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On some fractional problems

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I dedicate this modest work

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Abstract

The objective of this thesis is to give some existence results for various classes of initial value problem and boundary value problems for nonlinear fractional differential equations and coupled system involving a various kind of fractional-order derivatives in Banach Spaces. For this purpose, the technique used is to reduce the study of our problem to the research of a fixed point of an integral operator. The obtained results are based on some standard fixed point theorems and mönch fixed point theorem combined with the technique of measures of noncompactness, as well as the technique of topological degree theory. We have also provided a illustrative example to each of our considered problems to show the validity of conditions and justify the efficiency of our established results.

Key words and phrases : Fractional differential equation, Coupled fractional differential system, Fractional q-difference equation, Caputo fractional derivative, Caputo-Hadamard fractional derivative, Hilfer fractional derivative, fractional q-derivative, boundary value problems, initial value problem, Banach space, fixed-point, Kuratowski measure of noncompactness, existence, uniqueness, topological degree theory, condensing maps
AMS Subject Classification : 26A33, 34A08, 34B15.

Résumé

L'objectif de cette thèse est de donner des résultats d'existence pour diverses classes de problèmes de valeurs initiales et de problèmes de valeurs aux limites pour des équations différentielles fractionnaires non linéaires et un système couplé impliquant différents types de dérivées d'ordre fractionnaire dans les espaces de Banach. Pour cela, la technique utilisée est de réduire l'étude de notre problème à la recherche d'un point fixe d'un opérateur intégral. Les résultats obtenus sont basés sur des théorèmes de point fixe standard et le théorème de point fixe mönch combiné avec la technique des mesures de non-compacité, ainsi que la technique de la théorie des degrés topologiques. Nous avons également fourni un exemple illustratif à chaque de nos problèmes considérés pour montrer la validité des conditions et justifier l'efficacité de nos résultats établis.

الملخص:

الهدف من هذه الأطروحة هو إعطاء بعض نتائج الوجود لفئات مختلفة من مشاكل القيمة الأولية ومشاكل القيمة الحدية للمعادلات التفاضلية الكسرية غير الخطية والنظام المقترن الذي يتضمن نوعًا مختلفًا من المشتقات ذات الترتيب الكسري في فضاءات باناخ. لهذا الغرض ، فإن التقنية المستخدمة هي تحويل دراسة مشكلتنا إلى البحث عن نقطة ثابتة لمشغل متكامل. تستند النتائج التي تم الحصول عليها إلى بعض نظريات النقطة الثابتة القياسية ونظرية النقطة الثابتة مونش جنبًا إلى جنب مع تقنية مقاييس عدم التوافق ، بالإضافة إلى تقنية نظرية الدرجة الطوبولوجية. لقد قدمنا أيضًا مثالًا توضيحيًا لكل من مشاكلنا المدروسة لإظهار صحة الشروط وتبرير كفاءة نتائجنا المقررة.

List of symbols

We use the following notations throughout this thesis

Acronyms

- FC : Fractional calculus.
- FD : Fractional derivative.
- FDE : Fractional differential equation.
- FI : Fractional integral.
- IVP : Initial value problem.
- BVP : Boundary value problem.
- FHDE : Fractional hybrid differential equation.
- MNC : Measure of noncompactness.

Notation

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{R}^n$: Space of n-dimensional real vectors.
- $\in$: belongs to.
- sup : Supremum.
- max : Maximum.
- $n!$: Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.
- $\Gamma(\cdot)$: Gamma function.
- $\mathrm{B}(\cdot,\cdot)$: Beta function.
- $I_{a^+}^{\alpha,\psi}$: The fractional ψ–integral of order $\alpha > 0$.
- $I_{a^+}^{\alpha}$: The Riemann-Liouville fractional integral of order $\alpha > 0$.
- I_{q}^{α} : The q-Riemann-Liouville fractional integral of order $\alpha > 0$ and $0 < q \leq 1$.

- ${}_{H}I^{\alpha}_{a^+}$: The Hadamard fractional integral of order $\alpha > 0$.
- ${}^{\rho}I^{\alpha}_{a^+}$: The Katugampola fractional integral of order $\alpha > 0$, $\rho > 0$.
- $I^{\alpha,\delta}_{a^+,\eta}$: The Erdélyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, $\alpha \in \mathbb{R}$.
- ${}^{H}D^{\alpha,\beta;\psi}_{a^+}$: The ψ-Hilfer fractional derivative of orde $\alpha > 0$ and type β.
- ${}^{RL}D^{\alpha}_{a^+}$: The Riemann-Liouville fractional derivative of orde $\alpha > 0$.
- ${}_{RL}D^{\alpha}_{q}$: The Riemann-Liouville fractional q-derivative of orde $\alpha > 0$ and $0 < q \leq 1$.
- ${}^{C}D^{\alpha}_{a^+}$: The Caputo fractional derivative of orde $\alpha > 0$.
- D^{α}_{q} : The Caputo fractional q-derivative of orde $\alpha > 0$ and $0 < q \leq 1$.
- $D^{\alpha,\beta}_{a^+}$: The Hilfer fractional derivative of orde $\alpha > 0$ and type β.
- ${}^{H}D^{\alpha}_{a^+}$: The Hadamard fractional derivative of orde $\alpha > 0$.
- ${}^{C}_{H}D^{\alpha}_{a^+}$: The Caputo-Hadamard fractional derivative of orde $\alpha > 0$.
- $C(J,\mathbb{E})$: Space of continuous functions on J.
- $C^n(J,\mathbb{E})$: Space of n time continuously. differentiable functions on J.
- $C_{\gamma;\psi}(J,\mathbb{E})$ the weighted space of continuous functions on J.
- $AC(J,\mathbb{E})$: Space of absolutely continuous functions on J.
- $L^1(J,\mathbb{R})$: space of Lebesgue integrable functions on J.
- $L^1(J,\mathbb{E})$: space of Bochner integrable functions on J.
- $L^p(J,\mathbb{E})$: space of measurable functions u with $|u|^p$ belongs to $L^1(J,\mathbb{R})$.
- $L^{\infty}(J,\mathbb{E})$: space of functions u that are essentially bounded on J .

Table des matières

Introduction

In understanding and developing of a large class of systems, it is now well clear that researchers and scientists have taken their beginnings from nature. Natural things can be well understood in two possible ways, quantitative and qualitative. Mathematics plays a central role in this direction. It is the science of patterns and relationships. When we go back to understand the quantitative and qualitative behavior of nature, it seems that evolution is from integer to fraction. Quantitative behavior can be well explained using number theory, which started from integer and reached to fractional due to division operation and finally converged to real numbers. Calculus is a branch of mathematics describing how things change. It provides a framework for modeling systems undergoing change, and a way to deduce the predictions of such models. All these resulted in pointing a fact that integer order calculus is a subset of fractional calculus.

Fractional calculus can be defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer. The history of non-integer order derivatives spans from the end of the 17^{th} century to the present day. Specialists agree to trace its beginning to the dated 30 September 1695 when L'Hospital raised a question to Leibniz by questioning the meaning of $\frac{d^n f}{dt^n}$ when $n = \frac{1}{2}$. Leibniz, in his response, wanted to initiate a reflection on a possible theory of not whole derivation and wrote to L'Hospital : "... this would lead to a paradox from which, one day, we will have to draw useful consequences ". It was not until the $1990s$ that the first "useful consequences" appeared. the first serious attempt to give a logical definition for the fractional derivative is due to Liouville who published nine documents in this subject between 1832 and 1837. Independently, Riemann proposed an approach which proved essentially that of Liouville, and it is since she wears the no "Riemann-Liouville approach". Later, other theories appeared like that of Grunwald-Leitnikov, Weyl, Caputo, Hilfer, Hadamard, Caputo-Hadamard, ψ-Riemann-Liouville, ψ-Caputo . At that time there were almost no practical applications of this theory. And over time, new derivatives and fractional integrals arise. These integrals and fractional derivatives have a different kernel and this makes the number of definitions [89, 117, 75, 104, 83, 86] wide. Recently in 2017, Sousa and Oliviera [129] proposed interpolator of ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition so-called ψ-Hilfer fractional derivative, ie, a fractional derivative of a function with respect to another ψ-function. With this fractional derivative, we recover a wide class of fractional derivatives and integrals. A detailed historical account is given in the introduction of [115], more, this work is undoubtedly one of the first to collect scattered results and also can surveys of the history of the fractional theory derivative can be found in [65, 101, 105, 116, 118].

The theory of derivation and fractional integration has long been regarded as a branch of mathematics without any real or practical explanation and it is for this reason that it was considered as an abstract containing only little useful mathematical manipulations. During the past three decades, considerable interest was carried to fractional calculus by the application of these concepts in various fields of physics and engineering [115], the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. Starting from the sixties, many authors and the researches in this domain pointed out that the non-integer order derivative revealed to be a more adequate tool for the description of properties of various real materials as polymers. Various types of physical phenomena, in favor of the use of models with the help of fractional derivative, that is, fractality, recursivity, diffusion and/ or relaxation phenomena are given in [57]. Recent books [73, 89, 109, 117] provide a rich source of information on fractional-order calculus and its applications. The book by M. Caputo [56], published in 1969, in which he systematically used his original definition of fractional differentiation for formulating and solving problems of viscoelasticity and his lectures on seismology [55]. The transition from pure mathematical formulations to applications began to emerge since the 1990s, where fractional differential equations appeared in several fields such as physics, engineering, biology, mechanics ... For more details of fundamental works on various aspects of the fractional calculus and fundamental physical considerations in favour of the use of models based on derivatives of non-integer order we refer the monograph of Bagley [60], Engeita [66], Hilfer [76], Khare [88], Kilbas [89], Magin [95], Mainardi [96], Miller and Ross [101], Nishitomo [102], Oldham [107], Oldham and Spanier [105], Petras [108], Podlubny [109], Sabatier et al. [119], and the references therein.

It can be noted that most of the work on fractional calculus is devoted to the solvency of boundary problems generated by fractional differential equations. The resolution of these problems deals with the existence, the uniqueness of the solutions, and the multiplicity ... etc ; several methods are applied for the resolution of these problems as iterative techniques, degree topologies theory, hybrid fixed point theory, especially the methods based on the principle of the fixed point, we refer the reader to [8, 61, 89, 101, 110, 109, 38, 37, 117, 96]

Fixed point theory provides the tools to have existence theorems for many different nonlinear problems. Fixed point theorems are often based on certain properties (such as complete continuity, monotony, contraction, etc.) that the application under consideration must satisfy. The theory itself is a beautiful mixture of analysis, topology, and geometry. Over the last 80 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as Biology, Chemistry, Economics, Engineering, Game Theory, and Physics. Fixed point theory plays an important role in functional analysis, approximation theory, differential equations and applications such as boundary value problems etc. The famous fixed point theorems are Banach's theorem, the nonlinear alternative of Leary-Schauder and Schaefer's theorem. Another fixed point theory, Monch's fixed point theorem combined with Kuratowski's measure of non-compactness (the notion of a measure of noncompactness (MNC) plays an important role in the study of many nonlinear phenomen. This concept was introduced by Kuratowski [91] In 1955.).

In the other hand, the most commonly used techniques to study the existence of solutions to functional equations are based on fixed-point arguments. Although there are many standard fixed point theorem to analyse, under suitable conditions, the existence and uniqueness of solution of various problems for fractional differential equations, But, in the absence of compacity and the Lipschitz condition, the previously mentioned theorems are not applicable. In such cases, the measure of noncompactness argument appears as the most convenient and useful in applications. It is a method which was mainly initiated in the monograph of Banas and Goebel [33], then developed and used in many articles, see, for instance, Banas and Sadarangani [32], Mönch [99], and Szufla [125], Akhmerov et al.[16], Alvàrez [26], Benchohra, Henderson and Seba [36], Guo, Lakshmikantham and Liu [74], and the references therein. We also refer the readers to the recent book [31], where several applications of the measure of noncompactness can be found.

This thesis is arranged as follows :

In **Chapter** 1, we give a technically precise overview of definitions, notations, lemmas and notions of fractional calculus, measures of noncompactness, fixed point theorems that are used throughout this thesis.

Chapter 2, is reserved to expose some results of existence and uniqueness of solutions concerning a boundary problem for a fractional differential equation of the Caputo-Hadamard type. The first results are provided by the fixed point theorems of (Banach, Schaefer, Boyd and Wong and Leary-Schauder). The second result is proved by the fixed point theorem of mönch and the measure of noncompactness of Kuratowski. We are interested in the existence and uniqueness of solutions for the following fractional boundary value problem

$$ {}_{H}^{C}D^{r}_{1^+}x(t) = f(t,x(t)), \ \ t \in \mathrm{J} := [1,T], \tag{1}$$

with fractional boundary conditions :

$$ \alpha x(1) + \beta x(T) = \lambda \ {}_{H}I^{q}_{1^+}x(\eta) + \delta, \quad \ q \in (0,1]. \tag{2}$$

where ${}_{H}^{C}D^{r}_{1^+}$ denote the Caputo-Hadamard fractional derivative of order $0 < r \leq 1$ and ${}_{H}I^{q}_{1^+}$ denotes the standard Hadamard fractional integral. Throughout this paper, we always assume that $0 < r, q \leq 1$, $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is continuous. α, β, λ are real constants, and $\eta \in (1,T)$, $\delta \in \mathbb{E}$.

Finally, an example is given at the end of each section to illustrate the theoretical results.

In **Chapter** 3, we study the existence of solutions for certain classes of fractional hybrid differential equations. Our results are based on fixed point theorem for three operators in a Banach algebra due to Dhage. In Section 3.2 we look into the existence of solutions for the following hybrid Caputo-Hadamard fractional differential equation :

$$ \begin{cases} {}_{H}^{C}D^{r}_{1^+}\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_{H}I^{q_i}_{1^+}f_i(t,x(t))}{g(t,x(t))}\right] = h(t,x(t)), \ \ t \in \mathrm{J} := [1,T], \ \ 1 < r \leq 2, \\ \alpha_1 \left[\frac{x(t)-\sum_{i=1}^{m}\ {}_{H}I^{q_i}_{1^+}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1} + \beta_1\, {}_{H}^{C}D^{p}_{1^+}\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_{H}I^{q_i}_{1^+}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1} = \gamma_1, \\ \alpha_2 \left[\frac{x(t)-\sum_{i=1}^{m}\ {}_{H}I^{q_i}_{1^+}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T} + \beta_2\, {}_{H}^{C}D^{p}_{1^+}\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_{H}I^{q_i}_{1^+}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T} = \gamma_2. \end{cases} \tag{3}$$

where ${}_{H}^{C}D^{\varepsilon}_{1^+}$ and $I^{q_i}_{1^+}$ denotes the Caputo-Hadamard fractional derivatives of orders $\varepsilon, \varepsilon \in \{r, p\}$, $0 < p \leq 1$ and Hadamard integral of order q_i,respectively, $\alpha_i, \beta_i, \gamma_i, i = 1,2$, are real constants, $g \in C(\mathrm{J} \times \mathbb{R}, \mathbb{R} - \{0\})$, and $f, h \in C(\mathrm{J} \times \mathbb{R}, \mathbb{R})$.

Finally, an example is also constructed to illustrate our results.

In **Chapter 4**, we give existence and uniqueness results of solutions to two boundary problems concerning fractional differential equations with the Hilfer derivative, one subject to Riemman-Liouville integral boundary conditions and the other subject to a Multi-point Katugampola integral boundary conditions of multiple points, based on standard fixed point theorems and fixed point theorem of Mönch's combined with the Kuratowski measure of non-compactness. More specifically, in Section 4.2 we are interested in the existence of solutions for the following fractional differential equation

$$D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in \mathrm{J} := [0, T], \tag{4}$$

supplemented with the boundary conditions of the form :

$$a I_{0^+}^{1-\gamma} x(0) + b x(T) = \sum_{i=1}^{m} c_i \, {}^{\rho_i} I^{q_i} x(\eta_i) + d. \tag{5}$$

where $D^{\alpha,\beta}$ is the Hilfer fractional derivative $0 < \alpha \leq 1, 0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, ${}^{\rho_i} I_{0^+}^{q_i}$ is the Katugampola integral of $q_i > 0$ and $I_{0^+}^{1-\gamma}$ is the Riemann-Liouville integral of order $1-\gamma$, $f : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a, b, d, c_i, i = 1, \ldots, m$ are real constants, and $0 < \eta_i < T, i = 1, \ldots, m$.

This is followed in section 4.3 by another boundary value problem but this time with Katugampola integral boundary conditions of the form

$$D_{0^+}^{\alpha,\beta} y(t) = f(t, y(t)), t \in \mathrm{J} := [0, T]. \tag{6}$$

with the fractional boundary conditions

$$\begin{aligned} &I_{0^+}^{1-\gamma} y(0) = y_0, \;\; I_{0^+}^{3-\gamma-2\beta} y'(0) = y_1, \\ &I_{0^+}^{1-\gamma} y(\eta) = \lambda (I_{0^+}^{1-\gamma} y(T)), \;\; \gamma = \alpha + \beta - \alpha\beta. \end{aligned} \tag{7}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0 < \alpha \leq 1, 0 \leq \beta \leq 1$, $0 < \lambda < 1$, $0 < \eta < T$ and let $\mathbb{E}$ be a Banach space space with norm $\|.\|$, $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is given continuous function.

Finally, to illustrate the theoretical results, an example is given at the end of each section.

In **Chapter 5**, Sufficient conditions are established ensuring the existence and the uniqueness of solutions of the boundary problem for Caputo type Fractional Differential Equation with fractional integral boundary conditions. The first results are based on the fixed point theorem of Banach, Boyd and Wong, Schaefer and the nonlinear alternative of Leary-Schauder, the second result is obtained by using the Mönch's theorem combined with the measure of noncompactness of Kuratowski. More specifically, in Section 5.2 we are interested in the existence and uniqueness of solutions for the following fractional boundary value problem

$${}^{c}D_{0^+}^{\alpha} x(t) = f(t, x(t)), \;\; t \in \mathrm{J} := [0, T], \;\; 1 < \alpha \leq 2. \tag{8}$$

Subject with the integral boundary conditions

$$x(0) = 0, \;\; I_{0^+}^{\beta} x(\varepsilon) \;\; = \delta^{\rho} J_{0^+}^{\gamma} x(T). \tag{9}$$

where ${}^{c}D_{0^+}^{\alpha}$ denote the Caputo fractional derivative $1 < \alpha < 2$, $I_{0^+}^{\beta}$ denotes the standard Riemann-Liouville fractional integral and ${}^{\rho}I_{0^+}^{\gamma}$ Katugampola fractional integral $\gamma > 0$, $\rho > 0$, and let $\mathbb{E}$ is a reflexive Banach space with norm $\|.\|$, $f : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, δ are real constants.

As a second problem, we discuss in Section 5.3 the existence of solutions for the following boundary value problem

$$ {}^{C}D_{0^+}^{\alpha}x(t) = f(t, x(t)), \quad t \in \mathrm{J} := [0, T], \tag{10} $$

associated with the following Erdélyi-Kober fractional integral boundary conditions :

$$ \begin{aligned} x(T) &= \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} x(\beta_i), \quad 0 < \beta_i < T, \\ x'(T) &= \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} x'(\sigma_i), \quad 0 < \sigma_i < T, \\ x''(T) &= \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} x''(\varepsilon_i), \quad 0 < \varepsilon_i < T, \end{aligned} \tag{11} $$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $2 < \alpha \leq 3$ and $J_{\eta_i}^{\gamma_i,\delta_i}$ denote Erdélyi-Kober fractional integral of order $\delta_i > 0$, $\eta_i > 0$, $\gamma_i \in \mathbb{R}$. $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a continuous function, $a_i, b_i, d_i, i = 1, 2, \ldots, m$ are real constants. Recall that Erdélyi-Kober fractional integral operators play an important role especially in engineering, for more details on the Erdélyi-Kober fractional integrals, see [90, 67].

Finally, to illustrate the theoretical results, an example is given at the end of each section.

Chapter 6, is devoted to the existence results of solutions for certain classes of nonlinear Langevin fractional q-difference equation involving Caputo q-derivative in Banach space. The arguments are based on Mönch's fixed point theorem combined with the technique of kuratowski measures of noncompactness. More specifically, in Section 6.3 we are interested in the existence of solutions for the following Langevin fractional q-difference equation

$$ \begin{cases} D_q^{\beta}(D_q^{\alpha} + \lambda)x(t) & = f(t, x(t)), \quad t \in \mathrm{J} = [0, 1], \\ x(0) = \gamma, & x(1) = \eta, \end{cases} \tag{12} $$

where $0 < \alpha, \beta \leq 1$ and D_q is the fractional $q-$derivative of the Caputo type. $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a given function satisfying some assumptions that will be specified later and $\mathbb{E}$ is a Banach space with norm $\|x\|$, λ is any real number.

In Section 6.4, we give similar result to the following coupled fractional Langevin q-difference system

$$ \begin{cases} D_q^{\beta_1}(D_q^{\alpha_1} + \lambda_1)x_1(t) = f_1(t, x_1(t), x_2(t)), \\ \\ D_q^{\beta_2}(D_q^{\alpha_2} + \lambda_2)x_2(t) = f_2(t, x_1(t), x_2(t)), \end{cases} \quad t \in \mathrm{J}, \tag{13} $$

with the Dirichlet boundary conditions

$$ \begin{cases} x_1(0) = \gamma_1 & , x_1(1) = \eta_1 \\ x_2(0) = \gamma_2 & , x_2(1) = \eta_2. \end{cases} \tag{14} $$

where $\mathrm{J} := [0,1]$, $0 < \alpha, \beta \leq 1$, and D_q is the fractional $q-$derivative of the Caputo type. $f_i : \mathrm{J} \times \mathbb{E}^2 \to \mathbb{E}$ are given functions satisfying some assumptions that will be specified later, and $\mathbb{E}$ is a Banach space with norm $\|\cdot\|$, λ_i,$i = 1,2$, is any real number.

Finally, to illustrate the theoretical results, an example is given at the end of each section.

Chapter 7, we study existence and uniqueness results for a coupled system of nonlinear fractional q-difference subject to nonlinear more general four-point boundary condition. Our analysis relies on the topological degree for condensing maps via a priori estimate method and the Banach contraction principle fixed point theorem.

$$\begin{cases} D_q^{q_1} u_1(t) = f_1(t, u_1(t), u_2(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad , t \in \mathrm{J} := [0,1], \\ D_q^{q_2} u_2(t) = f_2(t, u_1(t), u_2(t)), \end{cases} \tag{15}$$

with the fractional boundary conditions

$$\begin{cases} u_1(0) = a_1\, I_q^{\beta_1} u(\eta_1), 0 < \eta_1 < 1,\ \beta_1 > 0, \\ u_1(1) = b_1\, I_q^{\alpha_2} u(\sigma_1), 0 < \sigma_1 < 1,\ \alpha_1 > 0, \\ u_2(0) = a_2\, I_q^{\beta_2} u(\eta_2), 0 < \eta_2 < 1,\ \beta_2 > 0, \\ u_2(1) = b_2\, I_q^{\alpha_2} u(\sigma_2), 0 < \sigma_2 < 1,\ \alpha_2 > 0, \end{cases} \tag{16}$$

For all $i = 1,2$ where $D_q^{q_i}$ is the fractional q-derivative of the Caputo type of order $1 < q_i \leq 2$, and $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given continuous function, $a_i, b_i, i = 1,2$ are suitably chosen real constants.

Finally, an illustrative example is presented to show the validity of the obtained results.

