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Zita Souad

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Global Asymptotic Behaviour of Lotka-Volterra competition systems with diffusion and time delays

in front of the jury:

Président :	Kina Abdelkrim	(M.A.A Univ. Ghardaia).
Framer :	Bahhedi Bahia	(M.A.B Univ. Ghardaia).
Examinateur :	Hadj Moussa Yacine	(M.A.A Univ. Ghardaia).

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ZITA Souad

Dédication

Dears to my heart

" **My parents**"

I thank all of my family because reconized my success is thanks to these invocations

My grandmother:Fatna

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Contents

Introduction

The Lotka-Volterra competition systems are mathematical models describing the evolution of the density of population (or the number of individuals) of multiple living species, competing with one another for the life resources. In this thesis we present the work of C.V. Pao [3] on the asymptotic behaviour of such populations in the long run. The Lotka-Volterra model of N competing species is given in the form

$$\begin{cases} \partial_t u_i(t,x) - L_i u_i(t,x) = & a_i u_i(t,x)\left(1 - u_i(t,x) - \sum_{j\neq i}^{N} b_{ij} u_j(t,x) - \sum_{j=1}^{N} c_{ij} u_j(t-\tau_j, x)\right), \\ & \qquad t>0,\ x\in\Omega \\ \frac{\partial u_i}{\partial \nu}(t,x) = 0, & \qquad t>0, \quad x\in\partial\Omega \\ u_i(t,x) = \eta_i(t,x), & \qquad -\tau_i \leqslant t \leqslant 0, x\in\Omega, \end{cases} \tag{1}$$

Ω represents the enviroment (in $\mathbb{R}, \mathbb{R}^2$, or $\mathbb{R}^3$) inside which occures all the interactions between populations, $u_i(t,x)$ is the density of population i at time $t \geq 0$ and in the position $x \in \overline{\Omega}$. The parameters are nonnegative constants where $a_i \neq 0$ is the self-growth rate of population i ; b_{ij} is relative rate of the effect of populations j on population i and c_{ij} is same as b_{ij} execept that the effect between populations is *delayed* with a delay of τ_j, and both are called competition rates. $\partial u_i/\partial\nu(t,x) = 0$ stats that no flux of all populations occures across $\partial\Omega$ the boundary of Ω.

$$L_i = D_i(x)\Delta + \sigma(x)\cdot\nabla$$

is a diffusion-convection operator ($D_i(x) > 0$) introduced to take into consideration the dispersion effect if exists for some or all populations, othewise L_i is allowed to be zero if the population shows no diffusion, hence the model is a coupled ordinary and parabolic system .

As mentioned earlier the aime is to study the asymptotic behaviour of the solution of (1) more precisely, in [3] the interst is given to the investigation of the conditions on the competition rates (b_{ij} and c_{ij}) underwhich the system has constant (independent of x)

asymptotic behaviour. The work is devides into three chapter, the first is preliminary, where all the necessary tools are set up such as elliptic maximum principle and results on semilinear parabolic systems. The second chapter is dedicated to steady state systems corresponding to time depending problems in the form

$$\begin{cases} \partial_t u_i - L_i u_i = u_i f_i(u, u_\tau), (t > 0; x \in \Omega), & \\ \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial\Omega \\ u_i(t,x) = \eta_i(t,x), & (-\tau_i \leqslant t \leqslant 0, x \in \Omega), \end{cases}$$

where $u_\tau(t,x) = (u_1(t_{\tau_1}, x), \cdots, u_N(t_{\tau_N}, x))$ In this case the steady state problem is

$$\begin{cases} -L_i u_i = u_i f_i(u,u), & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases}$$

the existence of constant quasisolutions and solutions is studied using upper and lower solutions method [3,4], pairs of quasisolutions (to be defined later) are important since they constitute attarcting sectors of solutions of the corresponding time depending problem as t tends to $+\infty$ for suitable set for initial functions η_i [4] liying between upper and lower solutions. Finaly the third chapter is devoted to the study of possible constant asymptotic behaviour of the solutions of (1) under conditions given on the competition rates only. Asymptotic behaviour is said to be *global* if it is proved to be the limit of $u(t,x)$, as t tends to infinity, for all nonnegative, non identically zero initial functions η_i.

Chapter 1

Preliminary

1.1 General notations

Here we present the notation used in this work, we have dialed the defferent lemmas and notes, ans theorems and the formulas in each chapter in sequential manner.

Notations

- $\mathbb{R}$: Set of real numbers.
- $\mathbb{N}$: Set of naturel numbers
- $\mathbb{R}^n$:is the set of all ordered n-tuples of real numbers,
  $\mathbb{R}^n$=$\{(x_1, x_2, ...x_n), x \in \mathbb{R}\}$
- $C^\alpha(\Omega)$: Set of Holder functions of exponent $\alpha \in [0, 1]$ in Ω.
- $\partial\bar{\Omega}$: The boundary of $\bar{\Omega}$
- $\bar{\Omega}$:The closure of Ω
- $\dfrac{\partial}{\partial\nu}$: The outward normal derivative on $\partial\Omega$.
- $(.)^T$:The transpose of a row vector .
- $[w]_i$: The $(N-1)$ vector obtained from $w \in \mathbb{R}^N$ by deleting the component w_i, that is $[w]_i = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N)$

1.2 Elliptic boundary value problems

We consider the linear elliptic boundary value problem

$$\begin{cases} -Lu + c(x)u = q(x) & \text{in } \Omega \\ Bu = h(x) & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Where L_i and B_i are given in the form

$$\begin{aligned} Lu &= \sum_{i,j=1}^{n} a_{i,j}(x)\partial^2 u/\partial x_i \partial x_j + \sum_{j=1}^{n} b_j(x)\partial u/\partial x_j \\ Bu &:= \alpha_0(x)\partial u/\partial \nu + \beta_0(x)u \end{aligned} \tag{1.2}$$

The operator L is uniformuly elliptic in $\bar{\Omega}$, this means that the matrix $(a_{ij}(x))$ is symmetric positive definite in $\bar{\Omega}$,. Its coeffcients and c are in $C^{\alpha}(\bar{\Omega})$.
Some well known examples of elliptic operators are the Laplacian $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, and the diffusion-convection operator $L := D(x)\Delta + \sigma(x) \cdot \nabla$ with $D(x) > c > 0$ for all $x \in \Omega$ and $\sigma : \Omega \to \mathbb{R}^n$ a function

1.2.1 Maximum principles

Theorem 1.2.1. *[8] Let u satisfy the differential inequality*

$$(L + h)[u] \geq 0 \tag{1.3}$$

with $h(x) \leq 0$,with L is uniformly elliptic in D,and with the coeffecients of L and h bounded. if u attains a nonnegative maximum M at an interior point of D, then $u = M$.

Theorem 1.2.2. *[8] Let u satisfy the differential inequality*

$$(L + h)[u] \geq 0$$

where L is the operator given in (1.2) ,and $h(x) \leq 0$ in D, where D is an connected set, suppose that $u \leq M$ in D, that $u = M$ at a boundary point P,and that $M \geq 0$.Assume that P lies on the boundary of a ball in D.If u is continuos in $D \cup P$, any outward directional derivative of u at P is positive unless $u = M$ in D.

