2 F_{B_2} V_2 N_2 - M_2 W_2 E_{C_2} N_2 = D_2 - D_1 + M_2 B_2^+ A_2 C_2^+ N_2 - M_1 B_1^+ A_1 C_1^+ N_1 \tag{3.48}$$

Eq (3.48) can be written as

$$\begin{bmatrix} M_1 F_{B_1} & -M_2 F_{B_2} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} + \begin{bmatrix} M_1 & -M_2 \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} E_{C_1} N_1 \\ E_{C_2} N_2 \end{bmatrix}$$

$$= D_2 - D_1 + M_2 B_2^+ A_2 C_2^+ N_2 - M_1 B_1^+ A_1 C_1^+ N_1 \tag{3.49}$$

By Lemma 4 , Eq (3.49) holds for some variables matrices V_1, V_2, W_1, *and* W_2 *if and only if all the next equalities*

$$r\begin{bmatrix} D_2 - D_1 + M_2 B_2^+ A_2 C_2^+ N_2 - M_1 B_1^+ A_1 C_1^+ N_1 & M_1 F_{B_1} & M_2 F_{B_2} & M_1 & M_2 \end{bmatrix} = r\begin{bmatrix} M_1 F_{B_1} & M_2 F_{B_2} & M_1 & M_2 \end{bmatrix}, \tag{3.50}$$

$$r\begin{bmatrix} D_2 - D_1 + M_2 B_2^+ A_2 C_2^+ N_2 - M_1 B_1^+ A_1 C_1^+ N_1 & M_1 F_{B_1} & M_2 F_{B_2} \\ E_{C_1} N_1 & 0 & 0 \\ E_{C_2} N_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} M_1 F_{B_1} & M_2 F_{B_2} \end{bmatrix} + r\begin{bmatrix} E_{C_1} N_1 \\ E_{C_2} N_2 \end{bmatrix}, \tag{3.51}$$

$$r\begin{bmatrix} D_2 - D_1 + M_2B_2^+A_2C_2^+N_2 - M_1B_1^+A_1C_1^+N_1 & M_1 & M_2 \\ N_1 & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \end{bmatrix} + r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \tag{3.52}$$

$$r\begin{bmatrix} D_2 - D_1 + M_2B_2^+A_2C_2^+N_2 - M_1B_1^+A_1C_1^+N_1 \\ N_1 \\ N_2 \\ E_{C_1}N_1 \\ E_{C_2}N_2 \end{bmatrix} = r\begin{bmatrix} N_1 \\ N_2 \\ E_{C_1}N_1 \\ E_{C_2}N_2 \end{bmatrix}. \tag{3.53}$$

hold.
The rank equality in (3.50) is equivalent to

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \end{bmatrix},$$

which proves (3.34).
We have

$$\begin{aligned}
& r\begin{bmatrix} D_2 - D_1 + M_2B_2^+A_2C_2^+N_2 - M_1B_1^+A_1C_1^+N_1 & M_1F_{B_1} & M_2F_{B_2} \\ E_{C_1}N_1 & 0 & 0 \\ E_{C_2}N_2 & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} D_2 - D_1 + M_2B_2^+A_2C_2^+N_2 - M_1B_1^+A_1C_1^+N_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2) \\
& - r(C_1) - r(C_2) \\
&= r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ A_1C_1^+N_1 & B_1 & 0 & 0 & 0 \\ -A_2C_2^+N_2 & 0 & B_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2) - r(C_1) - r(C_2) \\
&= r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & A_2 \end{bmatrix} - r(B_1) - r(B_2) - r(C_1) - r(C_2)
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
r\begin{bmatrix} M_1F_{B_1} & M_2F_{B_2} \end{bmatrix} + r\begin{bmatrix} E_{C_1}N_1 \\ E_{C_2}N_2 \end{bmatrix} =& r\begin{bmatrix} M_1 & M_2 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix} + r\begin{bmatrix} N_1 & C_1 & 0 \\ N_2 & 0 & C_2 \end{bmatrix} \\
& - r(B_1) - r(B_2) - r(C_1) - r(C_2).
\end{aligned} \tag{3.55}$$