Chapitre 1

Preliminaries and Background Materials

In this chapter, we introduce the necessary concepts for the good understanding of this thesis. We provide some essential properties of fractional differential operators. We also review some of the basic properties of coincidence degree theory for condensing maps and measures of noncompactness and fixed point theorems which are crucial in our results regarding fractional differential equations.

1.1 Functional spaces

Let $\mathbb{E}$ be a Banach space endowed with the norm $\|.\|_{\mathbb{E}}$ and let $\mathrm{J} := [a,b]$ be a compact interval of $\mathbb{R}$. We present some functional spaces :

1.1.1 Space of Continuous Functions

Definition 1.1. Let $C(\mathrm{J},\mathbb{E})$ be the Banach space of vector-valued continuous functions $u : \mathrm{J} \longrightarrow \mathbb{E}$, equipped with the norm

$$\|u\|_{\infty} = \sup\{\|u(t)\|/t \in \mathrm{J}\}.$$

Analogously, $C^n(\mathrm{J},\mathbb{E})$ is the Banach space of functions $u : \mathrm{J} \longrightarrow \mathbb{E}$, where u is n time continuously differentiable on J.

$$\|f\|_{C^n} := \sum_{k=0}^{n} \left\|f^{(k)}\right\|_C := \sum_{k=0}^{n} \max_{t\in \mathrm{J}} \left|f^{(k)}(t)\right|, n \in \mathbb{N}$$

In particulier if $n = 0$, $\quad C^0(\mathrm{J},\mathbb{E}) \equiv C(\mathrm{J},\mathbb{E})$.

1.1.2 Spaces of Lebesgue's Integrable Functions L^p

Denote by $L^1(\mathrm{J},\mathbb{R})$ the Banach space of functions u Lebesgues integrable with the norm

$$\|x\|_{L^1} = \int_{\mathrm{J}} |x(t)|\, dt.$$

while $L^p(\mathrm{J},\mathbb{R})$ denote the space of Lebesgue integrable functions on J where $|u|^p$ belongs to $L^1(\mathrm{J},\mathbb{R})$, endowed with the norm

$$\|u\|_{L^p} = \left[\int_0^T |u(t)|^p \mathrm{d}t\right]^{\frac{1}{p}}, \quad 1 < p < \infty.$$

In particular, if $p = \infty$, $L^\infty(\mathrm{J},\mathbb{R})$ is the space of all functions u that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{t\in \mathrm{J}} |u(t)| = \inf\{c \geq 0 : |u(t)| \leq c \text{ for a.e.} t\}.$$

1.1.3 Spaces of Absolutely Continuous Functions

Definition 1.2. A function $u\colon \mathrm{J} \to \mathbb{E}$ is said to be absolutely continuous on J if for all $\varepsilon > 0$ there exists a number $\delta > 0$ such that ; for all finite partitions $[a_i,b_i]_{i=1}^n \subset \mathrm{J}$ then $\sum_{k=1}^n (b_k - a_k) < \delta$ implies that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$

We denote by $AC(\mathrm{J},\mathbb{E})$ (or $AC^1(\mathrm{J},\mathbb{E})$) the space of all absolutely continuous functions defined on J. It is known that $AC(\mathrm{J},\mathbb{E})$ coincides with the space of primitives of Lebesgue summable functions :

$$u \in AC(\mathrm{J},\mathbb{E}) \Leftrightarrow u(t) = c + \int_a^t \phi(s)\mathrm{d}s, \quad \phi \in L^1(\mathrm{J},\mathbb{R}), \tag{1.1}$$

and therefore an absolutely continuous function u has a summable derivative $u'(t) = \phi(t)$ almost everywhere on J. Thus (1.1) yields

$$u'(t) = \phi(t) \text{ and } c = u(0).$$

Definition 1.3. For $n \in \mathbb{N}^*$ we denote by $AC^n(\mathrm{J},\mathbb{E})$ the space of functions $u : \mathrm{J} \longrightarrow \mathbb{E}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$ belongs to $AC(\mathrm{J},\mathbb{E})$:

$$\begin{aligned} AC^n(\mathrm{J},\mathbb{E}) &= \left\{u \in C^{n-1}(\mathrm{J},\mathbb{E}) : u^{(n-1)} \in AC(\mathrm{J},\mathbb{E})\right\} \\ &= \left\{u \in C^{n-1}(\mathrm{J},\mathbb{E}) : u^{(n)} \in L^1(\mathrm{J},\mathbb{E})\right\}. \end{aligned}$$

The space $AC^n(\mathrm{J},\mathbb{E})$ consists of those and only those functions u which can be represented in the form

$$u(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\phi(s)\mathrm{d}s + \sum_{k=0}^{n-1} c_k t^k, \tag{1.2}$$

where $\phi \in L^1(\mathrm{J},\mathbb{R}), c_j\ (k = 1,\ldots,n-1) \in \mathbb{R}$.
It follows from (1.2) that

$$\phi(t) = u^{(n)}(t) \text{ and } c_k = \frac{u^{(k)}(0)}{k!},\ (k = 1,\ldots,n-1).$$

and

$$AC^n_\delta([a,b],\mathbb{E}) = \left\{h : [a,b] \to \mathbb{E} : \delta^{n-1}h(t) \in AC([a,b],E)\right\}.$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative.
Interested reader can find more details in [83, 89].

1.1.4 Spaces of Weighted Continuous Functions $C_{\gamma;\psi}(\mathrm{J},\mathbb{E})$

Definition 1.4. [129] Let J be a finite interval and $0 \leq \gamma < 1$, we introduce the weighted space $C_{\gamma;\psi}(\mathrm{J},\mathbb{E})$ of continuous functions f on J. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, defined as follows

$$C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = \left\{ f : (a,b] \to \mathbb{E} : (\psi(t) - \psi(a))^{\gamma} f(t) \in C(\mathrm{J},\mathbb{E}) \right\}.$$

Obviously, $C_{\gamma;\psi}(\mathrm{J},\mathbb{E})$ is a Banach space endowed with the norm

$$\|f\|_{C_{\gamma;\psi}(\mathrm{J},\mathbb{E})} = \|\psi(t) - \psi(a))^{\gamma} f(t)\|_{C(\mathrm{J},\mathbb{E})} = \max_{t\in \mathrm{J}} |(\psi(t) - \psi(a))^{\gamma} f(t)|$$

In particular,

- If $\psi(t) = t$ then $C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = C_{\gamma}(\mathrm{J},\mathbb{E})$,
- If $\psi(t) = t$ and $\gamma = 0$ then $C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = C(\mathrm{J},\mathbb{E})$.

Definition 1.5. [129] The weighted space $C^{n}_{\gamma;\psi}(\mathrm{J},\mathbb{E})$ of function f on J is defined by

$$C^{n}_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = \left\{ f : \mathrm{J} \to \mathbb{E}; f(t) \in C^{n-1}(\mathrm{J},\mathbb{E}); f^{(n)}(t) \in C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) \right\}, 0 \leq \gamma < 1$$

with the norm

$$\|f\|_{C^{n}_{\gamma;\psi}(\mathrm{J},\mathbb{E})} = \sum_{k=0}^{n} \left\| f^{(k)} \right\|_{C(\mathrm{J},\mathbb{E})} + \left\| f^{(n)} \right\|_{C_{\gamma;\psi}(\mathrm{J},\mathbb{E})}$$

respectively. In particular, if $n = 0$,we have $C^{0}_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = C_{\gamma;\psi}(\mathrm{J},\mathbb{E})$

Definition 1.6. [129] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, the weighted space $C^{\alpha,\beta}_{\gamma}(\mathrm{J},\mathbb{E})$ is defined by

$$C^{\alpha,\beta}_{\gamma;\psi}(\mathrm{J},\mathbb{E}) = \{ f \in C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) : {}^{H}D^{\alpha,\beta;\psi}_{a^{+}} f \in C_{\gamma;\psi}(\mathrm{J},\mathbb{E}) \}, \gamma = \alpha + \beta - \alpha\beta.$$

Moreover, $C_{\gamma;\psi}(\mathrm{J},\mathbb{E})$ is complete metric space of all continuous functions mapping J into $\mathbb{E}$ with the metric d defined by

$$d(u_1,u_2) = \|u_1 - u_2\|_{C_{\gamma;\psi}(\mathrm{J},\mathbb{E})} := \max_{t\in \mathrm{J}} |(\psi(t) - \psi(a))^{\gamma} [u_1(t) - u_2(t)]|$$

For details see [129, 20, 71, 76].

1.2 Bochner Integral

Let $\mathbb{E}$ be a Banach space provided with the norm $\|.\|$, (a,b) an interval of $\mathbb{R}$ and μ a measure on (a,b) given by $d\mu(t) = \omega(t)dt$ where ω is a continuous positive function on (a,b) $\{A_1,\ldots,A_k\}$ is a finite collection of mutually disjoint subsets of (a,b) with finite measures μ and $\{x_1,\ldots,x_k\}$ is a set of elements of $\mathbb{E}$

Definition 1.7. A function ϕ defined by

$$\phi(t) = \sum_{i=1}^{n} \chi_{A_i}(t) x_i, \quad a < t < b$$

is called a simple function. We define the integral of ϕ with the measure μ on (a,b) by

$$\int_a^b \phi(t) d\mu(t) = \int_a^b \sum_{i=1}^{n} \chi_{A_i}(t) x_i = \sum_{i=1}^{n} \left(\int_{A_i} \omega(t) dt \right) x_i$$

Definition 1.8. A function $f\colon \mathrm{J} \to \mathbb{E}$ is called strongly measurable if there exists a sequence of simple functions $\{\phi_n\}$ of supports included in J such that

$$\lim_{n\to\infty} \|f(t) - \phi_n(t)\|_{\mathbb{E}} = 0,$$

for almost all $t \in \mathrm{J}$.

Definition 1.9. A function $f\colon \mathrm{J} \to \mathbb{E}$ is Bochner integrable on J if it is strongly measurable and such that

$$\lim_{n\to\infty} \int_a^b \|f(t) - \phi_n(t)\| d\mu(t) = 0,$$

for any sequence of simple functions $\{\phi_n\}$.
In this case the Bochner integral on (a,b) is defined by

$$\int_a^b f(t) d\mu(t) = \lim_{n\to\infty} \int_a^b \phi_n(n) d\mu(t)$$

Theorem 1.10. *A strongly measurable function $f\colon \mathrm{J} \to \mathbb{E}$ is Bochner integrable if and only if $\|f\|$ is integrable.*

1.2.1 Spaces of Bochner Integral Functions

Let $L^1(\mathrm{J}, \mathbb{E})$ be the Banach space of measurable functions $u : \mathrm{J} \to \mathbb{E}$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_{\mathrm{J}} \|x(t)\| \, dt.$$

Interested reader can find more details about Bochner integral in many books, e.g. [98, 131].

1.3 Elements From Fractional Calculus Theory

1.3.1 Introduction

Fractional calculus generalizes the integer-order integration and differentiation concepts to an arbitrary(real or complex) order. Fractional calculus is one of the most emerging areas of investigation and has attracted the attention of many researchers over the last few decades as it is a solid and growing work both in theory and in its applications [89, 117]. The importance of fractional calculus growth is notable not only in pure and applied mathematics but also in physics, chemistry, engineering. biology, and other [76, 77, 115, 89].

since the beginning of the fractional calculus in [94]1695, there are numerous definitions of integrals and fractional derivatives, and over time, new derivatives and fractional integrals arise. These integrals and fractional derivatives have a different kernel and this makes the number of definitions [89, 117, 75, 104, 83, 86] wide. On the other hand, we mention some recent formulation of the fractional derivative [20, 72, 104, 83].

With the wide number of definitions of integrals and fractional derivatives, it was necessary to introduce a fractional derivative of a function f with respect to another function, making use of the fractional derivative in the Riemann-Liouville sense, given by [89]

$$^{RL}D_{a^+}^{\alpha;\psi}f(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}f(t)$$

where $n-1<\alpha<n, n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$. However, such a definition only encompasses the possible fractional derivatives that contain the differentiation operator acting on the integral operator.

In the same way, recently, Almeida [20] using the idea of the fractional derivative in the Caputo sense, proposes a new fractional derivative called ψ-Caputo derivative with respect to another function ψ. which generalizes a class of fractional derivatives, whose definition is given by

$$^{C}D_{a+}^{\alpha:\psi}f(t)=I_{a^+}^{n-a:\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$$

where $n-1<\alpha<n, n=[\alpha]+1$ for $\alpha\neq N$ and $n=\alpha$ for $\alpha\in\mathbb{N}$

Although the definitions of fractional derivative ψ-Riemann-Liouville and ψ-Caputo are very general, there exist the possibility of proposing a fractional differentiable operator that unifies these above operators and can overcome the wide number of definitions. Motivated by the definition of Hilfer [76] fractional derivative that contains, as particular cases, the classical Riemann-Liouville and Caputo fractional derivative. Recently in 2017, Sousa and Oliviera [129] proposed interpolator of ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition so-called ψ-Hilfer fractional derivative, ie, a fractional derivative of a function with respect to another ψ function. With this fractional derivative, we recover a wide class of fractional derivatives and integrals, as we will show in Subsubsection 1.3.2 and 1.3.4. It's important to note that, a corresponding fractional integral is also discussed which generalized the Riemann-Liouville fractional integrals. Properties are presented and discussed ; some of them are also proven.

The advantage of the fractional operator ψ–Hilfer proposed here is the freedom of choice of the classical differentiation operator and the choice of the function ψ, i.e., from the choice of the function ψ, the operator of classical differentiation, can act on the fractional integration operator or else the fractional integration operator can act on the classical differentiation operator. This makes it possible to unify and obtain the properties of the two fractional operators mentioned above.

There are some definitions in fractional calculus which are very widely used and have importance in proving various results of fractional calculus. In this section, We will see a new class of integrals and fractional derivatives. Due to the huge amount of definitions, i.e., fractional operators, the following definition is a special approach when the kernel is unknown, involving a function ψ. That generalized all definitions of fractional integral and fractional differential operators that we use throughout this thesis.

1.3.2 Fractional ψ-Integral

Definition 1.11. [129] Let $(a,b)(-\infty\leq a<b\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $\psi(t)$ be an increasing and positive monotone function on (a,b), having a continuous derivative $\psi'(t)$ on (a,b). The left and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined by

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s)ds \tag{1.3}$$

Example 1.12. The fractional integral of the power function $f(t) = (\psi(t)-\psi(a))^\beta, \alpha > 0, \beta > 0$ with respect to another function ψ is defined by,

$$I_{a^+}^{\alpha,\psi}(\psi(t)-\psi(a))^\beta = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(a))^\beta \mathrm{ds}.$$

Using the change of variables $\psi(s) = \psi(a) + (\psi(t)-\psi(a))\psi(\tau), \quad \tau \in [0,1]$, we get,

$$\begin{aligned} I_{a^+}^{\alpha,\psi}(\psi(t)-\psi(a))^\beta &= \frac{(\psi(t)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(1-\psi(\tau))^{\alpha-1}\psi(\tau)^\beta d\tau \\ &= \frac{(\psi(t)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha,\beta+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(\psi(t)-\psi(a))^{\alpha+\beta}. \end{aligned}$$

Lemma 1.13 ([89, 129])**.** *The following basic properties of the fractional integral with respect to another function ψ hold :*

1. *The integral operator $I_{a^+}^{\alpha,\psi}$ is linear ;*
2. *The semi-group property of the fractional integration operator $I_{a^+}^{\alpha,\psi}$ is given by the following result*
$$I_{a+}^{\alpha;\psi} I_{a+}^{\beta;\psi} f(t) = I_{a+}^{\alpha+\beta;\psi} f(t), \quad \alpha,\beta > 0,$$
holds at every point if $f \in C_{\gamma;\psi}([a,b])$ and holds almost everywhere if $f \in L^1([a,b])$,
3. *Commutativity*
$$I_{a^+}^{\alpha,\psi}(I_{a^+}^{\beta,\psi} f(t)) = I_{a^+}^{\beta,\psi}(I_{a^+}^{\alpha,\psi} f(t)), \quad \alpha,\beta > 0;$$
4. *The fractional integration operator $I_{a^+}^{\alpha,\psi}$ is bounded from $C_{\gamma,\psi}[a,b]$ into $C_{\gamma,\psi}[a,b]$.*

On some class of fractional integrals

In the previous few decades, the area of fractional calculus over time has become an important tool for the development of new mathematical concepts in the theoretical sense and practical sense. So far, there are a variety of fractional operators, in the integral sense or in the differential sense. However, natural problems become increasingly complex and certain fractional operators presented with the specific kernel are restricted to certain problems. So [89, 117] it was proposed a fractional integral operator with respect to another function, that is, to a function ψ, making such a general operator, in the sense that it is enough to choose a function ψ and obtain an existing fractional integral operator. In this subsection, we present a class of fractional integrals, based on the choice of the arbitrary ψ function.

For the class of integrals that will be presented next, we suggest [89, 117, 86, 20, 67]

1. If we consider $\psi(t) = t$ in Eq. (1.3), we have
$$I_{a^+}^{\alpha,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds = {}_{RL}I_{a+}^{\alpha} f(t)$$
the Riemann-Liouville fractional integral.
2. Choosing $\psi(t) = \ln t$ and substituting in Eq. (1.3), we have
$$\begin{aligned} I_{a^+}^{\alpha,\ln t} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{1}{s}(\ln t - \ln s)^{\alpha-1} f(s)ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} \\ &= {}_{H}I_{a^+}^{\alpha} f(t) \end{aligned}$$
the Hadamard fractional integral.
3. If we consider $\psi(t) = t^{\sigma}, g(t) = t^{a\eta} f(t)$ and substituting in Eq. (1.3), we get
$$\begin{aligned} t^{-\sigma(\alpha+\eta)} I_{a^+}^{\alpha;t^{\sigma}} g(t) &= t^{-\sigma(\alpha+\eta)} I_{a^+}^{\alpha;t^{\sigma}} \left(t^{\sigma\eta} f(t)\right) \\ &= \frac{\sigma t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t s^{\sigma\eta+\sigma-1} \left(t^{\sigma} - s^{\sigma}\right)^{\alpha-1} f(s)ds \\ &= {}^{EK}I_{a^+,\sigma}^{\eta,\alpha} f(t) \end{aligned}$$
the Erdélyi-Kober fractional integral.
4. Choosing $\psi(t) = t^{\rho}$ and substituting in Eq. (1.3). we have
$$\begin{aligned} \frac{1}{\rho^{\alpha}} I_{a^+}^{\alpha;t^{\rho}} f(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} f(s)ds \\ &= {}^{\rho}I_{a^+}^{\alpha} f(t) \end{aligned}$$
the Katugampola fractional integral.

1.3.3 Fractional ψ-Derivative

In what follows, we begin to evoke two definitions of fractional derivatives with respect to another function, both definitions being motivated by the fractional derivative of Riemann-Liouville and Caputo, in that order, choosing a specific function ψ.

Definition 1.14. Let $\psi'(t) \neq 0(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0, n \in$ N. The Riemann-Liouville derivatives of a function f with respect to ψ of order α correspondent to the Riemann-Liouville, are defined by
$$\begin{aligned} \mathscr{D}_{a^+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds \end{aligned} \tag{1.4}$$

Definition 1.15. Let $\alpha > 0, n \in \mathbb{N}, I = |a,b]$ is the interval $-\infty \leq a < b \leq \infty, f, \psi \in C^n(|a,b|,\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in I$. The left ψ–Caputo fractional derivative of f of order α is given by
$$^{c}D_{a^+}^{\alpha;\psi} f(t) = I_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t) \tag{1.5}$$

1.3.4 Fractional ψ–Hilfer derivative

From the definition of fractional derivative in the Riemann-Liouville sense and the Caputo sense [89], was introduce the Hilfer fractional derivative [76], which unifies both derivatives. Motivated by the definition of Hilfer, we define a new generalized fractional operator so-called ψ–Hilfer fractional derivative of a function f with respect to another function. From the fractional derivative ψ–Hilfer, we present some essential lemma, properties and some relations between the ψ–fractional integral and the fractional derivative ψ–Hilfer.

Definition 1.16. [129] Let $n-1<\alpha<n$ with $n\in\mathbb{N}, \mathrm{J}=[a,b]$ is the interval such that $-\infty\leq a<b\leq\infty$ and $f,\psi\in C^n((a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in\mathrm{J}$. The ψ–Hilfer fractional derivative (left-sided and right-sided) ${}^HD_{a+}^{\alpha,\beta;\psi}(\cdot)$ of function of order α and type $0\leq\beta\leq 1$, are defined by

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}f(t)=I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(t) \tag{1.6}$$

The ψ–Hilfer fractional derivative as above defined, can be written in the following form

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}f(t)=I_{a^+}^{\gamma-\alpha;\psi}D_{a^+}^{\gamma;\psi}f(t)$$

with $\gamma=\alpha+\beta(n-\alpha)$ and $I_{a^+}^{\gamma-\alpha;\psi}(\cdot)$, $D_{a^+}^{\gamma;\psi}(\cdot)$ as defined in Eq. (1.4) and Eq. (1.5).

Lemma 1.17. *Given $\beta\in\mathbb{R}$, consider the functions*

$$f(t)=(\psi(t)-\psi(a))^{\delta-1}$$

where $\beta>n$. Then, for $\alpha>0$

$$ {}^HD_{a+}^{\alpha,\beta;\psi}f(t)=\frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}(\psi(t)-\psi(a))^{\delta-\alpha-1}$$

Proof. Using the [129, Lemma 2 and Lemma 3], we obtain

$$\begin{aligned} {}^HD_{+}^{\alpha,\beta;\psi}f(x) &= I_{a+}^{\gamma-\alpha;\psi}D_{a+}^{\gamma;\psi}f(x)\\ &= I_{a+}^{\gamma-\alpha;\psi}D_{a+}^{\gamma;\psi}(\psi(x)-\psi(a))^{\delta-1}\\ &= I_{a+}^{\gamma-\alpha;\psi}\left(\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma)}(\psi(x)-\psi(a))^{\delta-\gamma-1}\right)\\ &= \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma)}I_{a+}^{\gamma-\alpha;\psi}\left((\psi(x)-\psi(a))^{\delta-\gamma-1}\right)\\ &= \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}(\psi(x)-\psi(a))^{\delta-\alpha-1}\end{aligned}$$

□

Remark 1.18. [129] In particular, given $n\leq k\in\mathbb{N}$ and as $\delta>n$, we have

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}(\psi(t)-\psi(a))^k=\frac{k!}{\Gamma(k+1-\alpha)}(\psi(t)-\psi(a))^{k-\alpha}$$

On the other hand, for $n>k\in\mathbb{N}_0$, we have

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}(\psi(t)-\psi(a))^k=0$$

Theorem 1.19. *[129] If $f \in C^n|a,b|, n-1<\alpha<n$ and $0 \le \beta \le 1$, then*

$$I_{a+}^{\alpha;\psi H}D_{a+}^{\alpha,\beta;\psi}f(t) = f(t) - \sum_{k=1}^{n}\frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}f_{\psi}^{[n-k]}I_{a+}^{(1-\beta)(n-\alpha);\psi}f(a)$$

Theorem 1.20. *[129] Let $f,g \in C^n[a,b], \alpha>0$ and $0 \le \beta \le 1$. Then*

$${}^{H}D_{a+}^{\alpha,\beta;\psi}f(t) = {}^{H}D_{a+}^{\alpha,\beta:\psi}g(t) \Leftrightarrow f(t) = g(t) + \sum_{k=1}^{n}c_k(\psi(t)-\psi(a))^{\gamma-k}$$

where $c_k (k=1,...,n)$ are arbitrary constants.