1.3 Semilinear parabolic problem

We consider the semilinear parabolic problem

$$\begin{cases} \partial u_i/\partial t - L_i u_i = f_i(t,x,u(t,x),u_\tau(t,x)) \text{ in } D_T, i=1,...,n \\ B_i u_i := \alpha_i \partial u_i/\partial \nu + \beta_i(t,x)u_i = h_i(t,x) \text{ on } S_T \\ u_i(t,x) = \eta_i(t,x) \text{ in } J_i \times \Omega, \end{cases} \tag{1.4}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), ..., u_n(t,x)), u_\tau(t,x) = (u_1(t-\tau_1,x), ..., u_n(t-\tau_n,x))$,$\tau_i$ is the finite time delays of the density functions and for each i ,L_i is a uniformuly elliptic operator given in the form

$$L_i u = \sum_{j,k=1}^{n} a_{j,k}^{(i)}(x)\partial^2 u_i/\partial x_j \partial x_k + \sum_{j=1}^{n} b_j^{(i)}(x)\partial u_i/\partial x_j. \tag{1.5}$$

with Ω is a bounded domain of $\mathbb{R}^n$, and D_T, S_T, Q_t, J_i are given by

$$\begin{aligned} &D_T = (0,T] \times \Omega \quad S_T = (0,T] \times \partial\Omega \quad \bar{D}_T = [0,T] \times \bar{\Omega} \\ &J_i = [-r_i, 0] \quad Q_t^{(i)} = [-r_i, T] \times \bar{\Omega}, \quad Q_t = Q_t^{(1)} \times ... \times Q_t^{(n)}. \end{aligned} \tag{1.6}$$

For any $u = (u_i, [u]_{a_i}, [u]_{b_i})$ and $v = ([v]_{c_i}, [v]_{d_i})$ in a subset Λ of $\mathcal{C}(Q_t)$,a_i, b_i, c_i and d_i are nonegative integers with

$$a_i + b_i = n-1 \quad c_i + d_i = n \quad i = 1, ..., n, \tag{1.7}$$

such that for any $u = (u_i, [u]_{a_i}, [u]_{b_i})$ and $v = ([v]_{c_i}, [v]_{d_i})$ in Λ, $f_i(u,v)$ is nondecreasing in $[u]_{a_i}, [v]_{c_i}$, and is nonincreasing in $[u]_{b_i}, [v]_{d_i}$. In this case f_i is said to be mixed quasimonotone

1.3.1 Upper and lower solutions

Definition 1.3.1. *a pair of functions $\widetilde{u} = (\widetilde{u}_1, ..., \widetilde{u}_N)$, $\widehat{u} = (\widehat{u}_1, ..., \widehat{u}_N)$ are called coupled upper-lower solutions of (1.4), if $\widetilde{u} \geq \widehat{u}$ on $\bar{D}_T$ and if $\widetilde{u}_i$ and $\widehat{u}_i$ satisfy the differential inequalities for each i=1,...,N.*

$$\begin{cases} \partial\widetilde{u}_i/\partial t - L_i\widetilde{u}_i \geq f_i(t,x,\widetilde{u}_i,[\widetilde{u}]_{a_i},[\widehat{u}]_{b_i},[\widetilde{u}_\tau]_{c_i}\,[\widehat{u}_\tau]_{d_i}), \\ \partial\widehat{u}_i/\partial t - L_i\widehat{u}_i \leq f_i(t,x,\widehat{u}_i,[\widehat{u}]_{a_i},[\widetilde{u}]_{b_i},[\widehat{u}_\tau]_{c_i},[\widetilde{u}_\tau]_{d_i}) & (t,x \in D_T), \\ \alpha_i\partial\widetilde{u}_i/\partial\nu + \beta_i\widetilde{u}_i \geq 0 \geq \alpha_i\partial\widehat{u}_i/\partial\nu + \beta_i\widehat{u}_i, & (t,x \in S_T), \\ \widetilde{u}_i(t,x) \geq \eta_i(t,x) \geq \widehat{u}_i(t,x), & (t,x \ in \ J_i \times \Omega). \end{cases} \tag{1.8}$$

The exitence of pairs of upper and lower solution is powerful tool in studying semilinear parabolic (and elliptic) equations we can find various applications of upper and lower solutions method in, [4, 2, 3] and others.

1.3.2 Existance and uniquness solutions

One of the most important results of upper and lower solution methods is the folowing thoerem, which ensurs the existence and uniqueness of the solution of the parabolique system (1.4) having coupled upper and lower solutions, with mixed quasimonotone nonlinearities f_i

Theorem 1.3.1. *[2]*
Let $\widetilde{u}, \widehat{u}$ *be a pair of coupled upper and lower solution of (1.4) and Let* $f = (f_1, f_2, ..., f_n)$ *be mixed quasimonotone in* $<\widehat{u}, \widetilde{u}> = \{u \in \mathcal{C}(Q_t); \widehat{u} \leq u \leq \widetilde{u} \text{ in } Q_t\}$ *and are lipschizs . Then the system (1.4) has unique solution* $u = (u_1, ..., u_n)$ *in* $< \widehat{u}, \widetilde{u} >$ *.*

Chapter 2

Constant solutions and quasisolutions of a semilinear steady state problem

In this chapter we present the steady state problem:

$$\begin{cases} -L_i u_i = u_i f_i(u,u) & (\text{x } \in \Omega), \\ \partial u_i/\partial \nu = 0, & (\text{x } \in \partial\Omega), i=1,...,N, \end{cases} \tag{2.1}$$

corresponding to the time dependent problem

$$\begin{cases} \partial u_i/\partial t - L_i u_i = u_i f_i(u,u_\tau), (t>0; x\in\Omega), \\ \partial u_i/\partial \nu = 0, \\ u_i(t,x) = \eta_i(t,x), (-\tau_i \leqslant t \leqslant 0, x\in\Omega), \qquad i=1,...,N, \end{cases} \tag{2.2}$$

Assume that $f_i(u,v)$ is a C^1 function of u,v for u,v in a suitable subset ρ of $\mathbb{R}^N$, For each i=1,...,N, L_i is a diffusion-convection operator given by

$$L_i u_i = D_i(x)\triangle u_i + \sigma_i(x).\nabla u_i,$$

such that $D_i(x), \sigma_i(x), \eta_i(t,x)$ are C^α functions in thier domains $D_i(x)>0$ on $\bar{\Omega}$, $\eta_i(t,x)\geq 0$ on $I_i\times\Omega$ since it represents a density of population. The notation $\sigma_i(x)\cdot\nabla u_i$ is scalar product in $\mathbb{R}^n$. In the above system we allow $L_i=0$ this means that $D_i(x)=\sigma_i(x)=0$. The main assumption on $f_i(u,v)$ is the existance of a pair of constant vectors $\widetilde{c}=(\widetilde{c}_1,...,\widetilde{c}_N), \widehat{c}=(\widehat{c}_1,...,\widehat{c}_N)$ such that $\widetilde{c}\geqslant\widehat{c}\geqslant 0$ and

$$f_i(\widetilde{c}_i,[\widehat{c}]_i,\widehat{c}) \leqslant 0 \leqslant f_i(\widehat{c},[\widetilde{c}]_i,\widetilde{c}), \quad i=1,...,N. \tag{2.3}$$

And, as suggested in [3], the system is competitve in the sense that

$$\begin{cases} \dfrac{\partial f_i}{\partial u_j} \leqslant 0 & \text{for } j \neq i, \\ \dfrac{\partial f_i}{\partial v_j} \leqslant 0 & \text{for all } j, \\ \text{for all } i, j \text{ in } \mathbb{N} \text{ and } u, v \text{ in } \rho. \end{cases} \tag{2.4}$$