Substituting (3.54) and (3.55) into (3.51) leads to (3.35), Further the rank equalities in (3.52) and (3.53) respectively are equivalent to

$$r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 \\ N_1 & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} M_1 & M_2 \end{bmatrix} + r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix},$$
$$r\begin{bmatrix} D_2 - D_1 \\ N_1 \\ N_2 \end{bmatrix} = r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \tag{3.56}$$

where we establish (3.36) and (3.37) respectively, thus (a) is established.
(b) Eq (3.48) can be written as

$$M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2 =$$
$$D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1 + M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1 \tag{3.57}$$

by Lemma 4, Eq (3.57) holds for the two matrices V_2 and W_2 if and only if the next four equalities

$$E_G(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1 + M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1) = 0, \tag{3.58}$$

$$E_{M_2F_{B_2}}(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1 + M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1)F_{E_{C_2}N_2} = 0, \tag{3.59}$$

$$E_{M_2}(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1 + M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1)F_{N_2} = 0, \tag{3.60}$$

$$(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1 + M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1)F_L = 0. \tag{3.61}$$

hold for all matrices V_1 and W_1 , where $G = \begin{bmatrix} M_2F_{B_2} & M_2 \end{bmatrix}$ and $L = \begin{bmatrix} N_2 \\ E_{C_2}N_2 \end{bmatrix}$.
By Lemma 5, Eq (3.58) holds for all matrices V_1 and W_1 if and only if at least one of the next two conditions holds

$$E_G = 0 \ \text{ or } r\begin{bmatrix} E_G(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1) & E_GM_1F_{B_1} \\ E_{C_1}N_1 & 0 \end{bmatrix} = 0$$

further, we have

$$r(M_2) = k \ \text{ or } r\begin{bmatrix} D_2 - D_1 & M_1 & M_2 & 0 \\ N_1 & 0 & 0 & C_1 \\ 0 & -B_1 & 0 & A_1 \end{bmatrix} = r(B_1) + r(C_1) + r(M_2)$$

where we prove (3.38). Eq (3.59) holds for all matrices V_1 and W_1 if and only if at least one of the next conditions holds

$$r\begin{bmatrix} E_{M_2F_{B_2}}(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1)F_{E_{C_2}N_2} & E_{M_2F_{B_2}}M_1F_{B_1} \\ E_{C_1}N_1F_{E_{C_2}N_2} & 0 \end{bmatrix} = 0,$$

$$\text{or} \quad F_{E_{C_2}N_2} = 0 \quad \text{or} \quad E_{M_2F_{B_2}} = 0.$$

Further, are equivalent to

$$r\begin{bmatrix} D_2-D_1 & M_1 & M_2 & 0 & 0 \\ N_1 & 0 & 0 & C_1 & 0 \\ N_2 & 0 & 0 & 0 & C_2 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} M_2 \\ B_2 \end{bmatrix} + r\begin{bmatrix} N_2 & C_2 \end{bmatrix} + r\left(B_1\right) + r\left(C_1\right),$$
$$\text{or } r\begin{bmatrix} M_2 \\ B_2 \end{bmatrix} = r\left(B_2\right) + k \ \text{ or } r\begin{bmatrix} N_2 & C_2 \end{bmatrix} = r\left(C_2\right) + n.$$

which proves (3.39). Eq (3.60) holds for all matrices V_1 and W_1 if and only if at least one of the next conditions holds

$$E_{M_2} = 0 \ \text{ or } F_{N_2} = 0 \text{ or}$$
$$r\begin{bmatrix} E_{M_2}(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1)F_{N_2} & E_{M_2}M_1F_{B_1} \\ E_{C_1}N_1F_{N_2} & 0 \end{bmatrix} = 0.$$

Further, are equivalent to

$$r\left(M_2\right) = k \ \text{ or } r\left(N_2\right) = n \ \text{ or } r\begin{bmatrix} D_2-D_1 & M_1 & M_2 & 0 \\ N_1 & 0 & 0 & C_1 \\ N_2 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & A_1 \end{bmatrix} = r\left(M_2\right) + r\left(N_2\right) + r\left(C_1\right) + r\left(B_1\right).$$

Hence, we establish (3.40). Eq (3.61) holds for all matrices V_1 and W_1 if and only if at least one of the next two conditions holds

$$F_L = 0 \text{ or } r\begin{bmatrix} (D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1)F_L & M_1F_{B_1} \\ E_{C_1}N_1F_L & 0 \end{bmatrix} = 0$$