Lemma 1.21. *[129] Let $\gamma = \alpha+\beta-\alpha\beta$ where $\alpha \in (0,1), \beta \in [0,1]$, and $f \in C_{1-\gamma;\psi}^{\gamma}[a,b]$. Then*

$$I_{a^+}^{\gamma;\psi H}D_{a^+}^{\gamma;\psi}f = I_{a^+}^{\alpha;\psi H}D_{a^+}^{\alpha,\beta;\psi}f$$

and

$${}^{H}D_{a^+}^{\gamma;\psi}I_{a^+}^{\alpha;\psi}f = {}^{H}D_{a^+}^{\beta(1-\alpha):\psi}f$$

Lemma 1.22. *[129] If $f \in C^1[a,b], \alpha>0$ and $0 \le \beta \le 1$, then the following equalities*

$${}^{H}D_{a^+}^{\alpha,\beta;\psi}I_{a^+}^{\alpha;\psi}f(t) = f(t)$$

hold almost everywhere on $[a,b]$.

Theorem 1.23. *The ψ–Hilfer fractional derivatives are bounded operators for all $n-1<\alpha<n$ and $0 \le \beta \le 1$, given by*

$$\left\|{}^{H}D_{a+}^{\alpha,\beta;\psi}f\right\|_{C_{\gamma,\psi}} \le K\|f\|_{C_{\gamma,\psi}^{n}} \tag{1.7}$$

and

$$K = \frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n-\gamma+1)\Gamma(\gamma-\alpha+1)} \tag{1.8}$$

Lemma 1.24. *[129] Let $\alpha>0$, $0 \le \gamma < \alpha$ and $f \in C_{\gamma,\psi}[a,b](0<a<b<\infty)$. If $\gamma<\alpha$, then $I_{a^+}^{\alpha,\psi}: C_{\gamma;\psi}[a,b] \to C_{\gamma;\psi}[a,b]$ is continuous on $[a,b]$ and satisfies*

$$I_{a^+}^{\alpha,\psi}f(a) = \lim_{s\to a^+}I_{a^+}^{\alpha,\psi}f(s) = 0$$

On some class of fractional derivative

On the other hand, using the ψ–Hilfer fractional derivative operator Eq. (1.6). This derivative is quite general give rise to many cases by assigning different values to ψ, a, b and taking the limit of the parameters α and β.

For this large class of fractional derivatives that will be presented next, we suggest [89, 117, 86, 71, 20, 83]

1. Consider the $\psi(t) = t$ and taking the limit $\beta \to 1$ on both sides of Eq. (1.6), we get

$$
\begin{aligned}
{}^{H}D_{a^+}^{\alpha,1;t} f(t) &= I_{a+}^{n-\alpha;t} \left(\frac{d}{dt}\right)^n f(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(s)ds \\
&= {}^{C}D_{a^+}^{\alpha} f(t)
\end{aligned}
$$

the Caputo fractional derivative.

2. For $\psi(t) = t$ and taking the limit $\beta \to 0$ on both sides of Eq. (1.6), we have

$$
\begin{aligned}
{}^{H}D_{a+}^{\alpha,0;t} f(t) &= \left(t\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;t} f(t) \\
&= \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds \\
&= {}^{RL}D_{a^+}^{\alpha} f(t)
\end{aligned}
$$

the Riemann-Liouville fractional derivative.

3. For $\psi(t) = t$ and substituting in Eq. (1.6), we get

$$
\begin{aligned}
{}^{H}D_{a^+}^{\alpha,\beta;t} f(t) &= I_{a^+}^{\gamma-\alpha} \left(\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha)} f(t) \\
&= D_{a^+}^{\alpha,\beta} f(t)
\end{aligned}
$$

the Hilfer fractional derivative.

4. For $\psi(t) = \ln t$ and taking the limit $\beta \to 1$ on both sides of Eq. (1.6), we have

$$
\begin{aligned}
{}^{H}D_{a^+}^{\alpha,1;\ln t} f(t) &= I_{a^+}^{n-\alpha;\ln t} \left(t\frac{d}{dt}\right)^n f(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n f(s)\frac{ds}{s} \\
&= {}_{H}^{C}D_{a^+}^{\alpha} f(t)
\end{aligned}
$$

the Caputo-Hadamard fractional derivative.

For more details and proof. The reader can find the ψ–Hilfer and ψ–integral in the papers [129, 20].

1.4 Quantum Calculus (q-Difference)

1.4.1 Introduction

Fractional q-difference equations initiated at the beginning of the nineteenth century, exactly, In 1910 Jackson [82], the first researcher to develop q-calculus in a systematic way introduced the notion of the definite q-integral and some classical concepts. After that, at the beginning of the last century, studies on the q-difference equation appeared in much works, especially in [97, 2, 3, 128]. The fractional q-difference calculus was initially proposed by Al-Salam [25] and Agarwal [9], and one can find more details in [113, 27, 12]. whereas the preliminary concepts on q-fractional calculus can be found in [27].

In what follow, we recall some elementary definitions and properties related to the fractional q-calculus.

For $a \in \mathbb{R}$, we put

$$[a]_q = \frac{1-q^a}{1-q}.$$

Let 0 < q < 1. The q-analogue of the power $(a-b)^n$ is expressed by

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\left(a-bq^k\right), \quad a,b \in \mathbb{R}, n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right), \quad a,b,\alpha \in \mathbb{R}.$$

Definition 1.25. [84] The q-gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} - \{0,-1,-2,\dots\}.$$

The q-gamma function satisfies the classical recurrence relationship

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha)$$

Definition 1.26. [84] For any $\alpha, \beta > 0$, the q-beta function is defined by

$$B_q(\alpha,\beta) = \int_0^1 f^{(\alpha-1)}(1-qf)^{(\beta-1)} d_q f, \quad q \in (0,1)$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

Definition 1.27. [84] Let $f : \mathrm{J} \to \mathbb{R}$ be a suitable function. We define the q-derivative of order $n \in \mathbb{N}$ of the function by $D_q^0 f(t) = f(t)$,

$$D_q f(t) := D_q^1 f(t) = \frac{f(t)-f(qt)}{(1-q)t}, \quad t \neq 0, \qquad D_q f(0) = \lim_{t\to 0} D_q f(t),$$

and

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad t \in I, n \in \{1,2,\dots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 1.28. [84] For a given function $f : I_t \to \mathbb{R}$, the expression defined by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f\left(tq^n\right),$$

is called q-integral, provided that the series converges.
We note that $D_q I_q f(t) = f(t)$, while if f is continuous at 0, then

$$I_q D_q f(t) = f(t) - f(0).$$

1.4.2 Fractional q-Integral

q-Riemann-Liouville Integral

Definition 1.29. [112] Let $\alpha \geq 0$ and f be a function defined on $[a,b]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(t) = f(t)$, and

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s)\, d_q s, \quad \alpha \geq 0, \quad t \in [a,b].$$

Example 1.30. [112] The Riemann-Liouville fractional integral of the power function $(t-a)^\beta$, $\alpha \in \mathbb{R}_+$ and $\beta \in (-1,\infty)$. By definition,

$$I_q^\alpha (t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{\alpha+\beta}, \;\; \beta \in (-1,\infty), \; \alpha > 0, \; t > 0$$

In particular, for $\lambda = 0, a = 0$, we have $f \equiv 1$, then

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(\alpha+1)} t^\alpha, \quad \text{for all } t > 0.$$

In conclusion, we obtain

$$\begin{aligned} \int_0^t (t-qs)^{(\alpha-1)} d_q s &= \Gamma_q(\alpha) I_q^\alpha (1)(t) \\ &= \frac{1}{[\alpha]_q} t^\alpha \end{aligned}$$

Lemma 1.31. *[24] Let f be a function defined on* J *and suppose that α, β are two real nonegative numbers. Then the following hold :*

1. *The integral operator I_q^α is linear ;*
2. *The semi-group property of the fractional integration operator I_q^α is given by the following result*
$$I_q^\alpha I_q^\beta f(t) = I_q^{\alpha+\beta} f(t),$$
holds at every point if $f \in C([a,b])$ and holds almost everywhere if $f \in L^1([a,b])$,
3. *Commutativity*
$$I_q^\alpha I_q^\beta f(t) = I_q^\beta I_q^\alpha f(t),$$
4. *The fractional integration operator I_q^α is bounded in $L^p[a,b]$ $(1 \leq p \leq \infty)$;*
$$\|I_q^\alpha f\|_{L^p} \leq \frac{(b-qa)^\alpha}{\Gamma_q(\alpha+1)} \|f\|_{L^p}.$$

1.4.3 Fractional q-Derivative

q-Riemman-Liouville Derivative

Definition 1.32. [114] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : \mathrm{J} \to \mathbb{R}$ is defined by $D_q^0 f(t) = f(t)$ and

$$\begin{aligned} D_q^\alpha f(t) &= D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f(t) \\ &= \frac{1}{\Gamma_q(n-\alpha)} \int_0^t \frac{f(s)}{(t-qs)^{\alpha-n+1}} d_q s. \end{aligned}$$

where $[\alpha]$ is the integer part of α.

Example 1.33. Riemann-Liouville fractional derivative the power function $(t-a)^\beta, \alpha > 0, \beta > -1$

$$\begin{aligned} {}^{RL}D_q^\alpha (t-a)^\beta &= D_q^n I_q^{n-\alpha} (t-a)^\beta = \frac{d^n}{dt^n}\left[\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+1+n-\alpha)}(t-a)^{\beta+n-\alpha}\right] \\ &= \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+1+n-\alpha)} \frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha} \\ &= \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+1+n-\alpha)} \frac{\Gamma_q(\beta+n-\alpha+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \\ &= \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \end{aligned}$$

Remark 1.34. If we let $\beta = 0$ in the previous example, we see that the q–Riemann-Liouville fractional derivative of a constant is not 0. In fact,

$${}^{RL}D_q^\alpha 1(t) = \frac{(t-a)^{-\alpha}}{\Gamma_q(1-\alpha)}.$$

Remark 1.35. On the other hand, for $j = 1, 2, \cdots, [\alpha]+1$,

$${}^{RL}D_q^\alpha (t-a)^{\alpha-j}(t) = 0.$$

We could say that $(t-a)^{\alpha-j}$ plays the same role in Riemann-Liouville fractional differentiation as a constant does in classical integer-ordered differentiation.

As a result, we have the following fact :

Lemma 1.36 ([89, 109])**.** $\alpha > 0$*, and* $n = [\alpha]+1$ *then*

$${}^{RL}D_q^\alpha f(t) = 0 \Leftrightarrow f(t) = \sum_{j=1}^{n} c_j (t-a)^{\alpha-j},$$

where c_j $(j = 1, \ldots, n)$ *are arbitrary constants.*

Lemma 1.37. *[68] Let* $\alpha > 0$ *and* $n \in \mathbb{N}$ *where* $[\alpha]$ *denotes the integer part of* α*. Then, the following fundamental identity holds*

$${}_{RL}I_q^\alpha {}_{RL}D_q^n f(t) = {}_{RL}D_q^n {}_{RL}I_q^\alpha f(t) - \sum_{k=0}^{\alpha-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k h)(0).$$

The following lemma shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1.38 ([89, 109]). *If $\alpha > 0$ and $f \in L^p([a,b])$ $(1 \leq p \leq \infty)$, then the following equalities*

$$^{RL}D_q^{\alpha} I_q^{\alpha} f(t) = f(t), \tag{1.9}$$

hold almost everywhere on $[a,b]$.

Caputo q-Derivative

Definition 1.39. [114] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : \mathrm{J} \to \mathbb{R}$ is defined by

$$\begin{aligned}\left({}^cD_q^{\alpha} f\right)(t) &= \left(I_q^{[\alpha]-\alpha}\left(D_q^{[\alpha]} f\right)\right)(t)\\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t-qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f\right)(s) d_q s\end{aligned}$$

We put by convention

$$^cD_q^0 f(t) = f(t).$$

Example 1.40. The Caputo derivative of the power function $(t-a)^{\beta}, \alpha > 0, \beta > 1, n = [\alpha]+1$, then the following relation hold

$$^cD_q^{\alpha}(t-a)^{\beta} = \begin{cases} \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, & (\beta \in \mathbb{N} \text{ and } \beta \geq n \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > n-1), \\ \\ 0, & \beta \in \{0, \ldots, n-1\}. \end{cases} \tag{1.10}$$

Remark 1.41. We see that consistent with classical integer-ordered derivatives, for any constant C

$$^cD_q^{\alpha} C = 0.$$

We also recognize from (1.10) that :

Lemma 1.42 ([89, 109]). *Let $\alpha > 0$ and $n = [\alpha]+1$ then the differential equation*

$$^cD_{a^+}^{\alpha} f(t) = 0$$

has solutions

$$f(t) = \sum_{j=0}^{n-1} c_j (t-a)^j, \quad c_j \in \mathbb{R}, j = 0 \cdots n-1.$$

Lemma 1.43. *[114] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds :*

$$I_q^{\alpha} {}^cD_q^{\alpha} f(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} D_q^k f(0).$$

In particular, if $\alpha \in (0,1)$, then

$$I_q^{\alpha} {}^cD_q^{\alpha} f(t) = f(t) - f(0).$$

Lemma 1.44. *Let $\alpha > \beta > 0$, and $f \in L^1([a,b])$. Then we have :*

1. *The Caputo fractional q–derivative is linear ;*
2. ${}^cD_q^\alpha I_q^\alpha f(t) = f(t)$ *;*
3. ${}^cD_q^\beta I_q^\alpha f(t) = I_q^{\alpha-\beta} f(t)$.

1.5 On the measures of non-compactness

In this section, we present some definitions and we give several examples measures of noncompactness in certain specific spaces that will be used in the sequel

1.5.1 The general notion of a measure of noncompactness

Firstly, we need to fix the notation. In what follows, (E,d) will be a metric space, and $(X,\|\cdot\|)$ a Banach space. By $B(x,r)$ we denote the closed ball centered at x with radius r. By B_r we denote the ball $\mathrm{B}(0,r)$. If Q is non-empty subset of X , then $\overline{Q}$ and Conv$Q$ denote the closure and the closed convex closure of Q, respectively. When Q is a bounded subset, Diam(Q) denotes the diameter of Q. Also, we denote by $\mathscr{B}_E$ (resp. $\mathscr{B}_X$) the class of non-empty and bounded subsets of E (resp. of X),

We begin with the following general definition.

Definition 1.45 ([30, 33])**.** A mapping $\mu : \mathscr{B}_E \longrightarrow \mathbb{R}_+ = [0,\infty)$ will be called a measure of noncompactness in E if it satisfies the following conditions :

(1) Regularity : $\mu(Q) = 0$ if, and only if, Q is a precompact set.

(2) Invariant under closure : $\mu(Q) = \mu(\overline{Q})$, for all $Q \in \mathscr{B}_E$.

(3) Semi-additivity : $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.
To have a MNC in a Banach space X we need to add the two following additional properties :

(4) Semi-homogeneity : $\mu(\lambda Q) = |\lambda|\mu(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathscr{B}_X$.

(5) Invariant under translations : $\mu(x+Q) = \mu(Q)$, for all $x \in X$ and $Q \in \mathscr{B}_X$.

We note here that en literature, then are three main and most frequently used MNCs are the Kuratowski MNC κ , the Hausdorff MNC χ, and the De Blasi Measure of Weak Noncompactness β.

1.5.2 The Kuratowski and Hausdorff measure of noncompactness

we present a list of three important examples of measures of noncompactness which arise over and over in applications. The first example is the Kuratowski measure of noncompactness (or set measure of noncompactness).

Definition 1.46 ([91, 92])**.** Let (E,d) be a metric space and Q be a bounded subset of E. Then the Kuratowski measure of noncompactness (the set-measure of noncompactness, κ-measure) of Q, denoted by $\kappa(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q

can be covered by a finite number of sets with diameters$< \varepsilon$, i.e.,

$$\kappa(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset E, \mathrm{diam}(S_i) < \varepsilon, i = 1,2,\ldots,n, n \in \mathbb{N} \right\}.$$

In general, the computation of the exact value of $\kappa(Q)$ is difficult. Another measure of noncompactness, which seems to be more applicable, is so-called Hausdorff measure of noncompactness (or ball measure of noncompactness). It is defined as follows.

Definition 1.47. Let (E,d) be a complete metric space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of E, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of balls with radius $< \varepsilon$, i.e.,

$$\chi(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in E, r_i < \varepsilon, i = 1,2,\ldots,n, n \in \mathbb{N} \right\}.$$

If $(X, \|\cdot\|)$ is a Banach space, we have the following equivalent definition.

Definition 1.48. Let $(X, \|\cdot\|)$ be a Banach space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of X, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q has a finite ε-net in X, i.e.,

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \subset S + \varepsilon B(0,1), S \subset X, S \text{ is finite}\}.$$

We list below some properties are common to κ and χ and so we are going to use ϕ to denote either of them. These properties follow immediately from the definitions and show that both mappings are measures of noncompactness in the sense of Definition 1.45.

Properties 1. ([30, 91]) Let ϕ denote κ or χ. Then the following properties are satisfied in any complete metric space E :

(a) Regularity : $\phi(Q) = 0$ if, and only if, Q is a precompact set.

(b) Invariant under passage to the closure : $\phi(Q) = \phi(\overline{Q})$, for all $Q \in \mathscr{B}_E$.

(c) Semi-additivity : $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.

(d) Monotonicity :$Q_1 \subset Q_2 \Rightarrow \phi(Q_1) \leq \phi(Q_2)$.

(e) $\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.

(f) Non-singularity : If Q is a finite set, then $\phi(Q) = 0$.

(g) Generalized Cantor's intersection. If $\{Q_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded and closed nonempty subsets of E and $\lim_{n\to\infty} \phi(Q_n) = 0$ then $\bigcap_{n=1}^{\infty} Q_n$ is nonempty and compact in E. If X is a Banach space, then we also have :

(h) Semi-homogeneity : $\phi(\lambda Q) = |\lambda|\phi(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathscr{B}_X$.

(i) Algebraic semi-additivity : $\phi(Q_1 + Q_2) \leq \phi(Q_1) + \phi(Q_2)$, for all $Q_1, Q_2 \in \mathscr{B}_X$.

(j) Invariant under translations : $\phi(x + Q) = \phi(Q)$, for all $x \in X$ and $Q \in \mathscr{B}_X$.

(k) invariant under passage to the convex hull : $\phi(A) = \phi(conv(A))$,

(l) Lipschitzianity : $|\phi(Q_1) - \phi(Q_2)| \leq L_\phi d_H(Q_1, Q_2)$, where $L_\chi = 1$, $L_\kappa = 2$ and d_H denotes the Hausdorff semi-metric.

(m) Continuity : For every $Q \in \mathscr{B}_X$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|\phi(Q) - \phi(Q_1)| \leq \varepsilon$ for all Q_1satisfying $d_H(Q, Q_1) < \delta$.

Theorem 1.49 ([70, 103]). *Let $B(0,1)$ be the unit ball in a Banach space X. Then $\kappa(B(0,1)) = \chi(B(0,1)) = 0$ if X is finite dimensional, and $\kappa(B(0,1)) = 2, \chi(B(0,1)) = 1$ otherwise.*

The next result shows the equivalence between the Kuratowski's measure of noncompactness and the Hausdorff measure of noncompactness.

Theorem 1.50 ([34]). *Let (E,d) be a complete metric space and B be a nonempty and bounded subset of E. Then*

$$\chi(Q) \leq \kappa(Q) \leq 2\chi(Q).$$

1.5.3 The De Blasi Measure of Weak Noncompactness

The measure of weak noncompactness is an MNC in the sense of the general definition provided X is endowed with the weak topology. The important example of a measure of weak noncompactness was defined by De Blasi [64] in 1977 and it is the map $\beta : \mathscr{B}(X) \longrightarrow [0,\infty)$ defined by

$$\beta(Q) = \inf\left\{\varepsilon > 0 : \text{there exists } W \in \mathscr{K}^w(X) \text{ with } \ Q \subset W + \varepsilon B_1\right\}.$$

for every $Q \in \mathscr{B}(X)$. Here, $\mathscr{B}(X)$ means the collection of all nonempty bounded subsets of X and $\mathscr{K}^w(X)$ is the subset of $\mathscr{B}(X)$ consisting of all weakly compact subsets of X. Now, we are going to recall some basic properties of $\beta(\cdot)$.

Properties 2. Let Q_1, Q_2 be two elements of $\mathscr{B}(X)$. Then De Blasi measure of noncompactness has the following properties. For more details and the proof of these properties see [64]

(a) $Q_1 \subset Q_2 \Rightarrow \beta(Q_1) \leq \beta(Q_2)$,

(b) $\beta(Q) = 0 \Leftrightarrow Q$ is relatively weakly compact,

(c) $\beta(Q_1 \cup Q_2) = \max\{\beta(Q_1), \beta(Q_2)\}$,

(d) $\beta(\overline{Q}^w) = \beta(Q)$, where $\overline{Q}^w$ denotes the weak closure of Q,

(e) $\beta(Q_1 + Q_2) \leq \beta(Q_1) + \beta(Q_2)$,

(f) $\beta(\lambda Q) \leq |\lambda|\beta(Q), \lambda \in \mathbb{R}$

(g) $\beta(\text{conv}(Q)) = \beta(Q)$,

(h) $\beta(\cup_{|\lambda| \leq h} \lambda Q) = h\beta(Q)$.

The next lemma due to Ambrosetti has an important role in this work.

Lemma 1.51 ([74]). *Let $V \subset C(\mathrm{J}, X)$ be a bounded and equicontinuous subset. Then the function $t \to \beta(V(t))$ is continuous on* J,

$$\beta_C(V) = \max_{t \in \mathrm{J}} \beta(V(t)),$$

and

$$\beta\left(\int_{\mathrm{J}} u(s)ds\right) \leq \int_{\mathrm{J}} \beta(V(s))ds,$$

where $V(s) = \{u(s) : u \in V\}, s \in \mathrm{J}$ and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(\mathrm{J}, X)$.

On the other hand, the use of the Hausdorff measure χ in practice requires expressing of χ with the help of a handy formula associated with the structure of the underlying Banach space X in which the measure χ is considered. Unfortunately, it turns out that such formulas are known only in a few Banach spaces such as the classical space $C([a,b])$ of real functions defined and continuous on the interval $[a,b]$ or Banach sequence spaces c_0 and ℓ^p. We illustrate this assertion by a few examples.

1.5.4 Measures of Noncompactness in Some Spaces

The Hausdorff MNC in the Spaces $C[a,b]$

Let $C[a,b]$ denote the classical Banach space consisting of all real functions defined and continuous on the interval $[a,b]$. We consider $C[a,b]$ furnished with the standard maximum norm, i.e.,

$$\|x\| = \max_{t \in [a,b]} |x(t)|.$$

Keeping in mind the Arzelà-Ascoli criterion for compactness in $C[a,b]$ we can express the Hausdorff measure of noncompactness in the below described manner.

Namely, for $x \in C[a,b]$ denote by $\omega(x,\varepsilon)$ the modulus of continuity of the function x :

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in [a,b], |t-s| \leq \varepsilon\}.$$

Next, for an arbitrary set $Q \in \mathscr{B}_{C[a,b]}$ let us put :

$$\omega(Q,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in Q\},$$

and

$$\omega_0(Q) = \lim_{\varepsilon \to 0} \omega(Q,\varepsilon). \tag{1.11}$$

It can be shown [33] that for $Q \in \mathscr{B}_{C[a,b]}$ the following equality holds :

$$\chi(Q) = \frac{1}{2}\omega_0(Q).$$

This equality is very useful in applications.