Remark 2.0.1. ** The time depending problem (2.2) justifies the use of the notation $f_i(u,u)$ instead of $f_i(u)$ simply. Indeed, if $\lim_{t\longrightarrow+\infty} u(t,x) =: u(x)$ exists then $\lim_{t\longrightarrow+\infty} u(t_\tau, x) = \lim_{t\longrightarrow+\infty} u(t,x) =: u(x)$ and*

$$\lim_{t\longrightarrow+\infty} f_i(u, u_\tau) = f_i(u,u)$$

** The notation $(u_i, [u]_i)$ constitute the same vector u in $\mathbb{R}^N$, hence for example in (2.3) $f_i(\widetilde{c}_i, [\widehat{c}_i], \widehat{c}) := f_i((\widetilde{c}_i, [\widehat{c}_i]), \widehat{c})$, that is $u = (\widetilde{c}_i, [\widehat{c}_i])$ and $v = \widehat{c}$ in $f_i(u,v)$*
** By taking (2.4) into consideration one can find that a_i, b_i, c_i and d_i in (1.7) are as follows $a_i = 0$, $b_i = n-1$, $c_i = 0$ and $d_i = n$.*

2.1 Upper and Lower solutions method

We start by giving a definition of pair of upper and lower solutions

Definition 2.1.1. *[2]*

A pair of functions $(\widetilde{u}, \widehat{u})$ in $C(\Omega)\cap C^2(\Omega)$ are called ordered upper and lower solutions of

$$\begin{cases} -L_i u_i = f_i(x, u_i, [u]_{a_i}, [u]_{b_i}, [u]_{c_i}, [u]_{d_i}) & \text{in } \Omega \\ B_i u_i = h_i(x) & \text{on } \partial\Omega, i = 1,2,...N \end{cases} \tag{2.5}$$

if they satisfy the relation $\widetilde{u} \geq \widehat{u}$ and if

$$\begin{aligned} &-L_i\widetilde{u}_i \geq f_i(x, \widetilde{u}_i, [\widetilde{u}]_{a_i}, [\widehat{u}]_{b_i}, [\widetilde{u}]_{c_i}, [\widehat{u}]_{d_i}) && \text{in } \Omega \\ &-L_i\widehat{u}_i \leq f_i(x, \widehat{u}_i, [\widehat{u}]_{a_i}, [\widetilde{u}]_{b_i}, [\widehat{u}]_{c_i}, [\widetilde{u}]_{d_i}) && \text{in } \Omega \\ &B_i\widetilde{u}_i \geq h_i \geq B_i\widehat{u}_i && \text{on } \partial\Omega, \quad i = 1,2,...N \end{aligned}$$

Applying this definition to (2.1) shows that $(\widetilde{u}, \widehat{u})$ in $C(\Omega)\cap C^2(\Omega)$ is a pair of ordered upper and lower solutions if $\widetilde{u} \geq \widehat{u}$ and if

$$\begin{aligned} &-L_i\widetilde{u}_i \geq \widetilde{u}_i f_i(, \widetilde{u}_i, [\widehat{u}]_i, \widehat{u}) && \text{in } \Omega \\ &-L_i\widehat{u}_i \leq \widehat{u}_i f_i(\widehat{u}_i, [\widetilde{u}]_i, \widetilde{u}) && \text{in } \Omega \\ &\partial\widetilde{u}_i/\partial\nu \geq 0 \geq \partial\widehat{u}_i/\partial\nu && \text{on } \partial\Omega, \quad i = 1,2,...N \end{aligned} \tag{2.6}$$

$\widetilde{c}$ and $\widehat{c}$ are constant, so $L_i\widetilde{c}_i = L_i\widehat{c}_i = 0$ and $\partial\widetilde{c}_i/\partial\nu = \partial\widehat{c}_i/\partial\nu = 0$ which implies by assumption (2.3) that $(\widetilde{c}, \widehat{c})$ is a pair of ordred upper and lower solutions.

Theorem 2.1.1. *Under hypothesis* (2.3) $(\widetilde{c}, \widehat{c})$ *is a pair of ordred upper and lower solutions of* (2.1) .

Let $\rho := \{c \in \mathbb{R}^N | \widehat{c} \leq c \leq \widetilde{c})\}$. Function f is C^1 in ρ then there exists constant $K_i > 0$ such that

$$K_i \geq \max\left\{-\frac{\partial(u_i f_i(u,v)}{\partial u_i}; u, v \in \rho\right\}. \tag{2.7}$$

this implies that $u_i \to K_i u_i + u_i f_i(u_i, [u]_i, v)$ is nondecreasing Define also tow iterative sequences $\{\bar{u}^{(k)}\} = \{\bar{u}_1^{(k)}, ..., \bar{u}_N^{(k)}\}$ and $\{\underline{u}^{(k)}\} = \{\underline{u}_1^{(k)}, ..., \underline{u}_N^{(k)}\}$ by

$$\begin{cases} -L_i\bar{u}^{(k)} + K_i\bar{u}^{(k)} = K_i\bar{u}^{(k-1)} + \bar{u}^{(k-1)} f_i(\bar{u}_i^{(k-1)}, [\underline{u}^{(k-1)}]_i, \underline{u}^{(k-1)}) \\ -L_i\underline{u}^{(k)} + K_i\underline{u}^{(k)} = K_i\underline{u}^{(k-1)} + \underline{u}^{(k-1)} f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k-1)}]_i, \bar{u}^{(k-1)}) \\ \frac{\partial}{\partial\nu}\bar{u}_i^{(k)} = \frac{\partial}{\partial\nu}\underline{u}_i^{(k)} = 0, (i = 1, ..., N) \end{cases} \tag{2.8}$$

with $\bar{u}^{(0)} = \widetilde{c}$ and $\underline{u}^{(0)} = \widehat{c}$ we have the following theorem

Theorem 2.1.2. *[4, 3] Assume that f satisfies hypothesis* (2.3) *and* (2.4), *then the two sequneces $\bar{u}_i^{(k)} = (\bar{u}_1^k, ..., \bar{u}_N^k)$ and $\underline{u}_i^{(k)} = (\underline{u}_1^{(k)}, ...\underline{u}_N^{(k)})$ defined by* (2.8) *with $\bar{u}_i^{(0)} = (\widehat{c}_1, ..., \widehat{c}_N)$ and $\underline{u}_i^{(0)} = (\widetilde{c}_1, ..., \widetilde{c}_N)$ are constant, given by*

$$\begin{cases} \bar{u}^{(k)} = \bar{u}^{(k-1)} + (\bar{u}^{(k-1)}/K_i) f_i(\bar{u}_i^{(k-1)}, [\underline{u}^{(k-1)}]_i, \underline{u}^{(k-1)}), \\ \underline{u}^{(k)} = \underline{u}^{(k-1)} + (\underline{u}^{(k-1)}/K_i) f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k-1)}]_i, \bar{u}^{(k-1)}) \quad i = 1, ..., N \end{cases} \tag{2.9}$$

and they possess the monotone property

$$\widehat{c}_i \leq \underline{u}_i^{(1)} \leq ... \leq \underline{u}_i^{(k)} \leq \underline{u}_i^{(k+1)} \leq \bar{u}_i^{(k+1)} \leq \bar{u}_i^{(k)} \leq ... \leq \bar{u}_i^{(1)} \leq \widetilde{c}_i, \quad k = (0, 1, 2, ...), i = 1, ..., N. \tag{2.10}$$

before introducing the proof of this theorem a result known as positivity lemma [4] is needed to prove the monotone property

Lemma 2.1.1. *[4] Let c be bounded not identically zero functionc $= c(x) \geq 0$ in Ω, if $w \in C^2(\Omega)$ satisfies the relation*

$$\begin{aligned} -Lw + cw \geq 0 \in \Omega \\ \partial w/\partial\nu \geq 0 \qquad on \partial\Omega \end{aligned} \tag{2.11}$$

then $w \geq 0$ in $\bar{\Omega}$, Moreover,$w > 0$ in Ω unless $w = 0$.