Further, we get

$$r\left(N_2\right) = n \text{ or } r\begin{bmatrix} D_2-D_1 & M_1 & 0 \\ N_1 & 0 & C_1 \\ N_2 & 0 & 0 \\ 0 & -B_1 & A_1 \end{bmatrix} = r\left(N_2\right) + r\left(C_1\right) + r\left(B_1\right),$$

then we prove (3.41).
(c) Eq (3.48) can be written as

$$\begin{aligned} M_1F_{B_1}V_1N_1 + M_1W_1E_{C_1}N_1 = & D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1 \\ & + M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2. \end{aligned} \tag{3.62}$$

by Lemma 4, Eq (3.62) holds for the two matrices V_1 and W_1 if and only if the next four equalities

$$E_{G'}(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1 + M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2) = 0, \tag{3.63}$$

$$E_{M_1F_{B_1}}(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1 + M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2)F_{E_{C_1}N_1} = 0, \tag{3.64}$$

$$E_{M_1}(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1 + M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2)F_{N_1} = 0, \tag{3.65}$$

$$(D_2 - D_1 + M_2B_2^{+}A_2C_2^{+}N_2 - M_1B_1^{+}A_1C_1^{+}N_1 + M_2F_{B_2}V_2N_2 + M_2W_2E_{C_2}N_2)F_{L'} = 0 \tag{3.66}$$

hold for all matrices V_2 and W_2, where $G' = \begin{bmatrix} M_1F_{B_1} & M_1 \end{bmatrix}$ and $L' = \begin{bmatrix} N_1 \\ E_{C_1}N_1 \end{bmatrix}$.
By Lemma 5, Eq (3.63) holds for all matrices V_2 and W_2 if and only if at least one of the next two conditions holds

$$E_{G'} = 0 \text{ or } r\begin{bmatrix} E_{G'}(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1) & E_{G'}M_2F_{B_2} \\ E_{C_2}N_2 & 0 \end{bmatrix} = 0.$$

Further, we have

$$r(M_1) = k \text{ or } r\begin{bmatrix} D_1 - D_2 & M_2 & M_1 & 0 \\ N_2 & 0 & 0 & C_2 \\ 0 & -B_2 & 0 & A_2 \end{bmatrix} = r(M_1) + r(B_2) + r(C_2)$$

hence, we prove (3.42). Eq (3.64) holds for all matrices V_2 and W_2 if and only if at least one of the next conditions holds

$$r\begin{bmatrix} E_{M_1F_{B_1}}(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1)F_{E_{C_1}N_1} & E_{M_1F_{B_1}}M_2F_{B_2} \\ E_{C_2}N_2F_{E_{C_1}N_1} & 0 \end{bmatrix} = 0,$$

$$\text{or } E_{M_1F_{B_1}} = 0 \quad \text{or } F_{E_{C_1}N_1} = 0.$$

Further, are equivalent to

$$r\begin{bmatrix} D_1 - D_2 & M_2 & M_1 & 0 & 0 \\ N_2 & 0 & 0 & C_2 & 0 \\ N_1 & 0 & 0 & 0 & C_1 \\ 0 & -B_2 & 0 & A_2 & 0 \\ 0 & 0 & B_1 & 0 & A_1 \end{bmatrix} = r\begin{bmatrix} M_1 \\ B_1 \end{bmatrix} + r\begin{bmatrix} N_1 & C_1 \end{bmatrix} + r(C_2) + r(B_2),$$

$$\text{or } \; r\begin{bmatrix} M_1 \\ B_1 \end{bmatrix} = r(B_1) + k \quad \text{or } \; r\begin{bmatrix} N_1 & C_1 \end{bmatrix} = r(C_1) + n,$$

which proves (3.43). Eq (3.65) holds for all matrices V_2 and W_2 if and only if any one of the following conditions holds

$$r\begin{bmatrix} E_{M_1}(D_1 - D_2 - M_2B_2^{+}A_2C_2^{+}N_2 + M_1B_1^{+}A_1C_1^{+}N_1)F_{N_1} & E_{M_1}M_2F_{B_2} \\ E_{C_2}N_2F_{N_1} & 0 \end{bmatrix} = 0,$$