The Hausdorff MNC in the Space c_0

Let c_0 denote the space of all real sequences $x = \{x_n\}$ converging to zero and endowed with the maximum norm, i.e.,

$$\|x\| = \|\{x_n\}\| = \max\{|x_n| : n = 1,2,3,\ldots\}.$$

To describe the formula expressing the Hausdorff measure χ in the space c_0 fix arbitrarily a set $Q \in \mathscr{B}_{c_0}$. Then, it can be shown that the following equality holds (cf. [33]) :

$$\chi(Q) = \lim_{n \to \infty} \left\{ \sup_{x \in Q} \left(\max_{i \geq n} |x_i| \right) \right\}.$$

The formula expressing the Hausdorff measure of noncompactness is also known in the space ℓ^p for $1 \leq p < \infty$ [33]. On the other hand in the classical Banach spaces $L^p(a,b)$ and ℓ^∞ we only know some estimates of the Hausdorff measure of noncompactness with the help of formulas that define measures of noncompactness in those spaces. Refer to [33] for details.

1.6 Basic results from nonlinear functional analysis and Fixed points theory

Fixed point theorems are the basic mathematical tools that help establish the existence and uniqueness of solutions of various kinds of equations. The fixed point method consists of transforming a given problem into a fixed point problem. The fixed points of the transformed problem are thus the solutions of the given problem. In this section, we recall the famous fixed point theorems that we will use to obtain varied existence results. We start with the definition of a fixed point.

Definition 1.52. Let f be an application of a set $\mathbb{E}$ in itself. We call fixed point of f any point $u \in \mathbb{E}$ such that

$$f(u) = u.$$

1.6.1 Some Classical Fixed-Point Theorem

Banach's contraction principle, which guarantees the existence of a single fixed point of a contraction of a complete metric space with values in itself, is certainly the best known of the fixed point theorems. This theorem proved in 1922 by Stefan Banach is based essentially on the notions of Lipschitzian application and of contracting application.

Theorem 1.53. *[89, 126](Banach contraction principle)*
Let E be a complete metric space and let $F : E \to E$ be a contracting application, then F has a unique fixed point.

Definition 1.54. [54](Boyd-Wong Nonlinear Contraction) Assume that $\mathbb{E}$ is a Banach space and $T : \mathbb{E} \to \mathbb{E}$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property :

$$\|Tx - Ty\| \leq \psi(\|x - y\|), \forall x, y \in \mathbb{E}.$$

then, we say that T is a nonlinear contraction.

Theorem 1.55. *(Boyd-Wong Contraction Principle)[54]*
Suppose that $\mathbb{E}$ is a Banach space and $T : \mathbb{E} \to \mathbb{E}$ is a nonlinear contraction. Then T has a unique fixed point in $\mathbb{E}$.

Lemma 1.56. *[89, 126](Schaefer's Fixed-Point Theorem) Let $\mathbb{E}$ be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$\Omega = \{x \in \mathbb{E} : x = \mu Tx, 0 < \mu < 1\}$$

is bounded, Then T has a fixed point in $\mathbb{E}$.

Theorem 1.57. *[8] (Leray-Schauder nonlinear alternative) Let K be a convex subset of a Banach space, and let U be an open subset of K with $0 \in U$. Then every completely continuous map $N : \overline{U} \to K$ has at least one of the following two properties :*

1. *N has a fixed point in $\overline{U}$;*

2. *there is an* $x \in \partial U$ *and* $\lambda \in (0,1)$ *with* $x = \lambda N x$.

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 1.58. *([99, 125])(Mönch's Fixed-Point Theorem) Let D be a bounded, closed and convex subset of a Banach space such that* $0 \in D$*, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}\, N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0 \tag{1.12}$$

holds for every subset V of D, such that μ *is the measure of noncompactness of Kuratowski, then N has a fixed point.*

Lemma 1.59. *([125]) Let D be a bounded, closed and convex subset of the Banach space* $C(\mathrm{J},\mathbb{E})$*, G a continuous function on* $\mathrm{J} \times \mathrm{J}$ *and f a function from* $\mathrm{J} \times \mathbb{E} \longrightarrow \mathbb{E}$ *which satisfies the Caratheodory conditions, and suppose there exists* $p \in L^1(\mathrm{J},\mathbb{R}^+)$ *such that, for each* $t \in \mathrm{J}$ *and each bounded set* $B \subset \mathbb{E}$*, we have*

$$\lim_{h \to 0^+} \mu(f(\mathrm{J}_{t,h} \times B)) \leq p(t)\mu(B); \;\; here \quad \mathrm{J}_{t,h} = [t-h,t] \cap \mathrm{J}.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_{\mathrm{J}} G(s,t) f(s,y(s))ds : y \in V\right\}\right) \leq \int_{\mathrm{J}} \|G(t,s)\| p(s)\mu(V(s))ds.$$

1.6.2 A Fixed Point Result in a Banach Algebra

In this section, we recall some definitions and we give some results that we will need in the sequel.

Definition 1.60. An algebra $\mathscr{X}$ is a vector space endowed with an internal composition law noted by $(\cdot)$ that is,

$$\begin{cases} \mathscr{X} \times \mathscr{X} & \longrightarrow \mathscr{X} \\ (x,y) & \longrightarrow x \cdot y, \end{cases}$$

which is associative and bilinear.
A normed algebra is an algebra endowed with a norm satisfying the following property

$$\text{for all } x,y \in \mathscr{X} \;\; \|x \cdot y\| \leq \|x\|\|y\|.$$

A complete normed algebra is called a Banach algebra.

The following hybrid fixed point theorem for three operators in a Banach algebra $\mathscr{X}$ due to Dhage [63] will be used to prove the existence result for the nonlocal boundary value problem.

Lemma 1.61. *Let S be a closed convex , bounded and nonempty subset of a Banach algebra* $\mathscr{X}$*, and let* $\mathscr{A},\mathscr{C} : \mathscr{X} \longrightarrow \mathscr{X}$ *and* $\mathscr{B} : S \longrightarrow \mathscr{X}$ *be three operators such that*
(a) $\mathscr{A}$ *and* $\mathscr{C}$ *are Lipschitzian with Lipschitz constants* δ *and* ξ *, respectively ;*
(b) $\mathscr{B}$ *is compact and continuous ;*
(c) $x = \mathscr{A}x\mathscr{B}y + \mathscr{C}x \Rightarrow x \in S$ *for all* $y \in S$,
(d) $\delta M + \xi < 1$ *where* $M = \left\|\mathscr{B}(S)\right\|$.
Then the operator equation $\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x$ *has a solution in S.*

1.6.3 Topological Degree Theory For Condensing Maps

We start this subsection by introducing some necessary definitions, proposition of Isaia degree topology (See [6, 62]).

Definition 1.62. (Linear operator) Let $\mathscr{U}$ and $\mathscr{V}$ are two linear spaces over the same scalar field $\mathscr{F}$. Then a function $\mathscr{T}$ with domain $D(\mathscr{T})$ in $\mathscr{U}$ and range $R(\mathscr{T})$ in $\mathscr{V}$ is called linear operator if

$$\mathscr{T}(\alpha u+\beta v)=\alpha\mathscr{T}(u)+\beta\mathscr{T}(v), \text{ for all } u,v\in\mathscr{U}$$

Definition 1.63. Let $\mathscr{T}:A\longrightarrow\mathscr{U}$ be a continuous bounded map and $A\subset\mathscr{U}$. The operator $\mathscr{T}$ is said to be κ-Lipschitz if we can find a constant $\ell\geq 0$ satisfying the following condition,

$$\kappa(\mathscr{T}(B))\leq\ell\kappa(B), \text{ for every } B\subset A.$$

Moreover, $\mathscr{T}$ is called strict κ-contraction if $\ell<1$.

Definition 1.64. The function $\mathscr{T}$ is called κ-condensing if

$$\kappa(\mathscr{T}(B))<\kappa(B),$$

for every bounded and nonprecompact subset B of A.
In other words,

$$\kappa(\mathscr{T}(B))\geq\kappa(B), \text{ implies } \kappa(B)=0.$$

Further we have $\mathscr{T}:A\longrightarrow\mathscr{U}$ is Lipschitz if we can find $\ell>0$ such that

$$\|\mathscr{T}(u)-\mathscr{T}(v)\|\leq\ell\|u-v\|, \text{ for all } u,v\in A,$$

if $\ell<1$, $\mathscr{T}$ is said to be strict contraction.

For the following results, we refer to [81].

Proposition 1.65. *If $\mathscr{T},\mathscr{S}:A\longrightarrow\mathscr{U}$ are κ-Lipschitz mapping with constants ℓ_1 and ℓ_2 respectively, then $\mathscr{T}+\mathscr{S}:A\longrightarrow\mathscr{U}$ are κ-Lipschitz with constants $\ell_1+\ell_2$.*

Proposition 1.66. *If $\mathscr{T}:A\longrightarrow\mathscr{U}$ is compact, then $\mathscr{T}$ is κ-Lipschitz with constant $\ell=0$.*

Proposition 1.67. *If $\mathscr{T}:A\longrightarrow\mathscr{U}$ is Lipschitz with constant ℓ, then $\mathscr{T}$ is κ-Lipschitz with the same constant ℓ.*

Isaia [81] present the following results using topological degree theory.

Theorem 1.68. *Let $\mathscr{K}:A\longrightarrow\mathscr{U}$ be κ-condensing and*

$$\Theta=\{u\in\mathscr{U}: \textit{ there exist } \xi\in[0,1] \textit{ such that } x=\xi\mathscr{K}u\}.$$

If Θ is a bounded set in $\mathscr{U}$, so there exists $r>0$ such that $\Theta\subset\mathrm{B}_r(0)$, then the degree

$$\deg(I-\xi\mathscr{K},\mathrm{B}_r(0),0)=1, \textit{ for all } \xi\in[0,1].$$

Consequently, $\mathscr{K}$ has at least one fixed point and the set of the fixed points of $\mathscr{K}$ lies in $\mathrm{B}_r(0)$.

Chapitre 2

Caputo-Hadamard Fractional Differential Equations with Hadamard Integral Boundary Conditions

2.1 Introduction

In this chapter, we are concerned with the existence and uniqueness of solutions for certain classes of nonlinear fractional differential equations via Caputo-Hadamard fractional derivative. Sufficient and necessary conditions will be presented for the existence and uniqueness of the solution of fractional boundary value problem, First, we investigate the problem of existence and uniqueness for a boundary value problem for fractional differential equations with fractional integral boundary conditions. By applying some standard fixed point theorems. Next, we extend the study of the existence of solutions on an arbitrary Banach space. by applying the Mönch's fixed point theorem combined with the technique of measures of noncompactness. examples are given to illustrate our results. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.

In this chapter, we concentrate on the following boundary value problem,of nonlinear fractional differential equation with fractional integral as well as integer and fractional derivative :

$$ {}_{H}^{C}D_{1^+}^{r}x(t) = f(t,x(t)), \;\; t \in \mathrm{J} := [1,T], \;\; 0 < r \leq 1. \tag{2.1} $$

with fractional boundary conditions :

$$ \alpha x(1) + \beta x(T) = \lambda\, {}_{H}I_{1^+}^{q}x(\eta) + \delta, \qquad q \in (0,1] \tag{2.2} $$

where ${}_{H}^{C}D_{1^+}^{r}$ denote the Caputo-Hadamard fractional derivative and ${}_{H}I_{1^+}^{q}$ denotes the standard Hadamard fractional integral. Throughout this work, we always assume that $0 < r, q \leq 1$, $f : [1,T] \times \mathbb{E} \to \mathbb{E}$ is continuous. α, β, λ are real constants, and $\eta \in (1,T)$, $\delta \in \mathbb{E}$.

2.2 Existence of solutions

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo-Hadamard derivative.

Definition 2.1. A function $x(t) \in AC^1_\delta(\mathrm{J},\mathbb{E})$ is said to be a solution of (2.1) – (2.2) if x satisfies the equation ${}^C_H D^r_{1^+}x(t) = f(t,x(t))$ on J , and conditions (2.2).

For the existence of solutions for the problem (2.1) – (2.2), we need the following auxiliary lemma.

Lemma 2.2. *Let $h : [1,T] \to \mathbb{E}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = {}_H I^r_{1^+}h(t) + \frac{1}{\Lambda}\left\{\lambda\, {}_H I^{r+q}_{1^+}h(\eta) - \beta\, {}_H I^r_{1^+}h(T) + \delta\right\}, \quad \Lambda \neq 0. \tag{2.3}$$

where

$$\Lambda = \left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right),$$

if and only if x is a solution of the fractional BVP

$${}^C_H D^r_{1^+}x(t) = h(t), \quad t \in \mathrm{J}, \quad r \in (0,1] \tag{2.4}$$

$$\alpha x(1) + \beta x(T) = \lambda\, {}_H I^q_{1^+}x(\eta) + \delta, \quad q \in (0,1] \tag{2.5}$$

Proof. Assume that x satisfies (2.4). Then

$$x(t) = {}_H I^r_{1^+}h(t) + c_1. \tag{2.6}$$

By applying the boundary conditions (2.5) in (2.6), we obtain

$$\alpha c_1 + \beta\, {}_H I^r_{1^+}h(T) + \beta c_1 = \lambda\, {}_H I^{r+q}_{1^+}h(\eta)) + c_1\frac{\lambda(\log\eta)^q}{\Gamma(q+1)} + \delta$$

Thus,

$$c_1\left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right) = \lambda\, {}_H I^{r+q}_{1^+}h(\eta)) - \beta\, {}_H I^r_{1^+}h(T) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda}\left\{\lambda\, {}_H I^{r+q}_{1^+}h(\eta)) - \beta\, {}_H I^r_{1^+}h(T) + \delta\right\}.$$

Where,

$$\Lambda = \left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right)$$

Finally ,we obtain the solution (2.3)

$$x(t) = {}_H I^r_{1^+}h(t) + \frac{1}{\Lambda}\left\{\lambda\, {}_H I^{r+q}_{1^+}h(\eta) - \beta\, {}_H I^r_{1^+}h(T) + \delta\right\}$$

□

2.3 Caputo-Hadamard Fractional Differential Equations with Hadamard Integral Boundary Conditions

2.3.1 First result [1]

In the following subsections, we prove the existence and uniqueness results, for the boundary value problem (2.1) – (2.2) with $\mathbb{E} = \mathbb{R}$ by using a variety of fixed point theorems.

Existence and uniqueness result via Banach's fixed point theorem :

Theorem 2.3. *Assume the following hypothesis :*
(H1) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|,$$

then the problem (2.1)-(2.2) has a unique solution on J *if*

$$LM < 1, \tag{2.7}$$

with

$$M := \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\}.$$

.

Proof. Transform the problem (2.1)-(2.2) into a fixed point problem for the operator $\mathfrak{F}$ defined by

$$\mathfrak{F}x(t) = {}_H I_{1^+}^r h(t) + \frac{1}{\Lambda}\left\{ \lambda\, {}_H I_{1^+}^{r+q} h(\eta) - \beta\, {}_H I_{1^+}^r h(T) + \delta \right\} \tag{2.8}$$

Applying the Banach contraction mapping principle, we shall show that $\mathfrak{F}$ is a contraction.
Now let $x, y \in C(\mathrm{J}, \mathbb{R})$. Then, for $t \in \mathrm{J}$, we have

$$\begin{aligned}
|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t (\log\frac{t}{s})^{r-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&+ \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log\frac{\eta}{s})^{r+q-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&+ \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log\frac{T}{s})^{r-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&\leq L|x-y| \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \\
&:= LM\|x-y\|
\end{aligned}$$

Thus

$$\|(\mathfrak{F}x) - (\mathfrak{F}y)\|_\infty \leq LM\|x-y\|_\infty.$$

We deduce that $\mathfrak{F}$ is a contraction mapping. As a consequence of the Banach contraction principle. the problem (2.1)-(2.2) has a unique solution on J. This completes the proof.

□

1. **A. Boutiara**,M. Benbachir, K. Guerbati, Caputo-Hadamard fractional differential equations with Hadamard integral boundary conditions, *Facta Universitatis* **2020** (2020).

Existence result via Schaefer's fixed point theorem :

Theorem 2.4. *Assume the hypothesis :*
(H2) : The function $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
Then, the problem (2.1)-(2.2) *has at least one solution in* J.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathfrak{F}$ defined by (2.8) has a fixed point. The proof will be given in several steps.
Step 1 : $\mathfrak{F}$ is continuous Let x_n be a sequence such that $x_n \to x$ in $C(\mathrm{J},\mathbb{R})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\| &\leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s}\\ &+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s}\\ &+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s}\\ &\leq \left\{\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\|f(s,x_n(s))-f(s,x(s))\|.\end{aligned}$$

Since f is continuous, we have $\|(\mathfrak{F}x_n)-(\mathfrak{F}x)\|_\infty \to 0$ as $n\to\infty$.
Step 2 : $\mathfrak{F}$ maps bounded sets into bounded sets in $C(\mathrm{J},\mathbb{R})$
Indeed, it is enough to show that for any $r>0$, we take

$$u \in B_r = \{x \in C(\mathrm{J},\mathbb{R}), \|x\|_\infty \leq r\}.$$

From (H1), Then we have

$$|f(s,x(s))| \leq |f(s,x(s))-f(t,0)|+|f(t,0)| \leq Lr+K, \quad K=\sup_{t\in\mathrm{J}}|f(t,0)|.$$

For $x\in B_r$ and for each $t\in[1,T]$, we have

$$\begin{aligned}|(\mathfrak{F}x)(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}|f(s,x(s))|\frac{ds}{s}+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}|f(s,x(s))|\frac{ds}{s}\\ &+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}|f(s,x(s))|\frac{ds}{s}+\frac{|\delta|}{|\Lambda|}\\ &\leq \frac{Lr+K}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\frac{ds}{s}+\frac{|\lambda|(Lr+K)}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\frac{ds}{s}\\ &+\frac{|\beta|(Lr+K)}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}\frac{ds}{s}+\frac{|\delta|}{|\Lambda|}\\ &\leq (Lr+K)\left\{\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}+\frac{|\delta|}{|\Lambda|}\\ &\leq (Lr+K)M+\frac{|\delta|}{|\Lambda|}\end{aligned}$$

Thus,

$$\|\mathfrak{F}x\| \leq (Lr+K)M+\frac{|\delta|}{|\Lambda|}$$

Step 3 : $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C(\mathrm{J},\mathbb{R})$.
Let $t_1,t_2 \in \mathrm{J}, t_1 < t_2$, B_r be a bounded set of $C(\mathrm{J},\mathbb{R})$ as in Step 2, and let $x \in B_r$. Then

$$\begin{aligned}
\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\| &\leq \frac{1}{\Gamma(r)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{r-1}-(\log\frac{t_1}{s})^{r-1}\right]\|f(s,x(s))\|\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}\|f(s,x(s))\|\frac{ds}{s}\\
&\leq \frac{Lr+K}{\Gamma(r)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{r-1}-(\log\frac{t_1}{s})^{r-1}\right]\frac{ds}{s}+\frac{K}{\Gamma(r)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}\frac{ds}{s}\\
&\leq \frac{Lr+K}{\Gamma(r+1)}\left[(\log t_2)^r-(\log t_1)^r\right].
\end{aligned}$$

which implies $\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|_\infty \to 0$ as $t_1 \to t_2$, As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{F}$ is continuous and completely continuous.

Step 4 : A priori bounds.
Now it remains to show that the set

$$\Lambda = \{x \in C(\mathrm{J},\mathbb{R}) : x = \rho\mathfrak{F}(x) \text{ for some } 0 < \rho < 1\}$$

is bounded.

For such a $x \in \Lambda$. Thus, for each $t \in \mathrm{J}$, we have

$$\begin{aligned}
|x(t)| \leq \rho\Bigg\{&\frac{1}{\Gamma(r)}\int_1^t(\log\frac{t}{s})^{r-1}f(s,x(s))\frac{ds}{s}+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}f(s,x(s))\frac{ds}{s}\\
&+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\frac{T}{s})^{r-1}f(s,x(s))\frac{ds}{s}+\frac{|\delta|}{|\Lambda|}\Bigg\}
\end{aligned}$$

For $\rho \in [0,1]$, let x be such that for each $t \in \mathrm{J}$

$$\begin{aligned}
\|\mathfrak{F}x(t)\| &\leq \frac{1}{\Gamma(r)}\int_1^t(\log\frac{t}{s})^{r-1}|f(s,x(s))|\frac{ds}{s}+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}|f(s,x(s))|\frac{ds}{s}\\
&+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\frac{T}{s})^{r-1}|f(s,x(s))|\frac{ds}{s}+\frac{|\delta|}{|\Lambda|}\\
&\leq (Lr+K)M+\frac{|\delta|}{|\Lambda|}
\end{aligned}$$

Thus

$$\|\mathfrak{F}x\| < \infty$$

This implies that the set Λ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathfrak{F}$ has a fixed point which is a solution on J of the problem (2.1)-(2.2).

□

Existence via the Leray-Schauder nonlinear alternative :

Theorem 2.5. *Assume the following hypotheses :*
(H4) There exist $\omega \in L^1(\mathrm{J}, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,x)| \leq \omega(t)\psi(|x|), \textit{ for each } t \in \mathrm{J} \textit{ and each } x \in \mathbb{R}.$$

(H5) There exists a constant $\varepsilon > 0$ such that

$$\frac{\varepsilon}{\|\omega\|\psi(\varepsilon)M + \frac{|\delta|}{|\Lambda|}} > 1.$$

Then the boundary value problem (2.1)-(2.2) has at least one solution on J.

Proof. We shall use the Leray-Schauder theorem to prove that $\mathfrak{F}$ defined by (2.8) has a fixed point. It's shown in Theorem 2.4, we see that the operator $\mathfrak{F}$ is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem $\mathfrak{F}$ is completely continuous.
Let x be such that for each $t \in \mathrm{J}$, we take the equation $x = \lambda \Im x$ for $\lambda \in (0,1)$ and let x be a solution and following the similar computations as in the first step, we have that

$$\begin{aligned}
|x(t)| \leq & \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \omega(t)\psi(\|x\|) \frac{ds}{s} + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \omega(t)\psi(\|x\|) \frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} \omega(t)\psi(\|x\|) \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
& \leq \|\omega\|\psi(\|x\|)M + \frac{|\delta|}{|\Lambda|}.
\end{aligned}$$

and consequently

$$\frac{\|x\|_\infty}{\|\omega\|\psi(\|x\|_\infty)M + \frac{|\delta|}{|\Lambda|}} \leq 1.$$

Then by condition (H5), there exists ε such that $\|x\|_\infty \neq \varepsilon$. Let us set

$$\kappa = \{x \in C(\mathrm{J}, \mathbb{R}) : \|x\| < \varepsilon\}.$$

Obviously, the operator $\Im : \overline{\kappa} \to C(\mathrm{J}, \mathbb{R})$ is completely continuous. From the choice of κ , there is no $x \in \partial\kappa$ such that $x = \lambda \Im(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathfrak{F}$ has a fixed point $x \in \kappa$ which is a solution of the (2.1)-(2.2). The proof is completed.

□

Now we present another variant of existence-uniqueness result.