Proof 2.1.1. *We first prove that $w \geq 0$ in $\bar{\Omega}$ .Assume by contradution that w has a negative minimum m, at some point $x_0 \in \bar{\Omega}$ then $-w(x_0)$ is a positive maximum of $-w$. If $x_o \in \partial\Omega$ then by maximum principle, Theorem 1.2.2, we have $\partial -w(x_0)/\partial\nu > 0$ unless $w \equiv m$ but $\partial w(x_0)/\partial\nu < 0$ contradicts the boundary inequality in (2.11) we must have $w \equiv m < 0$, then $-Lw + cw = cm \geq 0$ again contradicts the fact that $m < 0$, which shows that $x_0 \in \Omega$. This ensure by maximum principle, Theorem 1.2.1, that w is a constant . But by the hypothesis on c, w can be a constant only when $w = 0$. This leads to contraduction, which shows that $w \geq 0$ in $\bar{\Omega}$.*
We observe that if $w(x_0) = 0$ at some point in Ω then this implies that $w = 0$ throughout Ω. This shows that either $w > 0$ or $w = 0$ in Ω.

Proof 2.1.2 (proof of Theorem 2.1.2)**.** *We have $\bar{u}^{(0)} = \widetilde{c}$ and $\underline{u}^{(0)} = \widehat{c}$ which are constant functions, hence the second hand of equation* (2.8) *for $k = 1$ is constant, this leads $\bar{u}^{(1)}$ and $\underline{u}^{(1)}$ to be constant in both cases of L_i (obviouse when $L_i = 0$ and when L_i is an elliptic operator, $\bar{u}^{(1)}$ and $\underline{u}^{(1)}$ are constant by uniqueness of the solution of linear elliptic boundary value problems. The same argument leads, by induction, to conclude that the tow sequences are constant, this implies that $L_i\underline{u}_i^{(k)} = L_i\bar{u}_i^{(k)} = 0$ whether $L_i = 0$ or not, hence the tow sequences are given by*

$$\begin{cases} \bar{u}^{(k)} = \bar{u}^{(k-1)} + (\bar{u}^{(k-1)}/K_i) f_i(\bar{u}_i^{(k-1)}, [\underline{u}^{(k-1)}]_i, \underline{u}^{(k-1)}), \\ \underline{u}^{(k)} = \underline{u}^{(k-1)} + (\underline{u}^{(k-1)}/K_i) f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k-1)}]_i, \bar{u}^{(k-1)}) \quad i = 1, ..., N \end{cases} \tag{2.12}$$

To show the monotone property, we proceed by induction, let $w_i^k = \underline{u}_i^{(k+1)} - \underline{u}_i^{(k)}$ we have by using the definition of upper and lower solutions in (2.6) *and the iteration process* (2.8)

$$\begin{aligned} -L_i w_i^0 + K_i w_i^0 &= -L_i(\underline{u}_i^{(1)} - \widehat{u}_i) + K_i(\underline{u}_i^{(1)} - \widehat{u}_i) \\ &\geq K_i(\widehat{u}_i - \widehat{u}_i) + \widehat{u}_i \left(f_i(\widehat{u}_i, [\widetilde{u}]_i, \widetilde{u}_i) - f_i(\widehat{u}_i, [\widetilde{u}]_i, \widetilde{u}_i) \right) \\ &\geq 0 \end{aligned}$$

and $\dfrac{\partial}{\partial\nu} w_i^0 = \dfrac{\partial}{\partial\nu}(\underline{u}_i^{(1)} - \widehat{c}_i) = 0$ which proves by positivity lemma that $w_i^0 \geq 0$ that is $\underline{u}_i^{(1)} \geq \widehat{c}_i$. Analogous argument shows that $\bar{u}_i^{(1)} \leq \widetilde{c}_i$

Now assume that $w^{k-1} \geq 0$ *and* $\bar{u}^{(k)} \leq \bar{u}^{(k-1)}$

$$\begin{aligned}
-L_i w_i^k + K_i w_i^k &= K_i w_i^{k-1} + \underline{u}_i^{(k)} f_i(\underline{u}_i^{(k)}, [\bar{u}^{(k)}]_i, \bar{u}^{(k)}) - \underline{u}_i^{(k-1)} f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k-1)}]_i, \bar{u}^{(k-1)}) \\
&\geq K_i w_i^{k-1} + \underline{u}_i^{(k)} f_i(\underline{u}_i^{(k)}, [\bar{u}^{(k)}]_i, \bar{u}^{(k)}) \\
&- \underline{u}_i^{(k-1)} f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k)}]_i, \bar{u}^{(k)}) \quad \textit{(by (2.4) and the induction assumption)} \\
&\geq K_i \underline{u}_i^{(k)} + \underline{u}_i^{(k)} f_i(\underline{u}_i^{(k)}, [\bar{u}^{(k)}]_i, \bar{u}^{(k)}) - K_i \underline{u}_i^{(k-1)} f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k)}]_i, \bar{u}^{(k)}) \\
&\geq 0
\end{aligned}$$

because $u_i \to K_i u_i + u_i f_i(u_i, [u]_i, v)$ *is nondecreasing function and* $\underline{u}_i^{(k)} \geq \underline{u}_i^{(k-1)}$. *So we have* $-L_i w_i^k + K_i w_i^k \geq 0$ *and* $\dfrac{\partial}{\partial \nu} w_i^k \geq 0$ *(in fact,* $\dfrac{\partial}{\partial \nu}(\underline{u}_i^{(k+1)} - \underline{u}_i^{(k)}) = 0$*) by positivity lemma* $w_i^k \geq 0$ *that is* $\underline{u}_i^{(k+1)} \geq \underline{u}_i^{(k)}$ *. Analogously, one obtains* $\bar{u}_i^{(k+1)} \leq \bar{u}_i^{(k)}$

2.2 Existence of solutions and quasisolutions

We start by introducing the notion of pairs of quasisolutions

Definition 2.2.1. *[4]A pair of functions* $(\bar{u} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_N), \underline{u} = (\underline{u}_1, \underline{u}_2, .., \underline{u}_N))$ *is called quasisolutions of*

$$\begin{cases} -L_i u_i = f_i(x, u_i, [u]_{a_i}, [u]_{b_i}, [u]_{c_i}, [u]_{d_i}) & \textit{in } \Omega \\ B_i u_i = h_i(x) & \textit{on } \partial\Omega, i = 1, 2, ..., N \end{cases} \tag{2.13}$$

if they satisfy

$$\begin{cases} -L_i \bar{u}_i = f_i(x, \bar{u}_i, [\bar{u}]_{a_i}, [\underline{u}]_{b_i}, [\bar{u}]_{c_i}, [\underline{u}]_{d_i}), (x \in \Omega), \\ -L_i \underline{u}_i = f_i(x, \underline{u}_i, [\underline{u}]_{a_i}, [\bar{u}]_{b_i}, [\underline{u}]_{c_i}, [\bar{u}]_{d_i}) \\ B_i \bar{u}_i = B_i \underline{u}_i = h_i, \textit{ on } \partial\Omega, \end{cases} \tag{2.14}$$

for each $i = 1, ..., N$

Remark 2.2.1. *As it can be seen from the definition,*

** pairs of quasisolutions are not necessarily solutions of* (2.13)*, but in the particular case when* $a_i = n - 1,\ b_i = 0, c_i = n,$ *and* $d_i = 0,$ *a pair of quasisolutions are both solutions.*