$$\text{or} \quad E_{M_1} = 0 \text{ or } F_{N_1} = 0.$$

which are equivalent to

$$r\begin{bmatrix} D_1-D_2 & M_2 & M_1 & 0 \\ N_2 & 0 & 0 & C_2 \\ N_1 & 0 & 0 & 0 \\ 0 & -B_2 & 0 & A_2 \end{bmatrix} = r(M_1)+r(N_1)+r(C_2)+r(B_2),$$
$$\text{or } r(M_1)=k \text{ or } r(N_1)=n.$$

then we prove (3.44). Eq (3.66) holds for all matrices V_2 and W_2 if and only if at least one of the next two conditions holds

$$F_{L'}=0 \text{ or } r\begin{bmatrix} (D_1-D_2-M_2B_2^+A_2C_2^+N_2+M_1B_1^+A_1C_1^+N_1)F_{L'} & M_2F_{B_2} \\ E_{C_2}N_2F_{L'} & 0 \end{bmatrix}=0,$$

therefore, we have

$$r(N_1)=n \text{ or } r\begin{bmatrix} D_1-D_2 & M_2 & 0 \\ N_2 & 0 & C_2 \\ N_1 & 0 & 0 \\ 0 & -B_2 & A_2 \end{bmatrix} = r(N_1)+r(B_2)+r(C_2).$$

which establishes (3.45).
(d) Immediate result from (b) and (c).

Next, we put $D_i=0$, $M_i=I_{n_i}$, $N_i=I_{p_i}$, $i=1,2.$, in Theorem (13) we achieve at the following well known results obtained in [24].

Corollary 8 *Let $A_1\in\mathbb{C}^{l\times n}$, $B_1\in\mathbb{C}^{l\times n_1}$, $C_1\in\mathbb{C}^{p_1\times n}$, $A_2\in\mathbb{C}^{l\times n}$, $B_2\in\mathbb{C}^{l\times n_2}$, $C_2\in\mathbb{C}^{p_2\times n}$ be given, $X_1\in\mathbb{C}^{n_1\times p_1}$, $X_2\in\mathbb{C}^{n_2\times p_2}$ be unknown matrices. We suppose that, the two linear matrix equations*

$$B_1X_1C_1=A_1 \text{ and } B_2X_2C_2=A_2$$

are solvable for X_1 and X_2 respectively, and we put

$$\mathcal{T}_3=\{X_1 \;/\; A_1=B_1X_1C_1\} \quad \text{and} \quad \mathcal{T}_4=\{X_2 \;/\; A_2=B_2X_2C_2\}.$$

Then, the following hold
(a) The two consistent equations $B_1X_1C_1=A_1$ and $B_2X_2C_2=A_2$ have a common solution if and only if

$$r\begin{bmatrix} 0 & C_1 & C_2 \\ B_1 & -A_1 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + r\begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

(b) $\mathcal{T}_3\subseteq\mathcal{T}_4$, that is all the solution of $A_1=B_1X_1C_1$ are solutions of $A_2=B_2X_2C_2$ if and only if

$$r\begin{bmatrix} 0 & C_1 & C_2 \\ B_1 & -A_1 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} = r(B_1)+r(C_1).$$

or equivalently

$$r\begin{bmatrix} 0 & C_1 & C_2 \\ B_1 & -A_1 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + r\begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{ and } \mathfrak{R}(B_2^*) \subseteq \mathfrak{R}(B_1^*) \text{ and } \mathfrak{R}(C_2) \subseteq \mathfrak{R}(C_1).$$

(c) $\mathcal{T}_3 = \mathcal{T}_4$ *if and only if*

$$r\begin{bmatrix} 0 & C_1 & C_2 \\ B_1 & -A_1 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} = r(B_1) + r(C_1) = r(B_2) + r(C_2).$$

or equivalently

$$r\begin{bmatrix} 0 & C_1 & C_2 \\ B_1 & -A_1 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + r\begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{ and } \mathfrak{R}(B_2^*) = \mathfrak{R}(B_1^*) \text{ and } \mathfrak{R}(C_2) = \mathfrak{R}(C_1).$$