Existence and uniqueness result via Boyd-Wong nonlinear contraction :

Theorem 2.6. *Assume that $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $H > 0$ satisfying the condition*

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{H + |x-y|}, \textit{ for } t \in \mathrm{J}, x, y \in \mathbb{R}. \tag{2.9}$$

Then the fractional BVP (2.1)-(2.2) has a unique solution on J .

Proof. We define an operator $\mathfrak{F}:\chi\to\chi$ as in (2.8) and a continuous nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\psi(\varepsilon)=\frac{H\varepsilon}{H+\varepsilon},\forall\varepsilon>0,$$

where $M\le H$. We notice that the function ψ satisfies $\psi(0)=0$ and $\psi(\varepsilon)<\varepsilon$ for all $\varepsilon>0$. For any $x,y\in\chi$, and for each $t\in$ J , we obtain

$$\begin{aligned}|(\mathfrak{F}x)(t)-(\mathfrak{F}y)(t)|&\le\frac{1}{\Gamma(r)}\int_1^t(\log\frac{t}{s})^{r-1}\|f(s,x(s))-f(s,y(s))\|\frac{ds}{s}\\&+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}\|f(s,x(s))-f(s,y(s))\|\frac{ds}{s}\\&+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\frac{T}{s})^{r-1}\|f(s,x(s))-f(s,y(s))\|\frac{ds}{s}\\&\le\frac{|x-y|}{H+|x-y|}\left\{\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\\&:=M\frac{|x-y|}{H+|x-y|}\\&\le\psi(\|x-y\|).\end{aligned}$$

Then, we get $\|\mathfrak{F}x-\mathfrak{F}y\|\le\psi(\|x-y\|)$. Hence, $\mathfrak{F}$ is a nonlinear contraction. Thus, by Theorem 1.55 (Boyd-Wong Contraction Principle) the operator $\mathfrak{F}$ has a unique fixed point which is the unique solution of the fractional BVP (2.1)-(2.2). The proof is completed.

□

2.3.2 Example

We consider the problem for Caputo-Hadamard fractional differential equations of the form :

$$\begin{cases}{}^C_HD^{\frac{2}{3}}_{1^+}x(t)=f(t,x(t)),(t,x)\in([1,e],\mathbb{R}^+),\\[2ex] x(1)+x(e)=\frac{1}{2}\left({}_HI^{\frac{1}{2}}_{1^+}x(2)\right)+\frac{3}{4}.\end{cases}\tag{2.10}$$

Here

$$r=\frac{2}{3},\qquad q=\frac{1}{2},\qquad \alpha=1,\qquad \beta=1,$$
$$\delta=\frac{3}{4},\qquad \lambda=\frac{1}{2},\qquad \eta=2,\qquad T=e.$$

With

$$f(t,x(t))=\frac{1}{t^2+4}(cos(x)+\frac{1}{4}),\qquad t\in[1,e]$$

Clearly, the function f is continuous.
For each $x\in\mathbb{R}^+$ and $t\in[1,e]$, we have

$$|f(t,x(t))-f(t,y(t))|\le\frac{1}{4}|x-y|$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{4}$.
Further,

$$M := \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \simeq 2.0286$$

and

$$LM \simeq 0.5071 < 1.$$

Therefore, by the conclusion of Theorem 2.3, It follows that the problem (2.10) has a unique solution defined on $[1,e]$.

2.4 Caputo-Hadamard Fractional Differential Equation with Three-Point Boundary Conditions in Banach Spaces

2.4.1 Second result [2]

This section is devoted to the study of the existence of solutions for problem (2.1)-(2.2), in which The function f is defined by $f : [1,T] \times \mathbb{E} \to \mathbb{E}$ such that $(\mathbb{E}, \|.\|)$ Banach space and $\delta \in \mathbb{E}$. In what follows, we present existence results for the problem (2.1)-(2.2) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type.

In the following,we prove existence results, for the boundary value problem (2.1)-(2.2) by using a Mönch fixed point theorem.
(H1) $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ satisfies the Caratheodory conditions ;
(H2) There exists $p \in C(\mathrm{J}, \mathbb{R}^+)$, such that,

$$\|f(t,x)\| \le p(t)\|x\|, \text{ for } t \in \mathrm{J} \text{ and each } x \in \mathbb{E};$$

(H3) For each $t \in \mathrm{J}$ and each bounded set $B \subset \mathbb{E}$, we have

$$\lim_{h \to 0^+} \mu(f(\mathrm{J}_{t,h} \times B)) \le p(t)\mu(B); \quad here \quad \mathrm{J}_{t,h} = [t-h,t] \cap \mathrm{J}.$$

Theorem 2.7. *Assume that conditions (H1)-(H3) hold. Let $p^* = \sup_{t \in \mathrm{J}} p(t)$. If*

$$p^* M < 1, \tag{2.11}$$

With

$$M := \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\}.$$

Then the BVP (2.1)-(2.2) has at least one solution.

Proof. Transform the problem (2.1)-(2.2) into a fixed point problem. Consider the operator $\mathfrak{F} : C(\mathrm{J}, \mathbb{E}) \to C(\mathrm{J}, \mathbb{E})$ defined by

$$\mathfrak{F}x(t) = {}_H I_{1^+}^r h(t) + \frac{1}{\Lambda} \left\{ \lambda\, {}_H I_{1^+}^{r+q} h(\eta) - \beta\, {}_H I_{1^+}^r h(T) + \delta \right\} \tag{2.12}$$

2. **A. Boutiara**, K. Guerbati, M. Benbachir, Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces, *AIMS Mathematics*, **2020** (2020) , 5(1) : 259–272.

Clearly, the fixed points of the operator $\mathfrak{F}$ are solutions of the problem (2.1)-(2.2). Let

$$R \geq \frac{|\delta|}{|\Lambda|(1-p^*M)}. \tag{2.13}$$

and consider

$$D = \{x \in C(\mathrm{J},\mathbb{E}) : \|x\| \leq R\}.$$

Clearly, the subset D is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

□

Step 1 : First we show that $\mathfrak{F}$ is continuous :
Let x_n be a sequence such that $x_n \to x$ in $C(\mathrm{J},\mathbb{E})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\| \leq & \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s} \\
& + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\frac{ds}{s} \\
& \leq \left\{\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\|f(s,x_n(s))-f(s,x(s))\|
\end{aligned}$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathfrak{F}(x_n)-\mathfrak{F}(x)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2 : Second we show that $\mathfrak{F}$ maps D into itself :
Take $x \in D$, by (H2), we have, for each $t \in \mathrm{J}$ and assume that $\mathfrak{F}x(t) \neq 0$.

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| \leq & \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\|f(s,x(s))\|\frac{ds}{s} + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\|f(s,x(s))\|\frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}\|f(s,x(s))\|\frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
& \leq \frac{1}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}p(s)\|x(s)\|\frac{ds}{s} + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}p(s)\|x(s)\|\frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}p(s)\|x(s)\|\frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
& \leq \frac{P^*R}{\Gamma(r)}\int_1^t (\log\frac{t}{s})^{r-1}\frac{ds}{s} + \frac{|\lambda|P^*R}{|\Lambda|\Gamma(r+q)}\int_1^\eta (\log\frac{\eta}{s})^{r+q-1}\frac{ds}{s} \\
& + \frac{|\beta|P^*R}{|\Lambda|\Gamma(r)}\int_1^T (\log\frac{T}{s})^{r-1}\frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
& \leq P^*R\left\{\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\} + \frac{|\delta|}{|\Lambda|} \\
& \leq P^*RM + \frac{|\delta|}{|\Lambda|} \\
& \leq R.
\end{aligned}$$

Step 3 : we show that $\mathfrak{F}(D)$ is equicontinuous :
By Step 2, it is obvious that $\mathfrak{F}(D) \subset C(\mathrm{J}, \mathbb{E})$ is bounded. For the equicontinuity of $\mathfrak{F}(D)$, let $t_1, t_2 \in \mathrm{J}$, $t_1 < t_2$ and $x \in D$, so $\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1) \neq 0$. Then

$$\begin{aligned}
\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\| \leq & \frac{1}{\Gamma(r)} \int_1^{t_1} \left[(\log \frac{t_2}{s})^{r-1} - (\log \frac{t_1}{s})^{r-1} \right] \|f(s, x(s))\| \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (\log \frac{t_2}{s})^{r-1} \|f(s, x(s))\| \frac{ds}{s} \\
& \leq \frac{R}{\Gamma(r)} \int_1^{t_1} \left[(\log \frac{t_2}{s})^{r-1} - (\log \frac{t_1}{s})^{r-1} \right] p(s) \frac{ds}{s} + \frac{R}{\Gamma(r)} \int_{t_1}^{t_2} (\log \frac{t_2}{s})^{r-1} p(s) \frac{ds}{s} \\
& \leq \frac{Rp^*}{\Gamma(r+1)} \left[(\log t_2)^r - (\log t_1)^r \right].
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
Hence $N(D) \subset D$.

Finally we show that the implication holds :
Let $V \subset D$ such that $V = \overline{conv}(\mathfrak{F}(V) \cup \{0\})$. Since V is bounded and equicontinuous, and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J. By assumption (H2), and the properties of the measure μ we have for each $t \in \mathrm{J}$.

$$\begin{aligned}
v(t) & \leq \mu(\mathfrak{F}(V)(t) \cup \{0\})) \leq \mu((\mathfrak{F}V)(t)) \\
& \leq \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} p(s) \mu(V(s)) \frac{ds}{s} + \frac{|\lambda|}{|\Lambda| \Gamma(r+q)} \int_1^{\eta} (\log \frac{\eta}{s})^{r+q-1} p(s) \mu(V(s)) \frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda| \Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} p(s) \mu(V(s)) \frac{ds}{s} \\
& \leq \frac{\|v\|}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} p(s) \frac{ds}{s} + \frac{|\lambda| \|v\|}{|\Lambda| \Gamma(r+q)} \int_1^{\eta} (\log \frac{\eta}{s})^{r+q-1} p(s) \frac{ds}{s} \\
& + \frac{|\beta| \|v\|}{|\Lambda| \Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} p(s) \frac{ds}{s} \\
& \leq p^* \|v\| \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda| (\log \eta)^{r+q}}{|\Lambda| \Gamma(r+q+1)} + \frac{|\beta| (\log T)^r}{|\Lambda| \Gamma(r+1)} \right\} \\
& := p^* \|v\| M.
\end{aligned}$$

This means that

$$\|v\|(1 - p^* M) \leq 0$$

By (2.11) it follows that $\|v\| = 0$, that is $v(t) = 0$ for each $t \in \mathrm{J}$, and then $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem 1.58, we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (2.1)-(2.2).

2.4.2 Example

Let

$$\mathbb{E} = l^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

with the norm

$$\|x\|_{\mathbb{E}} = \sum_{n=1}^{\infty} |x_n|$$

We consider the problem for Caputo-Hadamard fractional differential equations of the form :

$$\begin{cases} {}^{C}_{H}D^{\frac{2}{3}}_{1^+}x(t) = f(t,x(t)), (t,x) \in ([1,e],\mathbb{E}), \\ \\ x(1)+x(e) = \frac{1}{2}\left({}_{H}I^{\frac{1}{2}}_{1^+}x(2)\right)+\frac{3}{4}. \end{cases} \tag{2.14}$$

Here $r=\frac{2}{3}$, $q=\frac{1}{2}$, $\delta=\frac{3}{4}$, $\lambda=\frac{1}{2}$, $\eta=2$, $T=e$.
With

$$f(t,y(t)) = \frac{t\sqrt{\pi}-1}{16}(x(t)+1), t\in[1,e]$$

Clearly, the function f is continuous. For each $x\in\mathbb{R}^+$ and $t\in[1,e]$, we have

$$|f(t,x(t))| \leq \frac{t\sqrt{\pi}}{16}|x|$$

Hence, the hypothesis (H2) is satisfied with $p^* = \frac{t\sqrt{\pi}}{16}$. We shall show that condition (6.12) holds with $T=e$. Indeed,

$$p^*\left\{\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\} \simeq 0.6109 < 1$$

Simple computations show that all conditions of Theorem 2.7 are satisfied. It follows that the problem (2.14) has at least solution defined on $[1,e]$.

2.5 Conclusion

In this work, we obtained some existence results of nonlinear Caputo-Hadamard fractional differential equations with three-point boundary conditions by using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting some examples. Our results are quite general give rise to many new cases by assigning different values to the parameters involved in the problem. For an explanation, we enlist some special cases.

- We remark that when $\lambda=0$, problem (2.1)-(2.2), the boundary conditions take the form : $\alpha x(1)+\beta x(T)=\delta$ and the resulting problem corresponds to the one considered in [40, 41].
- If we take $\alpha=q=1$, $\beta=0$, in (2.2), then our results correspond to the case integral boundary conditions take the form : $x(1)=\lambda\int_1^e x(s)ds+\delta$ considered in [29].
- By fixing $\alpha=1$, $\beta=\lambda=0$, in (2.2), our results correspond to the ones for initial value problem take the form :$x(1)=\delta$.

- In case we choose $\alpha = \beta = 1$, $\lambda = \delta = 0$, in (2.2), our results correspond to periodic/anti-periodic type boundary conditions take the form : $x(1) = -(\beta/\alpha)x(T)$. In particular, we have the results for anti-periodic type boundary conditions when $(\beta/\alpha) = 1$. For more details on anti-periodic fractional order boundary value problems, see [7].
- Letting $\alpha = 1$, $\beta = \delta = 0$, in (2.2), then our results correspond to the case fractional integral boundary conditions take the form :$x(1) = \lambda I^q x(\eta)$.
- When, $\alpha = \beta = 1$, $\delta = 0$, in (2.2), our results correspond to fractional integral and anti-periodic type boundary conditions.

In the nutshell, the boundary value problem studied in this section is of fairly general nature and covers a variety of special cases.

Chapitre 3

Fractional Differential Equations in Banach Algebras Spaces

3.1 Introduction

The objective of this chapter is to prove the existence of solutions for a class of hybrid fractional differential equations with boundary hybrid conditions in the Banach algebra of all continuous functions on a bounded interval. Our approach mainly depends on a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [63]. Moreover, We present an example to show the validity of conditions and efficiency of our results. The chapter is inspired in the paper [48].

3.2 On the solvability of a system of Caputo-Hadamard fractional hybrid differential equations subject to some hybrid boundary conditions

3.2.1 Introduction

In the last few years, hybrid fractional differential equations have attracted many researchers and achieved a great deal of interest. By hybrid differential equations, we mean that the terms in the equation are perturbed either linearly or quadratically or through the combination of first and second types. Perturbation taking place in the form of the sum or difference of terms in an equation is called linear. On the other hand, if the equation is perturbed through the product or quotient of the terms in it, then it is called quadratic perturbation. So the study of the hybrid differential equation is more general and covers several dynamic systems for some developments on the existence results of hybrid fractional differential equations, we can refer to [63, 132, 59, 48] and the references therein.

This section deals with the existence of solutions on a bounded interval $\mathrm{J} = [0, T]$ for the following hybrid differential equation with boundary hybrid conditions

$$\begin{cases} {}_{H}^{C}D_{1^+}^{r}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]=h(t,x(t)),\ \ t\in \mathrm{J}:=[1,T],\ \ 1<r\leq 2,\\ \alpha_1\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}+\beta_1\,{}_{H}^{C}D_{1^+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}=\gamma_1,\\ \alpha_2\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}+\beta_2\,{}_{H}^{C}D_{1^+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}=\gamma_2 \end{cases} \tag{3.1}$$

where ${}_{H}^{C}D^{\varepsilon}$ and ${}_{H}I_{1^+}^{q_i}$ denotes the Caputo-Hadamard fractional derivatives of orders ε, $\varepsilon\in\{r,p\}$, $0<p\leq 1$ and Hadamard integral of order q_i,respectively, $\alpha_i,\beta_i,\gamma_i, i=1,2$, are real constants, $g\in C(\mathrm{J}\times\mathbb{R},\mathbb{R}-\{0\})$, and $f,h\in C(\mathrm{J}\times\mathbb{R},\mathbb{R})$.

3.2.2 Existence of solutions [1]

By $\hat{\mathbf{E}}=C(\mathrm{J},\mathbb{R})$ we denote the Banach space of all continuous functions from $\mathrm{J}:=[0,T]$ into $\mathbb{R}$ with the norm

$$\|x\|=\sup_{t\in \mathrm{J}}|x(t)|,$$

and a multiplication in $\hat{\mathbf{E}}$ by

$$(xy)(t)=x(t)y(t).$$

Clearly, $\hat{\mathbf{E}}$ is a Banach algebra with respect to the above supremum norm and the multiplication in it. In this section, we give our main existence result for problem (3.1). Before stating Let us define what we mean by a solution to the problem (3.1).

Definition 3.1. A function $x\in C(\mathrm{J},\mathbb{R})$, is said to be a solution of (3.1) if it satisfies the equation ${}_{H}^{C}D_{1^+}^{r}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]=h(t,x(t))$ on J, and the condition

$$\alpha_1\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}+\beta_1\,{}_{H}^{C}D_{1^+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}=\gamma_1,$$
$$\alpha_2\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}+\beta_2\,{}_{H}^{C}D_{1^+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}=\gamma_2$$

The integral form that is equivalent to problem (3.1) is given by the following. In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order on the closed bounded interval J.

Lemma 3.2. *Let h be continuous function on* J. *Then the solution of the boundary value problem*

$${}_{H}^{C}D_{1^+}^{r}\left[\frac{x(t)-\sum_{i=1}^{m}{}_{H}I_{1^+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]=h(t,x(t)),\ \ t\in \mathrm{J},\ \ 1<r\leq 2, \tag{3.2}$$

1. **A. Boutiara**, M. Benbachir, K. Guerbati, On the solvability of a system of Caputo-Hadamard fractional hybrid differential equations subject to some hybrid boundary conditions, Mathematica, (to appear).

with boundary conditions

$$\begin{aligned}&\alpha_1\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_HI_{1+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}+\beta_1\ {}_H^CD_{1+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_HI_{1+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=1}=\gamma_1,\\&\alpha_2\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_HI_{1+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}+\beta_2\ {}_H^CD_{1+}^{p}\left[\frac{x(t)-\sum_{i=1}^{m}\ {}_HI_{1+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]_{t=T}=\gamma_2\end{aligned}\tag{3.3}$$

satisfies the equation

$$\begin{aligned}x(t)=g(t,x(t))&\left[{}_HI_{1+}^{r}h(t)-\frac{(\log t)}{v_1}\left\{\alpha_{2H}I_{1+}^{r}h(T)+\beta_{2H}I_{1+}^{r-p}h(T)\right\}+\frac{\alpha_1v_2(\log t)+\gamma_1v_1}{\alpha_1v_1}\right]\\&+\sum_{i=1}^{m}{}_HI_{1+}^{q_i}f_i(t,x(t)).\end{aligned}\tag{3.4}$$

where

$$v_1=\left(\alpha_2(\log T)+\beta_2\frac{(\log T)^{1-p}}{\Gamma(2-p)}\right),\qquad v_2=\frac{\gamma_2\alpha_1-\gamma_1\alpha_2}{\alpha_1}$$

Proof. Applying the Hadamard fractional integral operator of order r to both sides of (3.2) and using Lemma 1.21,we have

$$\left[\frac{x(t)-\sum_{i=1}^{m}{}_HI_{1+}^{q_i}f_i(t,x(t))}{g(t,x(t))}\right]={}_HI_{1+}^{r}h(t)+c_1+c_2(\log t),\quad c_1,c_2\in\mathbb{R}.\tag{3.5}$$

Consequently, the general solution of (3.2)is

$$x(t)=g(t,x(t))\left({}_HI_{1+}^{r}h(t)+c_1+c_2(\log t)\right)+\sum_{i=1}^{m}{}_HI_{1+}^{q_i}f_i(t,x(t)),\quad c_1,c_2\in\mathbb{R}.\tag{3.6}$$

Applying the boundary conditions (3.3) in (3.5), a simple calculation gives

$$\begin{aligned}c_1&=\frac{\gamma_1}{\alpha_1},\\c_2&=\frac{1}{v_1}\left\{\gamma_2-\frac{\alpha_2\gamma_1}{\alpha_1}-\alpha_{2H}I_{1+}^{r}h(T)-\beta_2\ {}_HI_{1+}^{r-p}h(T)\right\}.\end{aligned}$$

Substituting the values of c_1,c_2 into (3.6), we get (3.4).

□

Now we list the following hypotheses.
(H1) The functions $g:\mathrm{J}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $h,f:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ are continuous.
(H2) There exist two positive functions ω_0,ϖ_1 with bounds $\|\omega_0\|$ and $\|\varpi_1\|$,respectively, such that

$$|g(t,x)-g(t,y)|\le\omega_0(t)|x-y|,\tag{3.7}$$

and

$$|f_i(t,x)-f_i(t,y)|\le\varpi_i(t)|x-y|.\tag{3.8}$$

for all $(t,x,y) \in \mathrm{J} \times \mathbb{R} \times \mathbb{R}$.
(H3) There exist a function $p \in L^{\infty}(\mathrm{J},\mathbb{R}^{+})$ and a continuous nondecreasing function $\varphi : [0,\infty) \to (0,\infty)$ such that

$$|h(t,x)| \leq p(t)\varphi(|x|), \qquad \text{for all } t \in \mathrm{J} \text{ and } x \in \mathbb{R}. \tag{3.9}$$

(H4) There exists $R > 0$ such that

$$R \geq \frac{Mg_0 + M_1}{1 - M\|\omega_0\| - M_2}, \tag{3.10}$$

and

$$\|\omega_0\|M + \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i+1)} < 1. \tag{3.11}$$

where $g_0 = \sup_{t\in\mathrm{J}} |g(t,0)|$, $f_i = \sup_{t\in\mathrm{J}} |f_i(t,0)|$, $i = 1,...,m$, and

$$\begin{aligned} M &= \|p\|\varphi(R)\left\{\frac{(\log T)^{r}}{\Gamma(r+1)} + \frac{|\alpha_2|}{|v_1|}\frac{(\log T)^{r+1}}{\Gamma(r+1)} + \frac{|\beta_2|}{|v_1|}\frac{(\log T)^{r-p+1}}{\Gamma(r-p+1)}\right\} + \frac{|\alpha_1 v_2|(\log t)}{|\alpha_1 v_1|} + \frac{|\gamma_1|}{|\alpha_1|}. \\ M_1 &= \sum_{i=1}^{m} \frac{f_i}{\Gamma(q_i+1)}, \quad M_2 = \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i+1)}. \end{aligned} \tag{3.12}$$

Theorem 3.3. *Assume that conditions (H1)-(H4) holds.Then problem (3.1) has at least one solution defined on* J.