** if* $(\bar{u}, \underline{u})$ *is a pair of quasisolutions and* $(\bar{u} = \underline{u}$ *then their common value is a solution of* (2.13)*.*

** In the case of problem* (2.1)*,* $(\bar{u}, \underline{u})$ *is a pair of quasisolutions if*

$$\begin{cases} -L_i \bar{u}_i = \bar{u}_i f_i(, \bar{u}_i, [\underline{u}]_i, \underline{u}) & \textit{in } \Omega \\ -L_i \underline{u}_i = \underline{u}_i f_i(\underline{u}_i, [\bar{u}]_i, \bar{u}) & \textit{in } \Omega \\ \partial \bar{u}_i / \partial \nu = \partial \underline{u}_i / \partial \nu = 0 \qquad \textit{on } \partial\Omega, \quad i = 1, 2, ...N \end{cases} \tag{2.15}$$

The following theorem shows the existance of constant quasisolutions for the steady state problem (2.1)

Theorem 2.2.1. *Suppose that the function f in* (2.1) *satisfies conditions* (2.4) *and* (2.3) *for* $\widetilde{c} \geqslant \widehat{c} > 0$. *Then,*
problem (2.1) has a pair of constant quasisolutions $\bar{\rho} = (\bar{\rho}_1, ..., \bar{\rho}_N)$, $\underline{\rho} = (\underline{\rho}_1, ..., \underline{\rho}_N)$ *such that*

$$\widetilde{c} \geqslant \bar{\rho} \geqslant \underline{\rho} \geqslant \widehat{c}.$$

In this case we have

$$f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) \quad = f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) \quad = 0 \tag{2.16}$$

Proof 2.2.1. *In view of Theorem 2.1.2, under hypothesis* (2.4) *and* (2.3) *the tow sequences defined by* (2.8) *are bounded monotone in* $\mathbb{R}^N$ *(constant) so they both converge as* k *tends to* $+\infty$, *set*

$$\underline{\rho}_i := \lim_{k \to +\infty} \underline{u}_i^k$$

and

$$\bar{\rho}_i := \lim_{k \to +\infty} \bar{u}_i^k$$

Since f *is continuous, by letting* $k \to +\infty$ *in* (2.10) *and*(2.8) *one obtains*

$$\widetilde{c} \geqslant \bar{\rho} \geqslant \underline{\rho} \geqslant \widehat{c}.$$

and

$$\begin{cases} L_i\bar{\rho}_i = \bar{\rho}_i f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) & \text{in } \Omega, \\ L_i\underline{\rho}_i = \underline{\rho}_i f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}\bar{\rho}_i = \dfrac{\partial}{\partial \nu}\underline{\rho}_i = 0, & \text{on } \partial\Omega, \quad (i = 1, ..., N) \end{cases}$$

which means that $(\bar{\rho} = (\bar{\rho}_1, ..., \bar{\rho}_N), \underline{\rho} = (\underline{\rho}_1, ..., \underline{\rho}_N))$ *is a pair of constant quasisolutions of* (2.1) . $\bar{\rho}$ *and* $\underline{\rho}$ *are constant and satisfy* $\bar{\rho} \geqslant \underline{\rho} \geqslant \widehat{c} > 0$ *then,*

$$f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) \quad = f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) \quad = 0$$

Theorem 2.2.2. *Under the same conditions in Theorem 2.1.1 we find*

(a) every solution u^* *of state problem (2.1) such that* $\hat{c} \leq u^*(x) \leq \widetilde{c}$ *for all* $x \in \bar{\Omega}$ *satisfies the relation* $\bar{\rho} \geqslant u^*(x) \geqslant \underline{\rho}$ *on* $\bar{\Omega}$

(b) if $\bar{\rho} = \underline{\rho} = \rho^*$ *then* ρ^* *is the unique positive solution of state system in* ρ.

Proof 2.2.2. *(a) Let u^* as in the theorem, define first two functions to be used in the proof $\bar{w}^{(k)}(x) = \bar{u}^{(k)} - u^*(x)$ and $\underline{w}^{(k)}(x) = u^*(x) - \underline{u}^{(k)}(x)$, the idea is to prove, by induction, that $\bar{w}^{(k)} \geq 0$ and $\underline{w}^{(k)} \geq 0$. These relations ar obviously true for $k = 0$ since $\hat{c} \leq u^*(x) \leq \tilde{c}$ for all $x \in \bar{\Omega}$. Suppose that $\bar{w}^{(k-1)} \geq 0$ and $\underline{w}^{(k-1)} \geq 0$, we have*

$$\begin{cases} -L_i\bar{w}_i^{(k)} + K_i\bar{w}_i^{(k)} = K_i\bar{w}_i^{(k-1)} + \bar{u}_i^{(k-1)} f_i(\bar{u}_i^{(k-1)}, [\underline{u}^{(k-1)}]_i, \underline{u}^{(k-1)}) - u_i^* f_i(u_i^*, [u^*]_i, u^*), \\ -L_i\underline{w}_i^{(k)} + K_i\underline{w}_i^{(k)} = K_i\underline{w}_i^{(k-1)} + u_i^* f_i(u_i^*, [u^*]_i, u^*) - \underline{u}_i^{(k-1)} f_i(\underline{u}_i^{(k-1)}, [\bar{u}^{(k-1)}]_i, \bar{u}^{(k-1)}), \\ \dfrac{\partial}{\partial \nu}\bar{w}_i^{(k)} = \dfrac{\partial}{\partial \nu}\underline{w}_i^{(k)} = 0, \quad (i = 1, ..., N). \end{cases} \tag{2.17}$$

$\underline{w}^{(k-1)} \geq 0$ and (2.4) imply that $K_i\bar{w}_i^{(k-1)} + \bar{u}_i^{(k-1)} f_i(\bar{u}_i^{(k-1)}, [\underline{u}^{(k-1)}]_i, \underline{u}^{(k-1)}) - u_i^* f_i(u_i^*, [u^*]_i, u^*) \geq K_i\bar{u}_i^{(k-1)} + \bar{u}_i^{(k-1)} f_i(\bar{u}_i^{(k-1)}, [u^*]_i, u^*) - u_i^* f_i(u_i^*, [u^*]_i, u^*)$ this leads, by taking into consideration the fact that the function $u_i \to K_i u_i + u_i f_i(u_i, [u]_i, v)$ is nondecreasing, to :
$K_i\bar{u}_i^{(k-1)} + \bar{u}_i^{(k-1)} f_i(\bar{u}_i^{(k-1)}, [u^*]_i, u^*) - K_i u_i^* - u_i^* f_i(u_i^*, [u^*]_i, u^*) \geq 0$ so we obtain

$$\begin{cases} -L_i\bar{w}_i^{(k)} + K_i\bar{w}_i^{(k)} \geq 0 \quad in\ \Omega \\ \dfrac{\partial}{\partial \nu}\bar{w}_i^{(k)} = 0, on\ \partial\Omega, \end{cases}$$

hence $\bar{w}_i^{(k)} \geq 0$ by positivity lemma. Analogously one finds that $\underline{w}_i^{(k)} \geq 0$ so we have for each $k = 1, 2, ...$ $\bar{u}^{(k)} \geq u^*(x) \geq \underline{u}^{(k)}$ on $\bar{\Omega}$. One can see, by letting $m \to \infty$ in this later relation, that

$$\bar{\rho} \geq u^*(x) \geq \underline{\rho} \tag{2.18}$$

(b) if $\bar{\rho} = \underline{\rho} = \rho^*$ (constant) then

$$f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) = f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) = f_i(\rho_i^*, [\rho^*]_i, \rho^*) = 0$$

that is

$$\begin{cases} L_i\rho_i^* = \rho_i^* f_i(\rho_i^*, [\rho^*]_i, \rho^*) \quad in\ \Omega \\ \dfrac{\partial}{\partial \nu}\rho_i^* = 0, on\ \partial\Omega, \end{cases}$$

this implies that ρ^* is a solution of (2.1), for any solution $u^*(x)$ of state problem, ρ^* is the unique solution of(2.1) because
$\bar{\rho} \geq u^*(x) \geq \underline{\rho}$, and $\underline{\rho} = \bar{\rho} = \rho^*$ so $u^* = \rho^*$.