Next, pre- and post-multiplying two matrices $P \in \mathbb{C}^{s\times l}$ and $Q \in \mathbb{C}^{n\times t}$ respectively on both sides of the matrix equation $BXC = A$ yields a transformed equation as follows

$$PBXCQ = PAQ$$

Hence, we achieve at the following results

Corollary 9 *Let* $A \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{l\times n}$, $C \in \mathbb{C}^{p\times n}$, *be given and* $X \in \mathbb{C}^{n\times p}$ *be unknown matrices. We suppose that, the linear matrix equation*

$$BXC = A \tag{3.67}$$

is consistent for X, and we put

$$\mathcal{T}_3 = \{MXN \ / \ A = BXC\} \ \text{ and } \ \mathcal{T}_4 = \{MXN \ / \ PAQ = PBXCQ\}.$$

where $M \in \mathbb{C}^{k\times n}$, $N \in \mathbb{C}^{p\times n}$ *are known matrices. Then, the following statements hold*
(a) $\mathcal{T}_3 \subseteq \mathcal{T}_4$ *is always holds*
(b) $\mathcal{T}_3 = \mathcal{T}_4$ *if and only if*

$$r\begin{bmatrix} M \\ PB \end{bmatrix} + r\begin{bmatrix} N & CQ \end{bmatrix} = r\begin{bmatrix} M \\ B \end{bmatrix} + r\begin{bmatrix} N & C \end{bmatrix} + r(CQ) + r(PB) - r(B) - r(C).$$

Proof *Immediate results from Theorem 13.*

Setting $M = I_n$ and $N = I_p$ in Corollary 9 yields the following:

Corollary 10 *Let A, B, C be as given in corollary (9), and assume that the equation* $BXC = A$ *is consistent, we denote*

$$\mathcal{T}_3 = \{X \ / \ A = BXC\} \ \text{ and } \ \mathcal{T}_4 = \{X \ / \ PAQ = PBXCQ\}. \tag{3.68}$$

Then, the following statements hold
(a) $\mathcal{T}_3 \subseteq \mathcal{T}_4$ *is always holds*
(b) $\mathcal{T}_3 = \mathcal{T}_4$ *if and only if*

$$r(CQ) = r(C) \text{ and } r(PB) = r(B)$$

Remark 2 In particular, if we put $P = B^*$ and $Q = C^*$ in (3.68) we get that $\mathcal{T}_4$ is the set of solutions of the normal equation $B^*AC^* = B^*BXCC^*$ of Eq (3.67), therefore $\mathcal{T}_3 = \mathcal{T}_4$ is always holds.

Establishing the connections between the generalized inverses of two matrices is one of the most significant research areas in the field of generalized inverses. As an illustration in this context, using Theorem 13 in the following result, we provide relationships between generalized inverses of two matrices.

Setting $A_i = B_i = C_i$, $i = 1, 2$, in Corollary 8 we get the following result

Corollary 11 *Let A_1, $A_2 \in \mathbb{C}^{l\times n}$ be given, X_1, $X_2 \in \mathbb{C}^{n\times l}$ be unknown matrices. we put*

$$\mathcal{T}_3 = \{X_1 \ / \ A_1 = A_1X_1A_1\} \ \text{ and } \ \mathcal{T}_4 = \{X_2 \ / \ A_2 = A_2X_2A_2\}$$

Then the following hold
(a) $\mathcal{T}_3 \cap \mathcal{T}_4 \neq \varnothing$, that is the two matrices A_1 and A_2 have a common generalized inverse if and only if

$$r\begin{bmatrix} 0 & A_1 & A_2 \\ A_1 & -A_1 & 0 \\ A_2 & 0 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\begin{bmatrix} A_1 & A_2 \end{bmatrix}.$$

(b) $\mathcal{T}_3 \subseteq \mathcal{T}_4$, that is all the generalized inverses of the matrix A_1 are generalized inverse of the matrix A_2 if and only if

$$r\begin{bmatrix} 0 & A_1 & A_2 \\ A_1 & -A_1 & 0 \\ A_2 & 0 & A_2 \end{bmatrix} = 2r\left(A_1\right).$$

The next corollory establishes more links across two domains of LMFs, we consider the two systems of linear matrix equations

$$\begin{cases} A_1X_1 = C_1 \\ X_1B_1 = D_1 \end{cases}, \quad \begin{cases} A_2X_2 = C_2 \\ X_2B_2 = D_2 \end{cases} \tag{3.69}$$

the general solutions for which can be mentioned as particular explicit linear matrix functions.