Proof. Define the set

$$S = \left\{x \in \hat{\mathbf{E}} : \|x\|_{\hat{\mathbf{E}}} \leq R\right\}.$$

Clearly, S is a closed convex bounded subset of the Banach space $\hat{\mathbf{E}}$. By Lemma 3.2 the boundary value problem (3.1) is equivalent to the equation

$$\begin{aligned} x(t) = \sum_{i=1}^{m} {}_{H}I_{1^+}^{q_i} f_i(t,x(t)) + g(t,x(t))\big[\, {}_{H}I_{1^+}^{r} h(s,x(s))(t) \\ -\frac{(\log t)}{v_1}\left\{\alpha_2\, {}_{H}I_{1^+}^{r} h(s,x(s))(T) + \beta_2\, {}_{H}I_{1^+}^{r-p} h(s,x(s))(T)\right\} + \frac{\alpha_1 v_2(\log t) + \gamma_1 v_1}{\alpha_1 v_1}\bigg] \end{aligned} \tag{3.13}$$

Define three operators $A, C : \hat{\mathbf{E}} \to \hat{\mathbf{E}}$ and $B : S \to \hat{\mathbf{E}}$ by

$$Ax(t) = g(t,x(t)),\ t \in \mathrm{J},$$

$$\begin{aligned} Bx(t) = {}& {}_{H}I_{1^+}^{r} h(s,x(s))(t) \\ & -\frac{(\log t)}{v_1}\left\{\alpha_2\, {}_{H}I_{1^+}^{r} h(s,x(s))(T) + \beta_2\, {}_{H}I_{1^+}^{r-p} h(s,x(s))(T)\right\} + \frac{\alpha_1 v_2(\log t) + \gamma_1 v_1}{\alpha_1 v_1}, t \in \mathrm{J}, \end{aligned}$$

and

$$Cx(t) = \sum_{i=1}^{m} {}_{H}I_{1^+}^{q_i} f_i(t,x(t)), t \in \mathrm{J}.$$

Then the integral equation (3.13) can be written in the operator form as

$$x(t) = Ax(t)Bx(t) + Cx(t), t \in \mathrm{J}.$$

We will show that the operators A, B,and C satisfy all the conditions of Lemma 1.61. This will be achieved in the following series of steps.

Step1 : First, we show that A and C are Lipschitzian on $\hat{\mathbf{E}}$.

Let $x, y \in \hat{\mathbf{E}}$. Then by (H2), for $t \in \mathrm{J}$, we have

$$|Ax(t) - Ay(t)| = |g(t, x(t)) - g(t, y(t))| \leq \omega_0(t)|x(t) - y(t)|.$$

for all $t \in \mathrm{J}$. Taking the supremum over t, we obtain

$$\|Ax - Ay\| \leq \|\omega_0\|\|x - y\|.$$

for all $x, y \to \hat{\mathbf{E}}$. Therefore A is Lipschitzian on $\hat{\mathbf{E}}$ with Lipschitz constant $\|\omega_0\|$.

Analogously, for $C : \hat{\mathbf{E}} \to \hat{\mathbf{E}}$, $x, y \in \hat{\mathbf{E}}$, we have

$$\begin{aligned}
|Cx(t) - Cy(t)| &= \left| \sum_{i=1}^{m} {}_H I_{1^+}^{q_i} f_i(t, x(t)) - \sum_{i=1}^{m} {}_H I_{1^+}^{q_i} f_i(t, y(t)) \right| \\
&\leq \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_1^t \left(\log \frac{t}{s}\right) \varpi_i(s)|x(s) - y(s)| \frac{ds}{s} \\
&\leq \|x(t) - y(t)\| \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i + 1)}.
\end{aligned}$$

which implies that

$$\|Cx - Cy\| \leq M_2\|x(t) - y(t)\|.$$

Hence $C : \hat{\mathbf{E}} \to \hat{\mathbf{E}}$ is Lipschitzian on $\hat{\mathbf{E}}$ with Lipschitz constant M_2.

Step 2 : The operator B is completely continuous on S. We first show that the operator B is continuous on $\hat{\mathbf{E}}$. Let x_n be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem, for all $t \in$ J, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \frac{1}{\Gamma(r)} \lim_{n\to\infty} \int_1^t \left(\log\frac{t}{s}\right) h(s,x_n(s))\frac{ds}{s} \\
&- \frac{(\log t)}{v_1}\left\{\frac{\alpha_2}{\Gamma(r)} \lim_{n\to\infty} \int_1^T \left(\log\frac{T}{s}\right) h(s,x_n(s))\frac{ds}{s}\right. \\
&\left.+\frac{\beta_2}{\Gamma(r-p)} \lim_{n\to\infty} \int_1^T \left(\log\frac{T}{s}\right) h(s,x_n(s))\frac{ds}{s}\right\} + \frac{\alpha_1 v_2(\log t)+\gamma_1 v_1}{\alpha_1 v_1} \\
&= \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right) \lim_{n\to\infty} h(s,x_n(s))\frac{ds}{s} \\
&- \frac{(\log t)}{v_1}\left\{\frac{\alpha_2}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) \lim_{n\to\infty} h(s,x_n(s))\frac{ds}{s}\right. \\
&\left.+\frac{\beta_2}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) \lim_{n\to\infty} h(s,x_n(s))\frac{ds}{s}\right\} + \frac{\alpha_1 v_2(\log t)+\gamma_1 v_1}{\alpha_1 v_1} \\
&= I^r h(s,x(s))(t) - \frac{(\log t)}{v_1}\left\{\alpha_2 I^r h(s,x(s))(T) + \beta_2 I^{r-p} h(s,x(s))(T)\right\} \\
&+ \frac{\alpha_1 v_2(\log t)+\gamma_1 v_1}{\alpha_1 v_1} \\
&= Bx(t).
\end{aligned}$$

for all $t \in$ J. This shows that B is a continuous operator on S.

Next, we will prove that the set $B(S)$ is a uniformly bounded in S. For any $x \in S$,we have

$$\begin{aligned}
|Bx(t)| \leq &\frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right) |h(s,x(s))|\frac{ds}{s} \\
&+ \frac{(\log t)}{|v_1|}\left\{\frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) |h(s,x(s))|\frac{ds}{s}\right. \\
&\left.+\frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) |h(s,x(s))|\frac{ds}{s}\right\} + \frac{|\alpha_1 v_2|(\log t)+|\gamma_1 v_1|}{|\alpha_1 v_1|}
\end{aligned}$$

Using (3.9), we can write

$$\begin{aligned}
&|Bx(t)| \leq \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right) p(s)\varphi\left(|x|\right)\frac{ds}{s} - \frac{(\log t)}{|v_1|}\left\{\frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log\frac{T}{s}\right) p(s)\varphi\left(|x|\right)\frac{ds}{s}\right. \\
&\left.+\frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log\frac{T}{s}\right) p(s)\varphi\left(|x|\right)\frac{ds}{s}\right\} + \frac{|\alpha_1 v_2|(\log t)+|\gamma_1 v_1|}{|\alpha_1 v_1|} \\
&\leq \|p\|\varphi(R)\left\{\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\alpha_2|}{|v_1|}\frac{(\log T)^{r+1}}{\Gamma(r+1)} + \frac{|\beta_2|}{|v_1|}\frac{(\log T)^{r-p+1}}{\Gamma(r-p+1)}\right\} + \frac{|\alpha_1 v_2|(\log t)+|\gamma_1 v_1|}{|\alpha_1 v_1|}
\end{aligned}$$

Thus $\|Bx\| \leq M$ for all $x \in S$ with M given in (3.12). This shows that B is uniformly bounded on S.

Let $t_1, t_2 \in$ J. Then for any $x \in S$, by (3.9) we get

$$
\begin{aligned}
|Bx(t_2) - Bx(t_1)| \leq & \frac{1}{\Gamma(r)} \left| \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} h(s) \frac{ds}{s} - \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} h(s) \frac{ds}{s} \right| \\
& + \frac{|(\log t_2) - (\log t_1)|}{|v_1|} \left\{ |\alpha_2| I^r h(T) + |\beta_2| I^{r-p} h(T) \right\} + \frac{|\alpha_1 v_2|}{|\alpha_1 v_1|} |(\log t_2) - (\log t_1)| \\
\leq & \frac{\varphi(R)\|p\|}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} + \frac{\varphi(R)\|p\|}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\
& + \frac{|(\log t_2) - (\log t_1)|}{|v_1|} \left\{ |\alpha_2| I^r h(T) + |\beta_2| I^{r-p} h(T) \right\} + \frac{|\alpha_1 v_2|}{|\alpha_1 v_1|} |(\log t_2) - (\log t_1)|.
\end{aligned}
\tag{3.14}
$$

Obviously, the right-hand side of inequality (3.14) tends to zero independently of $x \in S$ as $t_2 \to t_1$. As a consequence of the Ascoli-Arzela theorem, B is a completely continuous operator on S.

Step 3 : Hypothesis (c) of Lemma 1.61 is satisfied.

Let $x \in \hat{\mathbf{E}}$ and $y \in S$ be arbitrary elements such that $x = AxBy + Cx$. Then we have

$$
\begin{aligned}
|x(t)| \leq & |Ax(t)||By(t)| + |Cx(t)| \\
\leq & \sum_{i=1}^{m} {}_H I^{q_i} |f_i(t, x(t))| \\
& + |g(t, x(t))| \left[\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right) |h(s, x(s))| \frac{ds}{s} \right. \\
& + \frac{(\log t)}{|v_1|} \left\{ \frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right) |h(s, x(s))| \frac{ds}{s} \right.
\end{aligned}
$$

$$
\begin{aligned}
& \left. \left. + \frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log \frac{T}{s}\right) |h(s, x(s))| \frac{ds}{s} \right\} + \frac{|\alpha_1 v_2|(\log t) + |\gamma_1 v_1|}{|\alpha_1 v_1|} \right] \\
\leq & \sum_{i=1}^{m} \frac{1}{\Gamma(q_i)} \int_1^t \left(\log \frac{t}{s}\right)^{q_i+1} (|f_i(s, x(s)) - f_i(s, 0)| + |f_i(s, 0)|) \frac{ds}{s} \\
& + (|g(s, x(s)) - g(s, 0)| + |g(s, 0)|) \left[\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right) \varphi(R) p(s) \frac{ds}{s} \right. \\
& + \frac{(\log t)}{|v_1|} \left\{ \frac{|\alpha_2|}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right) \varphi(R) p(s) \frac{ds}{s} \right. \\
& \left. \left. + \frac{|\beta_2|}{\Gamma(r-p)} \int_1^T \left(\log \frac{T}{s}\right) \varphi(R) p(s) \frac{ds}{s} \right\} + \frac{|\alpha_1 v_2|(\log t) + |\gamma_1 v_1|}{|\alpha_1 v_1|} \right] \\
\leq & (\|\omega_0\||x(t)| + g_0)M + \sum_{i=1}^{m} \frac{\|\varpi_i\|}{\Gamma(q_i+1)} |x(t)| + \sum_{i=1}^{m} \frac{f_i}{\Gamma(q_i+1)} \\
\leq & (\|\omega_0\||x(t)| + g_0)M + M_2|x(t)| + M_1
\end{aligned}
$$

Thus

$$
|x(t)| \leq \frac{Mg_0 + M_1}{1 - M\|\omega_0\| - M_2}
$$

Taking the supremum over t,we get

$$\|x\| \leq \frac{Mg_0 + M_1}{1 - M\|\omega_0\| - M_2} \leq R$$

Step4 : Finally, we show that $\delta N + \rho < 1$, that is, (d) of Lemma 1.61 holds.
Since

$$N = \|B(S)\| = \sup_{x \in S} \left\{ \sup_{t \in \mathrm{J}} |Bx(t)| \right\} \leq M,$$

we have

$$\|\omega_0\| N + M_2 \leq \|\omega_0\| M + M_2 < 1$$

with $\delta = \|\omega_0\|$ and $\rho = M_2$ Thus all the conditions of Lemma 1.61 are satisfied, and hence the operator equation $x = AxBx + Cx$ has a solution in S. As a result, problem (6.2) has a solution on J.

$\square$

3.2.3 Example

Consider the following nonlocal hybrid boundary value problem :

$$\begin{cases} {}^{C}_{H}D^{r}_{0^+}\left[\frac{x(t) - {}_{H}I^{q_1} f_1(t,x(t))}{g(t,x(t))}\right] = \frac{e^{-2(\log t)}}{\sqrt{9+t}} \sin x(t), \quad t \in \mathrm{J} := [1,e], \\ \alpha_1 \left[\frac{x(t) - {}_{H}I^{q_1} f_1(t,x(t))}{g(t,x(t))}\right]_{t=1} + \beta_2 \, {}^{C}_{H}D^{p} \left[\frac{x(t) - {}_{H}I^{q_1} f_1(t,x(t))}{g(t,x(t))}\right]_{t=1} = \gamma_1, \\ \alpha_1 \left[\frac{x(t) - {}_{H}I^{q_1} f_1(t,x(t))}{g(t,x(t))}\right]_{t=T} + \beta_2 \, {}^{C}_{H}D^{p} \left[\frac{x(t) - {}_{H}I^{q_1} f_1(t,x(t))}{g(t,x(t))}\right]_{t=T} = \gamma_1 \end{cases} \tag{3.15}$$

We take

$$\begin{array}{llll} m = 1, & r = \frac{3}{2}, & p = \frac{1}{2}, & q_1 = \frac{1}{5}, \\ \alpha_1 = 5, & \alpha_2 = \frac{2}{5}, & \beta_1 = \frac{3}{8}, & \beta_2 = \frac{2}{5}, \\ \gamma_1 = 1, & \gamma_2 = 1, & T = e. & \end{array}$$

$$f_1(t,x(t)) = \frac{(\log t)^2}{100}\left(\frac{1}{2}\left(x(t) + \sqrt{x^2+1}\right) + \log t\right),$$

$$g(t,x(t)) = \frac{\sqrt{\pi}\cos(\pi \log t)}{(7\pi + 15t)^2}\frac{x(t)}{1+x(t)} + \frac{\log t}{10},$$

$$h(t,x(t)) = \frac{e^{-2(\log t)}}{\sqrt{9+t}} \sin x(t).$$

We can show that

$$|f_1(t,x) - f_1(t,y)| \leq \frac{(\log t)^2}{100}|x-y|$$

$$|g(t,x)-g(t,y)| \leq \frac{\sqrt{\pi}}{(7\pi+15(\log t)^2)^2}|x-y|$$

$$h(t,x(t)) \leq p(t)\varphi(|x|)$$

where

$$\varphi(|x|) = |x|, \qquad p(t) = e^{-2(\log t)}$$

Hence we have

$$\omega_0(t) = \frac{(\log t)^2}{100}, \qquad \varpi_1(t) = \frac{\sqrt{\pi}}{(7\pi+15(\log t)^2)^2}$$

Then

$$\|\omega_0\| = \frac{1}{100}, \qquad \|\varpi_1\| = \frac{\sqrt{\pi}}{(7\pi+15)^2}, \qquad \|p\| = 0.1353$$

and

$$g_0 = \sup_{t\in \mathrm{J}} |g(t,0)| = \frac{1}{10}, f_1 = \sup_{t\in \mathrm{J}} |f_1(t,0)| = \frac{1}{100}$$

Using these values, it follows by (3.10) and (3.11) that the constant R satisfies the inequality $0.0035 < R < 3.2552$. As all the conditions of Theorem 3.3 are satisfied, problem (3.15) has at least one solution on J.

Chapitre 4

Hilfer Fractional Differential Equation with Fractional Integral Boundary Conditions

4.1 Introduction

The aim of this chapter is to prove the existence of solutions for certain classes of nonlinear fractional differential equations in Banach Spaces via Hilfer fractional derivative. First, In section 4.2, we investigate the problem of existence and uniqueness for a boundary value problem for fractional differential equations with Katugampola Fractional Integral and Anti-Periodic Conditions in Real spaces. By applying some standard fixed point theorem. Next, In section 4.3, we study the existence of solutions for a boundary value problem for fractional differential equations involving three-point boundary conditions on an arbitrary Banach space. The used approach is based on Mönch's fixed point theorem combined with the technique of measures of noncompactness. Finally, we provide an illustrative example at the end of each section in support of our existence theorems.

4.2 Boundary Value Problems for Hilfer Fractional Differential Equations with Katugampola Fractional Integral and Anti-Periodic Conditions

4.2.1 Introduction

We study in this section some sufficient conditions for the existence and uniqueness of solutions to the following boundary value problem of nonlinear Hilfer fractional differential equation with Katugampola fractional integral and anti-periodic conditions :

$$D_{0^+}^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in \mathrm{J} := [0,T], \tag{4.1}$$

supplemented with the boundary conditions of the form :

$$aI_{0^+}^{1-\gamma}x(0) + bx(T) = \sum_{i=1}^{m} c_i \, {}^{\rho_i}I_{0^+}^{q_i}x(\eta_i) + d, \tag{4.2}$$

where $D^{\alpha,\beta}$ is the Hilfer fractional derivative $0<\alpha<1, 0\leq\beta\leq 1$, $\gamma=\alpha+\beta-\alpha\beta$, ${}^{\rho_i}I^{q_i}_{0^+}$ is the Katugampola integral of $q_i>0$ and $I^{1-\gamma}$ is the Riemann-Liouville integral of order $1-\gamma$, $f:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a,b,d,c_i, i=1,...,m$ are real constants, and $0<\eta_i<T, i=1,...,m$.

In the present work we initiate the study of boundary value problems like (4.1)-(4.2), in which we combine Hilfer fractional differential equations subject to the Katugampola fractional integral boundary conditions.

4.2.2 Existence of solutions [1]

Now, we shall present and prove a preparatory lemma for boundary value problem of linear fractional differential equations with Hilfer derivative.

Definition 4.1. A function $x(t)\in C_{1-\gamma}(\mathrm{J},\mathbb{R})$ is said to be a solution of (4.1)-(4.2) if x satisfies the equation $D^{\alpha,\beta}_{0^+}x(t)=f(t,x(t))$ on J , and the conditions (4.2).

For the existence of solutions for the problem (4.1)-(4.2), we need the following auxiliary lemma.

Lemma 4.2. *Let $h:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t)=I^{\alpha}_{0^+}h(t)+\frac{t^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m}c_i\,{}^{\rho_i}I^{q_i}_{0^+}I^{\alpha}_{0^+}h(\eta_i)-bI^{\alpha}_{0^+}h(T)+d\right\},\tag{4.3}$$

where

$$\Lambda=\left(a\Gamma(\gamma)+bT^{\gamma-1}-\sum_{i=1}^{m}c_i\frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_iq_i+\rho_i-1}{\rho_i})}\frac{\eta_i^{\gamma+\rho_iq_i-1}}{\rho_i^{q_i}}\right).$$

if and only if x is a solution of the fractional BVP

$$D^{\alpha,\beta}_{0^+}x(t)=h(t), t\in\mathrm{J},\tag{4.4}$$

$$aI^{1-\gamma}_{0^+}x(0)+bx(T)=\sum_{i=1}^{m}c_i^{\rho_i}I^{q_i}_{0^+}h(\eta_i)+d.\tag{4.5}$$

Proof. Assume x satisfies (4.4). Then Lemma 1.20 implies that

$$x(t)=I^{\alpha}_{0^+}h(t)+At^{\gamma-1}.\tag{4.6}$$

By applying the boundary conditions (4.5) in (4.6), we obtain

$$\begin{aligned}a A\Gamma(\gamma)+bI^{\alpha}_{0^+}h(T)+b\,A\,T^{\gamma-1}&=\sum_{i=1}^{m}c_i\,{}^{\rho_i}I^{q_i}_{0^+}I^{\alpha}_{0^+}h(\eta_i)\\&\quad+\sum_{i=1}^{m}c_iA\frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_iq_i+\rho_i-1}{\rho_i})}\frac{t^{\gamma+\rho_iq_i-1}}{\rho_i^{q_i}}+d.\end{aligned}$$

1. **A. Boutiara**, M. Benbachir, K. Guerbati, Boundary Value Problems for Hilfer Fractional Differential Equations with Katugampola Fractional Integral and Anti-Periodic Conditions, Mathematica, (to appear).

Thus,

$$A\left(a\Gamma(\gamma)+bT^{\gamma-1}-\sum_{i=1}^{m}c_i\frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_iq_i+\rho_i-1}{\rho_i})}\frac{\eta_i^{\gamma+\rho_iq_i-1}}{\rho_i^{q_i}}\right)=\sum_{i=1}^{m}c_i\,{}^{\rho_i}I_{0^+}^{q_i}I_{0^+}^{\alpha}h(\eta_i)$$
$$-bI_{0^+}^{\alpha}h(T)+d.$$

Consequently,

$$A=\frac{1}{\Lambda}\left\{\sum_{i=1}^{m}c_i\,{}^{\rho_i}I_{0^+}^{q_i}I_{0^+}^{\alpha}h(\eta_i)-bI_{0^+}^{\alpha}h(T)+d\right\},$$

where,

$$\Lambda=\left(a\Gamma(\gamma)+bT^{\gamma-1}-\sum_{i=1}^{m}c_i\frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_iq_i+\rho_i-1}{\rho_i})}\frac{\eta_i^{\gamma+\rho_iq_i-1}}{\rho_i^{q_i}}\right).$$

Finally, we obtain the desired equation (4.3).

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (4.1) – (4.2) by using a variety of fixed point theorems.

Existence and uniqueness result via Banach's fixed point theorem

Theorem 4.3. *Assume the following hypothesis :*
(H1) There exists a constant $L>0$ such that

$$|f(t,x)-f(t,y)|\leq L|x-y|.$$

If

$$L\Psi<1 \tag{4.7}$$

where

$$\Psi:=\left\{\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m}|c_i|\frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_iq_i+\rho_i}{\rho_i})}\frac{\eta_i^{\alpha+\rho_iq_i}}{\rho_i^{q_i}}+|b|\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}.$$

Then the problem (4.1) has a unique solution on J.

Proof. Transform the problem (4.1) – (4.2) into a fixed point problem for the operator Z defined by

$$Zx(t)=I_{0^+}^{\alpha}h(t)+\frac{t^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m}c_i\,{}^{\rho_i}I_{0^+}^{q_i}I_{0^+}^{\alpha}h(\eta_i)-bI_{0^+}^{\alpha}h(T)+d\right\} \tag{4.8}$$

Applying the Banach contraction mapping principle, we shall show that Z is a contraction.

We put $\sup_{t\in[0,T]}|f(t,0)|=M<\infty$, and choose

$$r\geq\frac{M\Psi}{1-L\Psi}.$$

To show that $ZB_r \subset B_r$, where $B_r = \left\{x \in C_{1-\gamma} : \|x\| \leq r\right\}$, we have for any $x \in B_r$

$$\begin{aligned}
|((Zx)(t))t^{1-\gamma}| \leq \sup_{t\in[0,T]} &\Big\{ t^{1-\gamma} I^{\alpha}_{0^+}|f(s,x(s))|(t) \\
&+ \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} c_i \,{}^{\rho_i}I^{q_i}_{0^+} I^{\alpha}_{0^+}|f(s,x(s))|(\eta_i) + bI^{\alpha}_{0^+}|f(s,x(s))|(T) + d \Big\}\Big\} \\
&\leq T^{1-\gamma} I^{\alpha}_{0^+}(|f(s,x(s)) - f(t,0)| + |f(t,0)|)(T) \\
&+ \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} |c_i| \,{}^{\rho_i}I^{q_i}_{0^+} I^{\alpha}_{0^+}(|f(s,x(s)) - f(t,o)| + |f(t,0)|)(\eta_i) \\
&+ |b| I^{\alpha}_{0^+}(|f(s,x(s)) - f(t,0)| + |f(t,0)|)(T) \Big\} + \frac{|d|}{|\Lambda|} \\
&\leq (Lr+M)\Big\{ T^{1-\gamma} I^{\alpha}_{0^+}(1)(T) + \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} |c_i| \,{}^{\rho_i}I^{q_i}_{0^+}(1)(\eta_i) + |b| I^{\alpha}_{0^+}(1)(T) \Big\}\Big\} + \frac{|d|}{|\Lambda|} \\
&:= (Lr+M)\Psi + \frac{|d|}{|\Lambda|} \leq r.
\end{aligned}$$

which implies that $ZB_r \subset B_r$.