2.3 Relation with asymptotic behaviour

We end this chapter with a theorem that shows the relation between the quasisolutions $\underline{\rho}$ and $\bar{\rho}$, and the asymptotic behaviour of the solution $u(t, x)$ of the time depending problem (2.2), this theorem will be needed in the next chapter

Theorem 2.3.1. *[3] Suppose that the function f in (2.2) satisfies conditions (2.4) and (2.3) for $\widetilde{c} \geqslant \widehat{c} > 0$. Then for any initial function $\eta_i(t,x) = (\eta_1(t,x), ...\eta_N(t,x))$ with $\widehat{c} \leqslant \eta(t,x) \leqslant \widetilde{c}$, problem (1.4) has a unique positive solution u(t, x) such that $\widehat{c} \leqslant \eta(t,x) \leqslant \widetilde{c}$ and*

$$\underline{\rho} \leqslant u(t,x) \leqslant \bar{\rho} \ as \ t \longrightarrow \infty \quad (x \in \bar{\Omega}), \tag{2.19}$$

where $\bar{\rho}$ and $\underline{\rho}$ are the constant vectors satisfying

$$f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) \quad = f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) \quad = 0$$

Moreover, if $\bar{\rho} = \underline{\rho} = \rho^*$ then

$$\lim_{t \longrightarrow \infty} u(t,x) = \rho^* \quad uniformly \ on \ \bar{\Omega}. \tag{2.20}$$

Chapter 3

Global asympotitic behaviour of competitive diffusive Lotka-Volterra system

3.1 Existence and uniqueness of solutions

We consider the lotka volterra system with N-competing species [3] given in the form

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - L_i u_i = a_i u_i \left(1 - u_i - \displaystyle\sum_{j \neq i}^{N} b_{ij} u_j - \sum_{j=1}^{N} c_{ij} (u_j)_\tau \right), (t > 0, x \in \Omega) \\ \dfrac{\partial u_i}{\partial v} = 0, (t > 0, x \in \partial\Omega) \\ u_i(t,x) = \eta_i(t,x), (-\tau_i \leqslant t \leqslant 0, x \in \Omega), i = 1, ..., N. \end{cases} \tag{3.1}$$

where Ω in $\mathbb{R}^n$ a domain, a_i, b_{ij} and c_{ij} are nonnegative constants for all $i, j = 1, ..., N$ with $a_i \neq 0$ and as in the previous chapter,

$$L_i = D_i(x)\Delta + \sigma(x) \cdot \nabla$$

where $D_i(x)$ and $\sigma_i(x)$ are C^α functions in thier domains $D_i(x) > 0$ on $\bar{\Omega}$. The notation $\sigma_i(x) \cdot \nabla u_i$ stands for scalar product in $\mathbb{R}^n$. In the above system we allow $L_i = 0$ this means that $D_i(x) = \sigma_i(x) = 0$.

The aim is to show that (3.1) has a unique nonnegative solution by using Theorem 1.3.1 since (3.1) is a particular case of (1.4). Theorem 1.3.1 requires the existence of a pair of coupled upper and lower solutions. Indeed, the pair $(\widetilde{u}, \widehat{u})$ with $\widetilde{u} = (C_1, ..., C_N), \widehat{u} =$

$(0,0,...,0)$ where $C_1,...,C_N$ are positive constants satisfing $C_i \geq \max(1, \|\eta\|_\infty)$, is a pair of coupled upper and lower solutions of (3.1), because they satisfy the inequalities in Definition 1.3.1 (in view of (1.7) and by observing the monotonecity with respect to u_j and u_τ, we have $a_i = c_i = 0,\ b_i = n-1$ and $d_i = n$ in Definition 1.3.1.)

$$\begin{cases} \partial\widetilde{u}_i/\partial t - L_i\widetilde{u}_i = 0 \geq f_i(t,x,\widetilde{u}_i,[\widehat{u}]_i,[\widehat{u}]) = a_iC_i(1-C_i), & \\ \partial\widehat{u}_i/\partial t - L_i\widehat{u}_i = 0 \leq f_i(t,x,\widehat{u}_i,[\widetilde{u}]_i,[\widetilde{u}]) = 0 & (t>0, x\in\Omega) \\ \partial\widehat{u}_i/\partial\nu = \partial\widetilde{u}_i/\partial\nu = 0 & (t>0, x\in\partial\Omega) \\ \widetilde{u}_i(t,x) \geq \eta_i(t,x) \geq \widehat{u}_i(t,x), & (-\tau_i \leqslant t \leqslant 0). \end{cases} \tag{3.2}$$

where

$$f_i(t,x,u,v) := a_iu_i\left(1 - u_i - \sum_{j\neq i}^{N} b_{ij}u_j - \sum_{j=1}^{N} c_{ij}v_j\right)$$

if we rewrite equations (3.2) we find

$$f_i(t,x,\widetilde{u}_i,[\widehat{u}]_i,[\widehat{u}]) \leq 0 \leq f_i(t,x,\widehat{u}_i,[\widetilde{u}]_i,[\widetilde{u}]) \tag{3.3}$$

f_i are lipschizs functions because are of the class C^1. by theorem 1.3.1 the system (3.1) has unique solution $u = (u_1,u_2,...,u_n)$ such that

$$0 \leq u_i(t,x) \leq C_i \quad \text{on } [0,\infty)\times\bar{\Omega}, \quad i = 1,...,N.$$

3.2 Asympotitic behaviour of solution

Studying the asymptotic behaviour of the solution of a time depending problem is the investigation of a possible limit of this solution as tends to infinity. This limit, if exists is always solution of the corresponding steady state problem. Note that a given solution of the steady state problem is called equilibrium state and it is not necessarily the asymptotic behaviour. In this section we show that under some conditions on the model parameters, the solution $u(t,x)$ of the Lotka Volterra competitive system (3.1) tends at $t\to+\infty$ to a non trivial constant equilibrium state ρ^*. That implies that ρ^* is a constant solution of the steady state problem :

$$\begin{cases} -L_iu_i = a_iu_i(1-u_i - \sum_{j\neq i}^{N} b_{ij}u_j - \sum_{j=1}^{N} c_{ij}(u_j), \quad (x\in\Omega), \\ \dfrac{\partial u_i}{\partial v} = 0 \text{ on } \partial\Omega, i = 1,...,N. \end{cases} \tag{3.4}$$