Corollary 12 *Let $A_i \in \mathbb{C}^{l\times n}$, $B_i \in \mathbb{C}^{p\times s}$, $C_i \in \mathbb{C}^{l\times p}$, $D_i \in \mathbb{C}^{n\times s}$ be given, $X_i \in \mathbb{C}^{n\times p}$ be unknown matrices, $i = 1, 2$. We suppose that, the two linear systems in (3.69) are solvable for X_1 and X_2 respectively, and we put*

$$\mathcal{T}_3 = \{X_1 \ / \ A_1X_1 = C_1,\, X_1B_1 = D_1\}, \tag{3.70}$$
$$\mathcal{T}_4 = \{X_2 \ / \ A_2X_2 = C_2,\, X_2B_2 = D_2\}. \tag{3.71}$$

(a) $\mathcal{T}_3 \cap \mathcal{T}_4 \neq \varnothing$, that is the two systems in (3.69) have a common solutions if and only if the next four conditions hold

$$r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \ r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$
$$A_1D_2 = C_1B_2, \text{ and } A_2D_1 = C_2B_1.$$

or equivalently

$$\mathfrak{R}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \ \mathfrak{R}\begin{bmatrix} D_1^* & D_2^* \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} B_1^* & B_2^* \end{bmatrix},$$
$$A_1 D_2 = C_1 B_2, \text{ and } A_2 D_1 = C_2 B_1.$$

(b) $\mathcal{T}_3 \subseteq \mathcal{T}_4$*, that is all the solutions of the system* $A_1X_1 = C_1$*,* $X_1B_1 = D_1$ *are solutions of the system* $A_2X_2 = C_2$*,* $X_2B_2 = D_2$ *if and only if the two next conditions hold*

$$r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r(A_1), \ r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = r(B_1).$$

(c) $\mathcal{T}_3 = \mathcal{T}_4$*, that is there are identical solutions in both systems in (3.69) if and only if the two following conditions hold*

$$r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r(A_1) = r(A_2) \text{ and } r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = r(B_1) = r(B_2).$$

or equivalently

$$\mathfrak{R}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \text{ and } \mathfrak{R}(A_1) = \mathfrak{R}(A_2),$$
$$\mathfrak{R}\begin{bmatrix} D_1^* & D_2^* \end{bmatrix} \subseteq \mathfrak{R}\begin{bmatrix} B_1^* & B_2^* \end{bmatrix} \text{ and } \mathfrak{R}(B_1^*) = \mathfrak{R}(B_2^*).$$

Proof *(a) By Lemma 9, the general solutions of systems in (3.69) are given respectively by:*

$$X_1 = A_1^+C_1 + F_{A_1}D_1B_1^+ + F_{A_1}V_1E_{B_1},$$

$$X_2 = A_2^+C_2 + F_{A_2}D_2B_2^+ + F_{A_2}V_2E_{B_2},$$

where V_1*,* V_2 *are arbitrary.*
So the two sets in (3.70) and (3.71) can be represented as :

$$\mathcal{T}_3 = \left\{A_1^+C_1 + F_{A_1}D_1B_1^+ + F_{A_1}V_1E_{B_1}\right\},$$
$$\mathcal{T}_4 = \left\{A_2^+C_2 + F_{A_2}D_2B_2^+ + F_{A_2}V_2E_{B_2}\right\}.$$