Now let $x, y \in C_{1-\gamma}(\mathrm{J}, \mathbb{R})$. Then, for $t \in \mathrm{J}$, we have

$$\begin{aligned}
|((Zx)(t) - (Zy)(t))t^{1-\gamma}| \leq \sup_{t\in[0,T]} &\Big\{ t^{1-\gamma} I^{\alpha}_{0^+}|f(s,x(s)) - f(s,y(s))|(t) \\
&+ \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} c_i \,{}^{\rho_i}I^{q_i}_{0^+} I^{\alpha}_{0^+}|f(s,x(s)) - f(s,y(s))|(\eta_i) + bI^{\alpha}_{0^+}|f(s,x(s)) - f(s,y(s))|(T) \Big\}\Big\} \\
&\leq L\|x-y\| \Big\{ T^{1-\gamma} I^{\alpha}_{0^+}(1)(T) + \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} |c_i| \,{}^{\rho_i}I^{q_i}_{0^+} I^{\alpha}_{0^+}(1)(\eta_i) + |b| I^{\alpha}_{0^+}(1)(T) \Big\}\Big\} \\
&\leq L\|x-y\| \Big\{ \frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|}\Big\{ \sum_{i=1}^{m} |c_i| \frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_i q_i+\rho_i}{\rho_i})} \frac{\eta_i^{\alpha+\rho_i q_i}}{\rho_i^{q_i}} + |b| \frac{T^{\alpha}}{\Gamma(\alpha+1)} \Big\}\Big\} \\
&:= L\Psi\|x-y\|.
\end{aligned}$$

Thus

$$\|((Zx) - (Zy))t^{1-\gamma}\|_\infty \leq L\Psi\|x-y\|_\infty.$$

We deduce that Z is a contraction mapping. As a consequence of Banach contraction principle. the problem (4.1)-(4.2) has a unique solution on J. This completes the proof.

□

Existence result via Schaefer's fixed point theorem

Theorem 4.4. *Assume the hypothesis :*
(H2) : The function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.
Then, the problem (4.1)-(4.2) has a least one solution in J.

Proof. We shall use Schaefer's fixed point theorem to prove that Z defined by (4.8) has a fixed point. The proof will be given in several steps.

Step 1 : Z is continuous Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}(\mathrm{J},\mathbb{R})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}
|((\mathrm{Z}x_n)(t)-(\mathrm{Z}x)(t))t^{1-\gamma}| &\leq t^{1-\gamma} I_{0^+}^{\alpha}\|f(s,x_n(s))-f(s,x(s))\|(t) \\
&+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}\|f(s,x_n(s))-f(s,x(s))\|(\eta_i)+bI_{0^+}^{\alpha}\|f(s,x_n(s))-f(s,x(s))\|(T)\right\} \\
&\leq \left\{T^{1-\gamma} I_{0^+}^{\alpha}(1)(T)+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}(1)(\eta_i)+bI_{0^+}^{\alpha}(1)(T)\right\}\right\}\|f(s,x_n(s))-f(s,x(s))\| \\
&:=\Psi\|f(s,x_n(s))-f(s,x(s))\|.
\end{aligned}$$

Since f is continuous, so $\|((\mathrm{Z}x_n)-(\mathrm{Z}x))t^{1-\gamma}\|_{\infty} \to 0$ as $n \to \infty$.

Step 2 : Z maps bounded sets into bounded sets in $C_{1-\gamma}(\mathrm{J},\mathbb{R})$
Indeed, it is enough to show that for any $r>0$, if we take $x \in B_r=\{x \in C(\mathrm{J},\mathbb{R}), \|x\|_{\infty} \leq r\}$, such that $\mathrm{Z}x(t)$ is bounded. Indeed, from (H3), then for $x \in B_r$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
|((\mathrm{Z}x)(t))t^{1-\gamma}| &\leq t^{1-\gamma} I_{0^+}^{\alpha}|f(s,x(s))|(t) \\
&+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}|f(s,x(s))|(\eta_i)+bI_{0^+}^{\alpha}|f(s,x(s))|(T)+d\right\} \\
&\leq (Lr+M)T^{1-\gamma} I_{0^+}^{\alpha}(1)(T)+\frac{L_1}{|\Lambda|}\left\{\sum_{i=1}^{m}|c_i| \,{}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}(1)(\eta_i)+|b| I_{0^+}^{\alpha}(1)(T)\right\}+\frac{|d|}{|\Lambda|} \\
&\leq (Lr+M)\left\{T^{1-\gamma} I_{0^+}^{\alpha}(1)(T)+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m}|c_i| \,{}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}(1)(\eta_i)+|b| I_{0^+}^{\alpha}(1)(T)\right\}\right\}+\frac{|d|}{|\Lambda|} \\
&:=(Lr+M)\Psi+\frac{|d|}{|\Lambda|}.
\end{aligned}$$

Thus,

$$\|((\mathrm{Z}x))T^{1-\gamma}\| \leq L_1\Psi+\frac{|d|}{|\Lambda|}.$$

Step 3 : G maps bounded sets into equicontinuous sets of $C_{1-\gamma}(\mathrm{J},\mathbb{R})$.
Let $t_1,t_2 \in \mathrm{J}, t_1<t_2$, B_r be a bounded set of $C_{1-\gamma}(\mathrm{J},\mathbb{R})$ as in Step 2, and let $x \in B_r$. Then

$$\begin{aligned}
\|(\mathrm{Z}x(t_2)-\mathrm{Z}x(t_1))t^{1-\gamma}\| &\leq I_{0^+}^{\alpha}|t_2^{1-\gamma}f(s,x(s))(t_2)-t_1^{1-\gamma}f(s,x(s))(t_1)| \\
&\leq \frac{(Lr+M)}{\Gamma(\alpha)}\left|t_2^{1-\gamma}\int_1^{t_1}(t_2-s)^{\alpha-1}(1)ds-t_1^{1-\gamma}\int_1^{t_1}(t_1-s)^{\alpha-1}(1)ds\right| \\
&+\frac{(Lr+M)}{\Gamma(\alpha)}\left|t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1)ds\right| \\
&\leq \frac{(Lr+M)}{\Gamma(\alpha+1)}(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1}).
\end{aligned}$$

which implies $\|\mathrm{Z}(t_2)-\mathrm{Z}x(t_1)\|_{\infty} \to 0$ as $t_1 \to t_2$, As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that Z is continuous and completely continuous.

Step 4 : A priori bounds.
Now it remains to show that the set $\Omega = \{x \in C(\mathrm{J}, \mathbb{R}) : x = \mu \mathrm{Z}(x) \text{ for some } 0 < \mu < 1\}$ is bounded.
For such $x \in \Omega$. Thus, for each $t \in \mathrm{J}$, we have

$$x(t) \leq \mu \left\{ I_{0^+}^{\alpha} h(t) + \frac{t^{\gamma-1}}{\Lambda} \left\{ \sum_{i=1}^{m} c_i \, {}^{\rho_i} I_{0^+}^{q_i} I_{0^+}^{\alpha} h(\eta_i) - b I^{\alpha} h(T) + d \right\} \right\}.$$

For $\mu \in [0,1]$, let x be such that for each $t \in \mathrm{J}$

$$\begin{aligned} \|(\mathrm{Z}x(t))t^{1-\gamma}\| &\leq (Lr+M)\left\{ T^{1-\gamma} I_{0^+}^{\alpha}(1)(T) + \frac{1}{|\Lambda|}\left\{ \sum_{i=1}^{m} |c_i| \, {}^{\rho_i} I_{0^+}^{q_i} I_{0^+}^{\alpha}(1)(\eta_i) + |b| I_{0^+}^{\alpha}(1)(T) \right\} \right\} + \frac{|d|}{|\Lambda|} \\ &:= (Lr+M)\Psi + \frac{|d|}{|\Lambda|}. \end{aligned}$$

Thus

$$\|(\mathrm{Z}x)t^{1-\gamma}\| < \infty$$

This implies that the set Ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Z has a fixed point which is a solution on J of the problem (4.1)-(4.2).

□

Existence result via the Leray-Schauder nonlinear alternative

Theorem 4.5. *Assume the following hypotheses :*
(H4) There exist $\omega \in L^1(\mathrm{J}, \mathbb{R}^+)$ and $\Phi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,x)| \leq \omega(t)\Phi(|x|), \text{ for a.e. } t \in \mathrm{J} \text{ and each } x \in \mathbb{R}.$$

(H5) There exists a constant $\varepsilon > 0$ such that

$$\frac{\varepsilon - \frac{|d|}{|\Lambda|}}{\|\omega\|\Phi(\varepsilon)\Psi} > 1.$$

Then the boundary value problem (4.1)-(4.2) has at least one solution on J.

Proof. We shall use the Leray-Schauder theorem to prove that Z defined by (4.8) has a fixed point. As shown in Theorem 4.4, we see that the operator Z is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem Z is completely continuous.
Let x be such that for each $t \in \mathrm{J}$, we take the equation $x = \rho \mathrm{Z}x$ for $\rho \in (0,1)$ and let x be a

solution. After that, the following is obtained.

$$
\begin{aligned}
&|(x(t))t^{1-\gamma}| \\
&\quad \leq t^{1-\gamma} I_{0^+}^{\alpha} |f(s,x(s))|(t) + \frac{1}{|\Lambda|} \left\{ \sum_{i=1}^{m} c_i \, {}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha} |f(s,x(s))|(\eta_i) + bI^{\alpha}|f(s,x(s))|(T) + d \right\} \\
&\quad \leq \Phi(\|x\|) T^{\gamma-1} I_{0^+}^{\alpha} \omega(s)(T) + \frac{\Phi(\|x\|)}{|\Lambda|} \left\{ \sum_{i=1}^{m} |c_i| \, {}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha} \omega(s)(\eta_i) + |b| I_{0^+}^{\alpha} \omega(s)(T) \right\} + \frac{|d|}{|\Lambda|} \\
&\quad \leq \Phi(\|x\|)\|\omega\| \left\{ I_{0^+}^{\alpha}(1)(T) + \frac{T^{\gamma-1}}{|\Lambda|} \left\{ \sum_{i=1}^{m} |c_i| \, {}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha}(1)(\eta_i) + |b| I_{0^+}^{\alpha}(1)(T) \right\} \right\} + \frac{|d|}{|\Lambda|} \\
&\quad := \Phi(\|x\|)\|\omega\|\Psi + \frac{|d|}{|\Lambda|},
\end{aligned}
$$

which leads to

$$
\frac{\|x\| - \frac{|d|}{|\Lambda|}}{\|\omega\|\Phi(\|x\|)\Psi} \leq 1.
$$

In view of (H5), there exists ε such that $\|x\| \neq \varepsilon$. Let us set $U = \{x \in C_{1-\gamma}(\mathrm{J},\mathbb{R}) : \|x\| < \varepsilon\}$. Obviously, the operator $\mathrm{Z} : \overline{U} \to C_{1-\gamma}(\mathrm{J},\mathbb{R})$ is completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda \mathrm{Z}(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, Z has a fixed point $x \in U$ which is a solution of the (4.1)-(4.2). The proof is completed.

□

Existence and uniqueness result via Boyd-Wong nonlinear contraction

Theorem 4.6. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous function and suppose that there exists $H > 0$ such that :*

$$
|f(t,x) - f(t,y)| \leq z(t) \frac{|x-y|}{H + |x-y|}, \ \text{for } t \in \mathrm{J}, x,y \in \mathbb{R}, \tag{4.9}
$$

where $z : [0,T] \to \mathbb{R}^+$ is continuous and H the constant defined by

$$
H = I_{0^+}^{\alpha} z(T) + \frac{T^{\gamma-1}}{\Lambda} \left\{ \sum_{i=1}^{m} |c_i| \, {}^{\rho_i}I_{0^+}^{q_i} I_{0^+}^{\alpha} z(\eta_i) + |b| I_{0^+}^{\alpha} z(T) \right\}.
$$

Then the fractional BVP (4.1)-(4.2) has a unique solution on J .

Proof. The operator Z is as defined in (4.8) and consider a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$
\psi(\varepsilon) = \frac{H\varepsilon}{H + \varepsilon}, \forall \varepsilon > 0.
$$

notice that the function ψ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in \tau$, and for each $t \in$ J, we obtain

$$
\begin{aligned}
|((\mathrm{Z}x)(t)-(\mathrm{Z}y)(t))t^{1-\gamma}| \leq & \sup_{t\in[0,T]}\Bigg\{t^{1-\gamma}I^{\alpha}_{0^+}|f(s,x(s))-f(s,y(s))|(t) \\
& + \frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i}_{0^+}I^{\alpha}_{0^+}|f(s,x(s))-f(s,y(s))|(\eta_i) + bI^{\alpha}_{0^+}|f(s,x(s))-f(s,y(s))|(T)\right\}\Bigg\} \\
& \leq T^{1-\gamma}I^{\alpha}_{0^+}(z(t)\frac{|x-y|}{H+|x-y|})(T) \\
& + \frac{1}{\Lambda}\left\{\sum_{i=1}^{m}|c_i|\, {}^{\rho_i}I^{q_i}_{0^+}I^{\alpha}_{0^+}(z(t)\frac{|x-y|}{H+|x-y|})(\eta_i) + |b|I^{\alpha}_{0^+}(z(t)\frac{|x-y|}{H+|x-y|})(T)\right\} \\
& \leq \frac{\psi(\|x-y\|)}{H}\left\{T^{1-\gamma}I^{\alpha}_{0^+}z(T) + \frac{1}{\Lambda}\left\{\sum_{i=1}^{m}|c_i|\, {}^{\rho_i}I^{q_i}_{0^+}I^{\alpha}_{0^+}z(\eta_i) + |b|I^{\alpha}_{0^+}z(T)\right\}\right\} \\
& := \psi(\|x-y\|).
\end{aligned}
$$

Then, we get $\|\mathrm{Z}x-\mathrm{Z}y\| \leq \psi(\|x-y\|)$. Hence, Z is a nonlinear contraction. Thus, by Boyd-Wong nonlinear contraction theorem, the operator Z has a unique fixed point which is the unique solution of the fractional BVP (4.1)-(4.2). The proof is completed.

□

4.2.3 Example

Example 4.7. We consider the problem for Hilfer fractional differential equations of the form

$$
\begin{cases}
D^{\frac{2}{3},\frac{1}{2}}_{0^+}x(t) = f(t,x(t)), (t,x) \in ([0,\pi],\mathbb{R}), \\
I^{\frac{2}{6}}_{0^+}x(0) + x(\pi) = \left(\frac{1}{6}I^{\frac{1}{4}}_{0^+}x(1)\right).
\end{cases} \tag{4.10}
$$

Here

$$
\begin{array}{llll}
a = 1, & b = 1, & c = 1, & d = 0, \\
\alpha = \frac{2}{3}, & \beta = \frac{1}{2}, & \gamma = \frac{4}{6}, & q = \frac{1}{4}, \\
\rho = \frac{1}{6}, & \eta = 1, & T = \pi, & m = 1.
\end{array}
$$

With

$$
f(t,x) = \left(\frac{sin^2(\pi t)}{(e^t+10)}\right)\left(\frac{|x|}{|x|+1}+1\right) + \left(\frac{\sqrt{3}}{4}\right), \quad t \in [0,\pi].
$$

Clearly, the function f is continuous.
For each $x \in \mathbb{R}^+$ and $t \in [0,\pi]$, we have

$$
|f(t,x(t)) - f(t,y(t))| \leq \frac{1}{10}|x-y|.
$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{10}$. Further,

$$\Psi := \left\{ \frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|} \left\{ \sum_{i=1}^{m} |c_i| \frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_i q_i+\rho_i}{\rho_i})} \frac{\eta_i^{\alpha+\rho_i q_i}}{\rho_i^{q_i}} + |b| \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right\} \right\} \simeq 5.003$$

and

$$L\Psi \simeq 0.5003 < 1.$$

Therefore, by the conclusion of Theorem (4.3), It follows that the problem (4.10) has a unique solution defined on $[0,\pi]$.

4.2.4 Conclusion

In this work, we have obtained some existence results for nonlinear Hilfer fractional differential equations with Katugampola integral boundary conditions by means of some standard fixed point theorems and nonlinear alternative of Leray-Schauder type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. Our results are new and generalize some available results on the topic.
In all this cases we choose $m = \rho_i = 1$;

- ✓ We remark that when $a = 1, b = c_1 = d = 0$, problem (4.1)-(4.2) reduces to the case initial value problem considered in [133].
- ✓ We remark that when $\beta = 1$, $c_1 = 0$, problem (4.1)-(4.2) reduces to the case initial value problem considered in [38].
- ✓ If we take $a = b = \beta = q = 1$, $d = 0$, in (4.1)-(4.2), then our results correspond to the case integral boundary conditions considered in [39].
- ✓ If we take $a = \beta = q = 1$, $b = 0$, in (4.1)-(4.2), then our results correspond to the case integral boundary conditions considered in [1].
- ✓ If we take $\alpha = 1$, $\beta = \delta = 0$, in (4.2), then our results correspond to the case Fractional integral boundary conditions considered in [15].
- ✓ By fixing ($a = 0, b = 1$) or ($a = 1, b = 0$) and $c_1 = 0, \beta = 1$ in (4.2), our results correspond to the ones for initial value problem take the form : $x(T) = d$ or $x(0) = d$.
- ✓ By fixing $a = 1$, $b = c_1 = 0$, in (4.2), our results correspond to the ones for initial value problem take the form : $I_{0^+}^{1-\gamma} x(0) = d$ considered in [52].
- ✓ In case we choose $a = b = \beta = 1$, $d = c_1 = 0$, in (4.2), our results correspond to anti-periodic type boundary conditions take the form : $x(0) = -x(T)$.
- ✓ When, $a = b = \beta = 1$, $d = 0$, the (4.2), our results correspond to Fractional integral and anti-periodic type boundary conditions.

In other hand, if $m \geq 1$ and $\rho = 1$, we have the case :

- ✓ When, $a = d = 0$, the (4.2), our results correspond to a initial value problem with m-point Fractional integral conditions.

4.3 Measure Of Noncompactness for Nonlinear Hilfer Fractional Differential Equation in Banach Spaces

4.3.1 Introduction

In this section, we will prove the existence of solutions of the following boundary value problem for a nonlinear fractional differential equation with fractional integral boundary conditions :

$$D_{0^+}^{\alpha,\beta}y(t)=f(t,y(t)),t\in \mathrm{J}:=[0,T]. \tag{4.11}$$

with the fractional boundary conditions

$$\begin{aligned}&I_{0^+}^{1-\gamma}y(0)=y_0,I_{0^+}^{3-\gamma-2\beta}y'(0)=y_1,\\&I_{0^+}^{1-\gamma}y(\eta)=\lambda(I_{0^+}^{1-\gamma}y(T)),\gamma=\alpha+\beta-\alpha\beta.\end{aligned} \tag{4.12}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0<\alpha<1$, $0\leq\beta\leq 1$, $0<\lambda<1$, $0<\eta<T$ and let $\mathbb{E}$ be a Banach space space with norm $\|.\|$, $f:\mathrm{J}\times\mathbb{E}\to\mathbb{E}$ is given continuous function.

We will present the existence results for the problem (4.11)-(4.12) which rely on Mönchs fixed point theorem combined with the technique of Kuratowski measure of noncompactness. We recall that when we analyze a problem involving functional operator, one of the best ways consists on the use of the technique of measure of noncompactness, see for instance [33, 42, 36, 8, 99, 125].

4.3.2 Existance Result [2]

First of all, we define what we mean by a solution of the BVP (4.11)-(4.12).

Definition 4.8. A function $y\in C_{1-\gamma}(\mathrm{J},\mathbb{E})$ is said to be a solution of the problem (4.11)-(4.12) if y satisfies the equation $D_{0^+}^{\alpha,\beta}y(t)=f(t,y(t))$ on J , and the conditions $I_{0^+}^{1-\gamma}y(0)=y_0,I_{0^+}^{3-\gamma-2\beta}y'(0)=y_1$, and $I_{0^+}^{1-\gamma}y(\eta)=\lambda(I_{0^+}^{1-\gamma}y(T))$.

Lemma 4.9. *Let $f:\mathrm{J}\times\mathbb{E}\to\mathbb{E}$ be a function such that $f\in C_{1-\gamma}(\mathrm{J},\mathbb{E})$ for any $y\in C_{1-\gamma}(\mathrm{J},\mathbb{E})$. A function $y\in C_{1-\gamma}^{\gamma}(\mathrm{J},\mathbb{E})$ is a solution of the integral equation*

$$\begin{aligned}y(t)=I_{0^+}^{\alpha}f(t,y(t))+\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2}+\zeta(\beta,\gamma,\eta,\lambda)\\ \left\{y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I_{0^+}^{\alpha-\gamma+1}f(T,y(T))-I_{0^+}^{\alpha-\gamma+1}f(\eta,y(\eta))\right\}t^{\gamma+4\beta-3}\end{aligned} \tag{4.13}$$

if and only if y is a solution of the Hilfer fractional BVP

$$D_{0^+}^{\alpha,\beta}y(t)=f(t,y(t)),t\in \mathrm{J}:=[0,T], \tag{4.14}$$

$$I_{0^+}^{1-\gamma}y(0)=y_0,I_{0^+}^{3-\gamma-2\beta}y'(0)=y_1,I_{0^+}^{1-\gamma}y(\eta)=\lambda(I_{0^+}^{1-\gamma}y(T)),\gamma=\alpha+\beta-\alpha\beta. \tag{4.15}$$

2. **A. Boutiara**, M. Benbachir, K. Guerbati, Measure Of Noncompactness for Nonlinear Hilfer Fractional Differential Equation in Banach Spaces, Ikonion Journal of Mathematics, 2019, 1(2).