We can see that $o = (0,...,0)$ is a trivial solution of this problem. To ensure the existence and uniqueness of a positive steady-state solution we need to impose some conditions on the reaction rates b_{ij} and c_{ij} (but not on a_i). Set $e = (1, 1, ..., 1)^T$ and define two $\mathbb{N} \times \mathbb{N}$ constant matrices A_0 and A_1 by

$$\begin{aligned} A_0 &= (b_{ij} + c_{ij}) \text{ with } \quad b_{ii} = 0, i, j = 1, ..., N, \\ A_1 &= (b_{ij} + c_{ij}) \text{ with } \quad b_{ii} = -1, i, j = 1, ..., N \end{aligned} \tag{3.5}$$

in terms of this two matricies ,we have the following result

Theorem 3.2.1. *Assume that A_1 is nonsingular and there exists a constant vector $M = (M_1, ..., M_N)^T$ such that*

$$M \geqslant e \text{ and } \quad A_0 M < e \tag{3.6}$$

then problem (3.4) has a unique positive constant solution $\rho^* = (\rho_1^*, ..., \rho_N^*)^T$ such that $\rho^* \leqslant M$.
Moreover, for any nonnegative initial function $\eta(t, x)$,with $\eta_i(t, x)$ not identically zero. the corresponding solution $u(t, x)$ of (3.1) possesses the property

$$\lim_{t \to \infty} u(t, x) = \rho^*, \quad x \in \bar{\Omega}. \tag{3.7}$$

Proof 3.2.1. *The idea is to prove that there exist vectors $(\widehat{c}, \widetilde{c})$ satisfying (2.3) where*

$$f_i(u_i, [u]_i, v) := a_i(1 - u_i - \sum_{j \neq i}^{N} b_{ij} u_j - \sum_{j=1}^{N} c_{ij}(u_j)$$

, then by using Theorem 2.2.1 we show the existance of a pair of constant quasisolutions $\bar{\rho}, \underline{\rho}$ which turn out to be equal thanks to the nonsingularity of A_1.
Condition $A_0 M < e$ means that for all $i = 1, \dots, N$

$$\sum_{j \neq i}^{N} b_{ij} M_j + \sum_{j=1}^{N} c_{ij} M_j < 1, \quad i = 1, ..., N, \tag{3.8}$$

It follows that there for all $i = 1, \dots, N$ the exist $\delta_i > 0$ such that

$$\delta_i \leq 1 - \sum_{j \neq i}^{N} b_{ij} M_j - \sum_{j=1}^{N} c_{ij} M_j$$

that is

$$0 \leq 1 - \delta_i - \sum_{j \neq i}^{N} b_{ij} M_j - \sum_{j=1}^{N} c_{ij} M_j$$

since $a_i > 0$ and $\delta_i > 0$ we obtain

$$0 \leq a_i \delta_i \left(1 - \delta_i - \sum_{j \neq i}^{N} b_{ij} M_j - \sum_{j=1}^{N} c_{ij} M_j \right)$$

and since $M_i \geq 1$ *we have*

$$a_i(1 - M_i - \sum_{j\neq i}^{N} b_{ij}\delta_j - \sum_{j=1}^{N} c_{ij}\delta_j) \leqslant 0,$$

so

$$f_i(\delta_i, [M]_i, M) \leq 0 \leq f_i(M_i, [\delta]_i, \delta), \quad i = 1, \ldots, N$$

with f_i *as defined above, that is* f_i *satisfy condition* (2.4) *with* $(\widehat{c}, \widetilde{c}) = (M, \delta)$ *this implies in view of Theorem 2.1.1 that* (M, δ) *is ordred upper and lower solutions of* (3.4). f_i *satisfies also* (2.4), *always with* $(\widehat{c}, \widetilde{c}) = (M, \delta)$ *it follows from Theorem 2.2.1 that problem* (3.4) *has a pair of constant quasisolutions* $\bar{\rho}$ *and* $\underline{\rho}$ *such that*

$$0 \leqslant \delta \leqslant \underline{\rho} \leqslant \bar{\rho} \leqslant M$$

and

$$f_i(\bar{\rho}_i, [\underline{\rho}]_i, \underline{\rho}) = f_i(\underline{\rho}_i, [\bar{\rho}]_i, \bar{\rho}) = 0,$$

since $a_i > 0$ *we obtain*

$$\begin{cases} 1 - \bar{\rho}_i - \sum_{j\neq i}^{N} b_{ij}\underline{\rho}_j - \sum_{j=1}^{N} c_{ij}\underline{\rho}_j = 0, \\ 1 - \underline{\rho}_i - \sum_{j\neq i}^{N} b_{ij}\bar{\rho}_j - \sum_{j=1}^{N} c_{ij}\bar{\rho}_j = 0. \end{cases} \tag{3.9}$$

Set $\rho_i = \bar{\rho}_i - \underline{\rho}_i, i = 1, ..., N$. *Then by substraction of the equations in (3.9)*

$$-\rho_i + \sum_{j\neq i}^{N} b_{ij}\rho_j - \sum_{j=1}^{N} c_{ij}\rho_j = 0, \quad i = 1....N. \tag{3.10}$$

this means that $A_1\rho = 0$, *but* A_1 *is nonsingular, then* $\rho = 0$ *which leads to* $\bar{\rho} - \underline{\rho} = 0$, *then we find that* $\bar{\rho} = \underline{\rho}$ *let* ρ^* *their common value.*

It follows from Theorem 2.2.2 that ρ^* *is the unique positive solution of(3.4) satisfying* $\delta \leq \rho^* \leq M$. δ *can always be choosen small enaugh to ensure the uniqueness of positive solutions of* (3.4) *in* $]0, M]$. *Moreover, by Theorem 3.2.1, the solution* $u(t, x)$ *of* (3.1) *converges to* ρ^* *as* $t \to +\infty$ *whenever* $\delta \leq \eta(t, x) \leq M$.

Remains to show the convergence of the solution (3.1) for any nonnegative $\eta(t, x)$ *with* $\eta(0, \cdot) \neq 0$

we consider the scalar parabolic equation

$$\begin{cases} \partial U_i/\partial t - L_i U_i = a_i U_i(1 - U_i), & (t > 0, x \in \Omega) \\ \partial U_i/\partial v = 0, & (t > 0, x \in \partial\Omega), \\ U_i(0, x) = \eta_i(0, x) & (x \in \Omega), i = 1, ..., N. \end{cases} \tag{3.11}$$

We can see that for any nonnegative not identically zero $\eta_i(0,\cdot)$, $(0, C_i)$ *is a pair of upper and lower solutions of* (3.11) *therfore theorem 1.3.1 ensures the existence of a unique positive solution* $U_i(t,x)$ *on* $(0,\infty)\times\bar{\Omega}$ *and by theorem 2.3.1, one can see that* $U_i(t,x)$ *converges to the unique positive steady-state solution* $U_is = 1$ *as* $t \longrightarrow \infty$.
Moreover, by noticing that $(0,U)$ *is a pai of upper and lower solutions of* (3.4) *we obtain* $u_i(t,x) \le U_i(t,x)$ *by the positive property of* $u_i(t,x)$ *and a maximum principle of parabolic equations* $u_i(t,x) < U_i(t,x)$ *on* $(0,\infty)\times\bar{\Omega}$. *Hence there exists* $t_1 > 0$ *such that* $u_i(t,x) \leqslant 1$ *on* $[t_1,\infty)\times\bar{\Omega}$.
Let $w_i(t,x)$ *is the solution of the linear parabolic equation*

$$\begin{cases} \partial w_i/\partial t - L_i w_i = q_i(t,x) w_i & (t>0, x\in\Omega) \\ \partial w_i/\partial v = 0, & (t>0, x\in\partial\Omega), \\ w_i(0,x) = \eta_i(0,x) & (x\in\Omega), i=1,...,N. \end{cases} \tag{3.12}$$

where

$$q_i(t,x) := a_i(1 - u_i - \sum_{j\neq i}^{N} b_{ij}u_j - \sum_{j=1}^{N} c_{ij}(u_j)_\tau), \tag{3.13}$$