By Lemma 12 , $\mathcal{T}_3 \cap \mathcal{T}_4 \neq \varnothing$ *if and only if all the following rank equalities hold*

$$r\left[\left(A_1^+C_1 + F_{A_1}D_1B_1^+\right) - \left(A_2^+C_2 + F_{A_2}D_2B_2^+\right) \quad F_{A_1} \quad F_{A_2}\right] = r\left[F_{A_1} \quad F_{A_2}\right], \tag{3.72}$$
$$r\begin{bmatrix} \left(A_1^+C_1 + F_{A_1}D_1B_1^+\right) - \left(A_2^+C_2 + F_{A_2}D_2B_2^+\right) & F_{A_1} \\ E_{B_2} & 0 \end{bmatrix} = r\left(F_{A_1}\right) + r\left(E_{B_2}\right), \tag{3.73}$$
$$r\begin{bmatrix} \left(A_1^+C_1 + F_{A_1}D_1B_1^+\right) - \left(A_2^+C_2 + F_{A_2}D_2B_2^+\right) & F_{A_2} \\ E_{B_1} & 0 \end{bmatrix} = r\left(F_{A_2}\right) + r\left(E_{B_1}\right), \tag{3.74}$$
$$r\begin{bmatrix} \left(A_1^+C_1 + F_{A_1}D_1B_1^+\right) - \left(A_2^+C_2 + F_{A_2}D_2B_2^+\right) \\ E_{B_1} \\ E_{B_2} \end{bmatrix} = \begin{bmatrix} E_{B_1} \\ E_{B_2} \end{bmatrix}. \tag{3.75}$$

We need to simplify these rank equalities by Lemma 1, and three types of EBMOs and simplifying by $A_iA_i^+C_i = C_i$, $D_iB_i^+B_i = D_i$ and $A_iD_i = C_iB_i$, $i = 1,2$, the rank equalities in (3.72)-(3.75) are equivalent respectively to

$$\begin{gathered} r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \\ A_1D_2 = C_1B_2, \\ A_2D_1 = C_2B_1, \\ r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}. \end{gathered} \tag{3.76}$$

From (3.76) we prove (a).
(b) By Lemma 12, $\mathcal{T}_3 \subseteq \mathcal{T}_4$ if and only if the following two rank equalities hold

$$r\begin{bmatrix} (A_1^+C_1 + F_{A_1}D_1B_1^+) - (A_2^+C_2 + F_{A_2}D_2B_2^+) & F_{A_1} & F_{A_2} \end{bmatrix} = r\left(F_{A_2}\right), \tag{3.77}$$

$$r\begin{bmatrix} (A_1^+C_1 + F_{A_1}D_1B_1^+) - (A_2^+C_2 + F_{A_2}D_2B_2^+) \\ E_{B_1} \\ E_{B_2} \end{bmatrix} = r\left(E_{B_2}\right), \tag{3.78}$$

Thus Eqs (3.77) and (3.78) are equivalent respectively to

$$r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\left(A_1\right), \quad r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = r\left(B_1\right), \tag{3.79}$$

then (b) is proved.
(c) Similarly, $\mathcal{T}_3 \supseteq \mathcal{T}_4$ if and only if

$$r\begin{bmatrix} C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\left(A_2\right), \quad r\begin{bmatrix} D_1 & D_2 \\ B_1 & B_2 \end{bmatrix} = r\left(B_2\right), \tag{3.80}$$

Combining (3.79) with (3.80) we prove (c).

Conclusion

In this work, our objective is based on some fundamental questions associated to the algebraic connections such as the inclusion and the intersection between two domains of linear matrix functions, as well as specific types of linear matrix equations , the general solutions to which can be expressed via particular explicit linear matrix functions to establish some connections between its domains by using of some basic tools. Among them are matrix rank and the matrix range method. The rank of a matrix is one of the most basic quantities and useful methods that are widely used in linear algebra; specifically, in matrix theory and its applications. Thus, they show that a variety of matrix equality and matrix set inclusion topics may be solved with the help of the matrix rank and range method.

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جامعة غرداية

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شهادة الترخيص بالإيداع

أنا الأستاذ: قرارة سهام

بصفتي مشرف والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـ

Relationship between some Linear Matrix Functions

من انجاز الطالب(ة): عليهم سعاد

الكلية: العلوم والتكنولوجيا.
القسم: الرياضيات والاعلام الالي.
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أشهد ان الطالبة قد قامت بالتعديلات والتصحيحات المطلوبة من طرف لجنة المناقشة وان المطابقة بين النسخة الورقية والالكترونية استوفت جميع شروطها.

امضاء المسؤول عن التصحيح

مصادقة رئيس القسم

نائب عميد مكلف بالدراسات والمسائل المرتبطة بالطلبة
كلية العلوم والتكنولوجيا
عريف محمد