then $w_i(t,x) > 0$ *on* $(0,\infty)\times\bar{\Omega}$ *by maximum principle of parabolic boundary-value problems.* $u_i(t,x)$ *is also a solution of (3.11) , the uniqueness property of the solution ensures that* $u_i(t,x) = w_i(t,x) > 0$ *on* $(0,\infty)\times\bar{\Omega}$.
Therefore if we choose a constant δ_i *satisfying*

$$\delta_i \le \min\left(u_i(t,x); t_1 \le t \le t_1+\tau_i, x\in\bar{\Omega}\right) \tag{3.14}$$

then we find
$\delta_i \le u_i(t,x) \le 1$ *on* $[t^*-\tau_i, t^*]\times\bar{\Omega}$ *where* $t^* = t_1 + \bar{\tau}$ *and* $\bar{\tau} = \max\{\tau_i, i=1,....N\}$.*and if we use* $u_i(t,x) = \eta_i(t,x)$ *where* $\eta_i(t,x)$ *is the initial function in the domain* $[t^* - \tau_i, 0)\times\Omega$, *by theorem 2.3.1 we conclude that the solution* $u(t,x)$ *of (3.1) corresponding to any nonnegative* $\eta(t,x)$ *with* $\eta_i(0,x)$ *not identically zero, converges to the constant* p^* *as* $t \longrightarrow \infty$.

Remark 3.2.1. *We can see that if*

$$\sum_{j\neq i}^{N} b_{ij} + \sum_{j=1}^{N} c_{ij} < 1 \quad \textit{for } i=1,...,N, \tag{3.15}$$

by another write of condition (3.15) we find $A_0e < e$ *then condition (3.6) is satisfied with* $M = e$,*As a consequence of Theorem 3.2.1 we have the following conclusion*

Corollary 3.2.1. *Suppose A_1 is nonsingular and condition (3.15) is satisfied then then state problem (3.4) has a unique positive constant solution $\rho^* = (\rho_1^*, ...\rho_N^*)^T$ such that $\rho^* \leqslant e$.*

Moreover, for any nonnegative initial function $\eta(t,x)$,with $\eta_i(t,x)$ not identically zero, the corresponding solution $u(t,x) = (u_1(t,x), ...u_N(t,x))$ of (3.1) possesses the property

$$\lim_{t\to\infty} u(t,x) = \rho^*, \quad x \in \bar{\Omega}. \tag{3.16}$$

3.3 On the ordinary system case

In special case of $L_i = 0$ for all $i = 1, ..., N$. The problem (3.1) is reduced to the ordinary differential system

$$\begin{cases} \grave{u}_i(t) = a_i u_i(t)(1 - u_i(t) - \sum_{j\neq i}^{N} b_{ij} u_j(t) - \sum_{j=1}^{N} c_{ij} u_j(t-\tau) \quad (t > 0) \\ u_i(t) = \eta_i(t), i = (1, ..., N). \end{cases} \tag{3.17}$$

For the ordinary system (3.17) we have a similar result as that in theorem 3.2.1 since $L_i = 0$ is allowed

Theorem 3.3.1. *Assume that A_1 is nonsingular and condition (3.6) is satisfied. Then problem (3.17) has a unique positive equilibrium solution $p^* \leqslant M$. Moreover, for any nonnegative initial function $\eta(t)$ with $\eta_i(0) > 0$, i = 1,..., N,*

the corresponding solution $u(t) = (u_1(t), ..., u_N(t))$ of (3.17) converges to ρ^* as $t \longrightarrow \infty$.

Remark 3.3.1. *In particular, the conclusions holds true if condition (3.6) is replaced by (3.15) then problem (3.17) has a unique positive constant solution $\rho^* = (\rho_1^*, ..., \rho_N^*)^T$ such that $\rho^* \leqslant e$.*

Moreover, for any nonnegative initial function $\eta(t)$,with $\eta_i(0) \neq 0$, the corresponding solution $u(t) = (u_1(t), ..., u_N(t))$ of general system (3.17) possesses the property

$$\lim_{t\to\infty} u(t) = \rho^*. \tag{3.18}$$

3.4 Example of application to the three species case

We consider the following Lotka Volterra system with 3-competing species

$$\begin{cases} \partial u/\partial t - D_1\triangle u = \alpha_1 u(1-u-\beta_1 v_{\tau_2} - \gamma_1 w_{\tau_3}), \\ \partial v/\partial t - D_2\triangle v = \alpha_2 v(1-v-\beta_2 u_{\tau_1} - \gamma_2 w_{\tau_3}), \\ \partial w/\partial t - D_3\triangle w = \alpha_3 w(1-w-\beta_3 u_{\tau_1} - \gamma_3 v_{\tau_2}), (t>0, x\in\Omega) \\ \partial u/\partial\nu = \partial v/\partial\nu = \partial w/\partial\nu = 0, (t>0, x\in\partial\Omega) \\ u=\eta_1, (t\in I_1\times\Omega), v=\eta_2(t\in I_2\times\Omega), w=\eta_3, (t\in I_3\times\Omega) \end{cases} \tag{3.19}$$

and the correspoding ordinary differential system

$$\begin{cases} \dot{u} = \alpha_1 u(1-u-\beta_1 v_{\tau_2} - \gamma_1 w_{\tau_3}), \\ \dot{v} = \alpha_2 v(1-v-\beta_2 u_{\tau_1} - \gamma_2 w_{\tau_3}), \\ \dot{w} = \alpha_3 w(1-w-\beta_3 u_{\tau_1} - \gamma_3 v_{\tau_2}), \\ u(t)=\eta_1(t), (t\in I_1), v(t)=\eta_2(t)(t\in I_2), w(t)=\eta_3(t), (t\in I_3) \end{cases} \tag{3.20}$$

where τ_i is allowed to be zero for some or all i. In this case

$$A_0 = \begin{pmatrix} 0 & \beta_1 & \gamma_1 \\ \beta_2 & 0 & \gamma_2 \\ \beta_3 & \gamma_3 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} -1 & \beta_1 & \gamma_1 \\ \beta_2 & -1 & \gamma_2 \\ \beta_3 & \gamma_3 & -1 \end{pmatrix}$$

As a consequence of Theorems 3.2.1 and 3.2.2 we have the following results

Theorem 3.4.1. *Assume that A_1 is nonsingular and there exists a constant vector $M = (M_1, M_2, M_3)^T$ such that*

$$M \geqslant e \text{ and } \quad A_0 M < e \tag{3.21}$$

where $e = (1,1,1)$ then

problem (3.19) has a unique positive equilibrium solution $\rho^* \leqslant M$. Moreover, for any nonnegative initial function $\eta(t,x)$ with $\eta_i(0,x)$ not identically zero,for i = 1,2,3, the corresponding solution $(u(t,x), v(t,x), w(t,x))$ converge to ρ^* as $t \longrightarrow \infty$.

Remark 3.4.1. *The conclusion in theorem 3.4.1 hold true with $M = e$ if condition (3.21)is replaced by*

$$\beta_i + \gamma_i < 1 \quad \text{for } i = 1,2,3. \tag{3.22}$$

and we have $\rho^* \leqslant 1$

Corollary 3.4.1. *Problem (3.20) has a unique positive equilibrium solution* $\rho^* \leqslant M$*. Moreover, for any nonnegative initial function* $\eta(t)$ *with* $\eta_i(0) > 0$*, the corresponding solution* $(u(t), v(t), w(t))$ *of ordinary problem converges to* ρ^* *as* $t \longrightarrow \infty$*.*

Remark 3.4.2. *The conclusions in Corollary 3.4.1 hold true with* $M = e$ *if condition (3.21) is replaced by condition (3.22). The unique positive equilibrium solution* $\rho^* \leqslant 1$*.*

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