Proof. Assume y satisfies (4.13). Then Lemma (1.20) implies that

$$y(t) = c_0 t^{\gamma-1} + c_1 t^{\gamma+2\beta-2} + c_2 t^{\gamma+4\beta-3} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds.$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$.
From (4.15), we have
$I_{0^+}^{1-\gamma}y(0) = y_0$ implies that $c_0 = \frac{y_0}{\Gamma(\gamma)}$
$I_{0^+}^{3-\gamma-2\beta}y'(0) = y_1$ implies that $c_1 = \frac{y_1}{\Gamma(\gamma+2\beta-1)}$
$I_{0^+}^{1-\gamma}y(1) = \lambda(I_{0^+}^{1-\gamma}y(T))$ implies that

$$\begin{aligned}
\left(I_{0^+}^{1-\gamma}y\right)(\eta) &= \left(I_{0^+}^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}\right)(\eta) + \left(I_{0^+}^{1-\gamma}\frac{y_1}{\Gamma(\gamma)}t^{\gamma+2\beta-2}\right)(\eta) \\
&\quad + c_2\left(I_{0^+}^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(\eta) + I_{0^+}^{\alpha-\gamma+1}f(\eta,y(\eta)) \\
&= y_0 + \frac{y_1}{\Gamma(2\beta)}\eta^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}\eta^{4\beta-2} + I_{0^+}^{\alpha-\gamma+1}f(\eta,y(\eta)) \\
\left(I_{0^+}^{1-\gamma}y\right)(T) &= \left(I_{0^+}^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}\right)(T) + \left(I_{0^+}^{1-\gamma}\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2}\right)(T) \\
&\quad + c_2\left(I_{0^+}^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(T) + I_{0^+}^{\alpha-\gamma+1}f(T,y(T)) \\
&= y_0 + \frac{y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + I_{0^+}^{\alpha-\gamma+1}f(T,y(T)) \\
\lambda\left(I_{0^+}^{1-\gamma}y\right)(T) &= \lambda y_0 + \frac{\lambda y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\lambda\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + \lambda I_{0^+}^{\alpha-\gamma+1}f(T,y(T))
\end{aligned}$$

that is,

$$c_2 = \Lambda\left\{y_0(\lambda-1) + \frac{\lambda T^{2\beta-1} - \eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I_{0^+}^{\alpha-\gamma+1}f(T,y(T)) - I_{0^+}^{\alpha-\gamma+1}f(\eta,y(\eta))\right\}$$

□

In the sequel, we set the following notations for the sake of computational convenience.

$$\Lambda = \frac{\Gamma(4\beta-1)}{\Gamma(\gamma+4\beta-2)(\eta^{4\beta-2}-\lambda T^{4\beta-2})} \tag{4.16}$$

$$\Delta = \left\{\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{\Lambda}{\Gamma(\alpha-\gamma+2)}\left[\lambda T^{\alpha-\gamma+1} + \eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right\} \tag{4.17}$$

$$\Lambda_1 = \frac{p^* T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + |\Lambda|\left[\frac{\lambda p^* T^{\alpha-\gamma+4\beta-1}}{\Gamma(\alpha-\gamma+2)} + \frac{p^*\eta^{\alpha-\gamma+1}T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}\right] \tag{4.18}$$

$$\Lambda_2 = \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|T^{2\beta-1}}{\Gamma(\gamma+2\beta-1)} + |\Lambda|\left[\|y_0\|(\lambda-1) + \frac{\lambda T^{2\beta-1} - \eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \tag{4.19}$$

The following hypotheses will be used in the sequel.
(H1) $f : \mathrm{J}\times\mathbb{E} \to \mathbb{E}$ satisfies the Caratheodory conditions,
(H2) There exists $p \in C(\mathrm{J},\mathbb{R}^+)$, such that,

$$\|f(t,y)\| \leq p(t)\|y\|, \text{ for } t \in \mathrm{J} \text{ and each } y \in \mathbb{E};$$

(H3) For each $t \in \mathrm{J}$ and each bounded set $B \subset \mathbb{E}$, we have

$$\lim_{h\to 0^+} \mu(f(\mathrm{J}_{t,h} \times B)) \leq t^{1-\gamma} p(t)\mu(B)\,; \text{ here } \mathrm{J}_{t,h} = [t-h,t] \cap \mathrm{J}.$$

Theorem 4.10. *Assume that conditions (H1)-(H3) hold. Let*

$$p^* = \sup_{t\in \mathrm{J}} p(t).$$

If

$$p^*\Delta < 1 \tag{4.20}$$

then the BVP (4.11)-(4.12) has at least one solution.

Proof. Transform the problem (4.11)-(4.12) into a fixed point problem. Consider the operator $\mathcal{N} : C_{1-\gamma}(\mathrm{J},\mathbb{E}) \to C_{1-\gamma}(\mathrm{J},\mathbb{E})$ defined by

$$\mathcal{N}(y)(t) = I_{0^+}^{\alpha} f(t,y(t)) + \frac{y_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2} + \Lambda$$
$$\left[y_0(\lambda-1) + \frac{\lambda T^{2\beta-1} - \eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I_{0^+}^{\alpha-\gamma+1} f(T,y(T)) - I_{0^+}^{\alpha-\gamma+1} f(\eta,y(\eta))\right] t^{\gamma+2(2\beta)-3}$$

Clearly, the fixed points of the operator $\mathcal{N}$ are solutions of the problem (4.11)-(4.12). Let

$$R \geq \frac{\Lambda_2}{1-\Lambda_1} \tag{4.21}$$

and consider

$$D = \{y \in C_{1-\gamma}(\mathrm{J},\mathbb{E}) : \|y\| \leq R\}.$$

Clearly, the subset D is closed, bounded and convex. We shall show that the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

□

Step1 : we show that $\mathcal{N}$ is continuous :
Let y_n be a sequence such that $y_n \to y$ in $C_{1-\gamma}(\mathrm{J},\mathbb{E})$. Then for each $t \in \mathrm{J}$,

$$
\begin{aligned}
&\|t^{1-\gamma}(\mathcal{N}(y_n)(t) - \mathcal{N}(y)(t))\| \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_n(s)) - f(s,y(s))\|ds + \frac{|\Lambda|t^{4\beta-2}}{\Gamma(\alpha-\gamma+1)} \\
&\times \left\{\lambda\int_0^T (T-s)^{\alpha-\gamma}\|f(s,y_n(s)) - f(s,y(s))\|ds + \int_0^{\eta}(\eta-s)^{\alpha-\gamma}\|f(s,y_n(s)) - f(s,y(s))\|ds\right\} \\
&\leq \left(\frac{t^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{|\Lambda|t^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(\lambda T^{\alpha-\gamma+1} + \eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s)) - f(s,y(s))\| \\
&\leq \left(\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{|\Lambda|T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(\lambda T^{\alpha-\gamma+1} + \eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s)) - f(s,y(s))\|
\end{aligned}
$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathcal{N}(y_n)-\mathcal{N}(y)\|_\infty \to 0 \text{ as } n\to\infty.$$

Step2 : we show that $\mathcal{N}$ maps D into itself :
Take $y\in D$, by (H2), we have, for each $t\in$ J and assume that $Ny(t)\neq 0$.

$$\begin{aligned}
\|t^{1-\gamma}\mathcal{N}(y)(t)\| &\leq t^{1-\gamma}I_{0^+}^{\alpha+1}|f(t,y(t))|+\frac{|y_0|}{\Gamma(\gamma)}+\frac{|y_1|}{\Gamma(\gamma+2\beta-1)}t^{2\beta-1}+|\Lambda|t^{4\beta-2}\\
&\times\left\{|y_0|(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+\lambda I_{0^+}^{\alpha-\gamma+1}|f(T,y(T))|+I_{0^+}^{\alpha-\gamma+1}|f(\eta,y(\eta))|\right\}\\
&\leq t^{1-\gamma}I_{0^+}^{\alpha+1}p(t)|y|+\frac{|y_0|}{\Gamma(\gamma)}+\frac{|y_1|}{\Gamma(\gamma+2\beta-1)}t^{2\beta-1}+|\Lambda|t^{4\beta-2}\\
&\times\left\{|y_0|(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+\lambda I_{0^+}^{\alpha-\gamma+1}p(T)|y|+I_{0^+}^{\alpha-\gamma+1}p(\eta)|y|\right\}\\
&\leq T^{1-\gamma}Rp^*I_{0^+}^{\alpha+1}(1)(T)+\frac{|y_0|}{\Gamma(\gamma)}+\frac{|y_1|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1}+|\Lambda|T^{4\beta-2}\\
&\times\left\{|y_0|(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+Rp^*\left\{\lambda I_{0^+}^{\alpha-\gamma+1}(1)(T)+I_{0^+}^{\alpha-\gamma+1}(1)(\eta)\right\}\right\}\\
&\leq\frac{Rp^*T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+|\Lambda|\left[\frac{\lambda Rp^*T^{\alpha-\gamma+4\beta-1}}{\Gamma(\alpha-\gamma+2)}+\frac{Rp^*\eta^{\alpha-\gamma+1}T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}\right]\\
&+\frac{|y_0|}{\Gamma(\gamma)}+\frac{|y_1|T^{2\beta-1}}{\Gamma(\gamma+2\beta-1)}+|\Lambda|\left[|y_0|(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|\right]T^{4\beta-2}\\
&=R\Lambda_1+\Lambda_2\leq R.
\end{aligned}$$

Step3 : we show that $\mathcal{N}(D)$ is equicontinuous :
By Step 2, it is obvious that $\mathcal{N}(D)\subset C_{1-\gamma}(\mathrm{J},\mathbb{E})$ is bounded. For the equicontinuity of $\mathcal{N}(D)$, let $t_1,t_2\in$ J, $t_1<t_2$ and $y\in D$, so $t_2^{1-\gamma}Ny(t_2)-t_1^{1-\gamma}Ny(t_1)\neq 0$. Then

$$\begin{aligned}
\| &\quad t_2^{1-\gamma}Ny(t_2)-t_1^{1-\gamma}Ny(t_1)\|\leq I_{0^+}^{\alpha}|t_2^{1-\gamma}f(s,y(s))(t_2)-t_1^{1-\gamma}f(s,y(s))(t_1)|\\
+ &\quad \frac{1}{\Gamma(\gamma+2\beta-1)}\left|y_1t_2^{2\beta-1}-y_1t_1^{2\beta-1}\right|+|\Lambda|\left(t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)}\right)\\
\times &\quad \left\{\left|y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I_{0^+}^{\alpha-\gamma+1}f(T,y(T))-I_{0^+}^{\alpha-\gamma+1}f(\eta,y(\eta))\right|\right\}\\
\leq &\quad \frac{1}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}|f(s,y(s))|ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}|f(s,y(s))|ds\right.\\
+ &\quad \left.t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,y(s))|ds\right]+\frac{|y_1|}{\Gamma(\gamma+2\beta-1)}\left(t_2^{2\beta-1}-t_1^{2\beta-1}\right)+|\Lambda|\left(t_2^{4\beta-2}-t_1^{4\beta-2}\right)\\
\times &\quad \left\{|y_0(\lambda-1)|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+|\lambda|I_{0^+}^{\alpha-\gamma+1}|f(T,y(T))|-I_{0^+}^{\alpha-\gamma+1}|f(\eta,y(\eta))|\right\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(\alpha)}\Big\{t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}|y|p(s)ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}|y|p(s)ds \\
& +t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|y|p(s)ds\Big\} + \frac{|y_1|}{\Gamma(\gamma+2\beta-1)}\left(t_2^{2\beta-1}-t_1^{2\beta-1}\right)+|\Lambda|\left(t_2^{4\beta-2}-t_1^{4\beta-2}\right) \\
\times\ & \left\{|y_0(\lambda-1)|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+|\lambda|I_{0^+}^{\alpha-\gamma+1}|y|p(s)(T)-I_{0^+}^{\alpha-\gamma+1}|y|p(s)(\eta)\right\} \\
\leq\ & \frac{Rp^*}{\Gamma(\alpha+1)}\left(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1}\right)+\frac{|y_1|}{\Gamma(\gamma+2\beta-1)}\left(t_2^{2\beta-1}-t_1^{2\beta-1}\right)+|\Lambda|\left(t_2^{4\beta-2}-t_1^{4\beta-2}\right) \\
\times\ & \left\{|y_0|(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}|y_1|+\frac{Rp^*(\lambda T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})}{\Gamma(\alpha-\gamma+2)}\right\}
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
Hence $\mathcal{N}(D) \subset D$.

Finally, we show that the implication holds :
Let $V \subset D$ such that $V = \overline{conv}(\mathcal{N}(V) \cup \{0\})$.
We have $V(t) \subset \overline{conv}(\mathcal{N}(V) \cup \{0\})$ for all $t \in$ J. $NV(t) \subset ND(t)$, $t \in$ J is bounded in $\mathbb{E}$.
By assumption (H2), and the properties of the measure μ we have for each $t \in$ J.

$$
\begin{aligned}
t^{1-\gamma}v(t) \leq & \mu(t^{1-\gamma}\mathcal{N}(V)(t)\cup\{0\})) \leq \mu(t^{1-\gamma}(\mathcal{N}V)(t)) \\
& \leq \mu\left(t^{1-\gamma}I_{0^+}^{\alpha}f(s,V(s))(t)\right. \\
& \left.+|\Lambda|t^{4\beta-2}\left\{|\lambda|I_{0^+}^{\alpha-\gamma+1}f(s,V(s))(T)+I_{0^+}^{\alpha-\gamma+1}f(s,V(s))(\eta)\right\}\right) \\
& \leq t^{1-\gamma}I_{0^+}^{\alpha}\mu(f(s,V(s)))(t) \\
& +|\Lambda|t^{4\beta-2}\left\{|\lambda|I_{0^+}^{\alpha-\gamma+1}\mu(f(s,V(s)))(T)+I_{0^+}^{\alpha-\gamma+1}\mu(f(s,V(s)))(\eta)\right\} \\
& \leq t^{1-\gamma}I_{0^+}^{\alpha}(p(s)\mu(V(s)))(t) \\
& +|\Lambda|t^{4\beta-2}\left\{|\lambda|I_{0^+}^{\alpha-\gamma+1}(p(s)\mu(V(s)))(T)+I_{0^+}^{\alpha-\gamma+1}(p(s)\mu(V(s)))(\eta)\right\} \\
& \leq p^*\|v\|\left(T^{1-\gamma}I_{0^+}^{\alpha}(1)(T)+|\Lambda|T^{4\beta-2}\left\{|\lambda|I_{0^+}^{\alpha-\gamma+1}(1)(T)+I_{0^+}^{\alpha-\gamma+1}(1)(\eta)\right\}\right) \\
& \leq p^*\|v\|\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\Lambda|}{\Gamma(\alpha-\gamma+2)}\left(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right)T^{4\beta-2}\right]
\end{aligned}
$$

This means that

$$\|v\| \leq p^*\Delta\|v\|$$

By $p^*\Delta < 1$ it follows that $\|v\| = 0$, that is $v(t) = 0$ for each $t \in$ J , and then $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem 1.58, we conclude that $\mathcal{N}$ has a fixed point which is a solution of the problem (4.11)-(4.12).

4.4 Example

Example 4.11. :
Let $\mathbb{E} = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots) : \sum_{n=1}^{\infty} |y_n| < \infty\}$ with the norm

$$\|y\|_{\mathbb{E}} = \sum_{n=1}^{\infty} |y_n|$$

We consider the problem for Hilfer fractional differential equations of the form :

$$\begin{cases} D_{0^+}^{\alpha,\beta} y(t) = f(t, y(t)), (t, y) \in ([0,1], \mathbb{R}), \\ \\ I_{0^+}^{1-\gamma} y(0) = y_0, I_{0^+}^{3-\gamma-2\beta} y'(0) = y_1, I_{0^+}^{1-\gamma} y(\eta) = \lambda \left(I_{0^+}^{1-\gamma} y(T) \right) \end{cases} \tag{4.22}$$

Here $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{3}{4}$, $\lambda = \frac{1}{2}$, $T = 1$.
With

$$f(t, y(t)) = \frac{1}{4} + \frac{ct^2}{e^{t+4}} (|y(t)| + 1), t \in [0,1]$$

and

$$c = \frac{e^3}{10} \sqrt{\pi}$$

Clearly, the function f is continuous. For each $y \in \mathbb{E}$ and $t \in [0,1]$, we have

$$\|f(t, y(t))\| \leq \frac{ct^2}{e^{t+4}} \|y\|$$

Hence, the hypothesis (H2) is satisfied with $p^* = ce^{-3}$. We shall show that condition (4.20) holds with $T = 1$. Indeed,

$$p^* \left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{\Lambda}{\Gamma(\alpha-\gamma+2)} \left[\lambda T^{\alpha-\gamma+1} + \eta^{\alpha-\gamma+1} \right] T^{2(2\beta)-2} \right] \simeq 0.6 < 1$$

Simple computations show that all conditions of Theorem 4.10 are satisfied. It follows that the problem (4.22) has at least solution defined on $[0,1]$.

Chapitre 5

Caputo type Fractional Differential Equation with Fractional Integral Boundary Conditions

5.1 Introduction

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications, it is worthwhile mentioning the applications of those conditions in the study of population dynamics [53] and cellular systems [5]. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors such as, for instance, Ahmad and Ntouyas [11, 14] Arara and Benchohra [28], Benchohra et al. [37, 38], Infante [80] and the references therein.

In this chapter, we introduce a new approach to study a class of fractional differential equations called : differential equations of fractional orders with integral boundary conditions. More precisely we study the existence and the uniqueness of a solution of a fractional problem involving the Caputo fractional-order with fractional integral boundary conditions in Banach space. The techniques we are going to use consist in showing the existence and uniqueness of the solution on a bounded interval of type $[0,T]$, $T \in \mathbb{N}^*$, by the standard fixed point theorems and Mönch's fixed point theorem combined with the technique of measures of noncompactness.

5.2 Caputo type Fractional Differential Equation with Katugampola Fractional Integral Conditions

In this section, we study the existence and uniqueness of solutions for a boundary value problem, posed in a given Banach space. More specifically, we pose the following boundary value problem, of nonlinear fractional differential equation with fractional integral boundary conditions :

$$^{C}D^{\alpha}_{0^+}x(t) = f(t,x(t)), \quad t \in \mathrm{J} := [0,T], \tag{5.1}$$

With the following boundary conditions

$$x(0) = 0, \quad I^{\beta}_{0^+}x(\varepsilon) \ = \delta^{\rho}\mathrm{J}^{\gamma}_{0^+}x(T). \tag{5.2}$$

where $^{C}D^{\alpha}_{0^+}$ denote the Caputo fractional derivative $1 < \alpha \leq 2$, $I^{\beta}_{0^+}$ denotes the standard Riemann-Liouville fractional integral and $^{\rho}\mathrm{J}^{\gamma}_{0^+}$ Katugampola fractional integral $\gamma > 0$, $\rho > 0$, $\varepsilon \in (0,T)$, and let $\mathbb{E}$ is a reflexive Banach space with norm $\|.\|$, $f : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, δ are real constants.

5.2.1 Existence of solutions [1]

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo derivative.

Definition 5.1. A function $x(t) \in AC^{1}_{\delta}(\mathrm{J},\mathbb{R})$ is said to be a solution of (5.1) – (5.2) if x satisfies the equation $^{C}D^{\alpha}_{0^+}x(t) = f(t,x(t))$ on J, and the conditions (5.2).

For the existence of solutions for the problem (5.1) – (5.2), we need the following auxiliary lemma.

Lemma 5.2. *Let $h : \mathrm{J} \to \mathbb{R}$ be a continuous function . A function x is a solution of the fractional integral equation*

$$x(t) = I^{\alpha}_{0^+}h(t) + \frac{1}{\Lambda}\left\{\delta\ ^{\rho}\mathrm{J}_{0^+}{}^{\gamma}I^{\alpha}_{0^+}h(T) - I^{\alpha+\beta}_{0^+}h(\varepsilon)\right\}, \quad \Lambda \neq 0, \tag{5.3}$$

where

$$\Lambda = \left(\frac{\Gamma(2)}{\Gamma(2+\beta)}\varepsilon^{\beta+1} - \delta\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\gamma+\rho}{\rho})}\frac{T^{\rho\gamma+1}}{\rho^{\gamma}}\right). \tag{5.4}$$

If and only if x is a solution of the fractional BVP

$$^{C}D^{\alpha}_{0^+}x(t) = f(t,x(t)), \quad \textit{for a.e. } t \in \mathrm{J} := [0,T], 1 < \alpha \leq 2. \tag{5.5}$$

$$\begin{cases} x(0) & = 0, \\ I^{\beta}_{0^+}x(\varepsilon) & = \delta^{\rho}\mathrm{J}^{\gamma}_{0^+}x(T). \end{cases} \tag{5.6}$$

Proof. Assume x satisfies (5.5). Then Lemma 1.20 implies that

$$x(t) = I^{r}_{0^+}h(t) + c_1 + c_2 t. \tag{5.7}$$

By applying the boundary conditions (5.6) in (5.7), we obtain
From the condition $x(0) = 0$, we deduce that $c_1 = 0$. Therefore, differentiating (5.7) gives

$$x(t) = I^{r}_{0^+}h(t) + c_2 t. \tag{5.8}$$

1. **A. Boutiara**, M. Benbachir and K. Guerbati, "Caputo type Fractional Differential Equation with Katugampola fractional integral conditions," 2020 2nd International Conference on Mathematics and Information Technology (ICMIT), Adrar, Algeria, 2020, pp. 25-31.

moreover we have

$$I_{0^+}^{\beta}x(\varepsilon) = I_{0^+}^{\alpha+\beta}(\varepsilon)) + c_2\frac{\Gamma(2)}{\Gamma(2+\beta)}\varepsilon^{\beta+1}$$

and

$${}^{\rho}\mathrm{J}_{0^+}^{\gamma}x(T) = {}^{\rho}\,\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}(T)) + c_2\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\gamma+\rho}{\rho})}\frac{T^{\rho\gamma+1}}{\rho^{\gamma}}$$

Thus,

$$c_2\left(\frac{\Gamma(2)}{\Gamma(2+\beta)}\varepsilon^{\beta+1} - \delta\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\gamma+\rho}{\rho})}\frac{T^{\rho\gamma+1}}{\rho^{\gamma}}\right) = \delta^{\rho}\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}(T)) - I_{0^+}^{\alpha+\beta}(\varepsilon))$$

Consequently,

$$c_2 = \frac{1}{\Lambda}\left\{\delta^{\rho}\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}(T)) - I_{0^+}^{\alpha+\beta}(\varepsilon))\right\}.$$

Where,

$$\Lambda = \left(\frac{\Gamma(2)}{\Gamma(2+\beta)}\varepsilon^{\beta+1} - \delta\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\gamma+\rho}{\rho})}\frac{T^{\rho\gamma+1}}{\rho^{\gamma}}\right), \quad \textit{and} \quad \Lambda \neq 0$$

Finally ,we obtain the solution (5.3)

□

In the following subsections, we prove existence (uniqueness) results for the boundary value problem (5.1)-(5.2) by using Banach's fixed point theorem, Schaefer's fixed point theorem, the Leray-Schauder nonlinear alternative, and Boyd-Wong Contraction Principle.

Existence and uniqueness result via Banach's fixed point theorem :

Theorem 5.3. *Assume the following hypotheses :*
(H1) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

(H2) $LM < 1$, where M is defined by

$$M := \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}} + \frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\},$$

where Λ given by (5.4). Then the problem (5.1)-(5.2) *has a unique solution on* J.

Proof. Transform the problem (5.1)-(5.2) into a fixed point problem for the operator $\mathscr{G}$ defined by

$$\mathscr{G}x(t) = I_{0^+}^{\alpha}h(t) + \frac{1}{\Lambda}\left\{\delta^{\rho}\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}h(T) - I_{0^+}^{\alpha+\beta}h(\varepsilon)\right\} \tag{5.9}$$

Applying the Banach contraction mapping principle, we shall show that $\mathscr{G}$ is a contraction.

Now let $x, y \in C(\mathrm{J}, \mathbb{R})$. Then, for $t \in \mathrm{J}$, we have

$$\begin{aligned}
|(\mathscr{G}x)(t)-(\mathscr{G}y)(t)| \leq & \sup_{t\in \mathrm{J}}\{I_{0^+}^{\alpha}|f(s,x(s))-f(s,y(s))|(t) \\
& +\frac{t}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}|f(s,x(s))-f(s,y(s))|(T)+I_{0^+}^{\alpha+\beta}|f(s,x(s))-f(s,y(s))|(\varepsilon)\right\}\Big\} \\
& \leq L\|x-y\|I_{0^+}^{\alpha}(1)(T)+\frac{L\|x-y\|T}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma}I_{0^+}^{\alpha}|(1)(T)+I_{0^+}^{\alpha+\beta}(1)(\varepsilon)\right\} \\
& \leq L\|x-y\|\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{L\|x-y\|T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\} \\
& \leq L\|x-y\|\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\} \\
& := LM\|x-y\|
\end{aligned}$$

Thus

$$\|(\mathscr{G}x)-(\mathscr{G}y)\|_{\infty} \leq LM\|x-y\|_{\infty}.$$

We deduce that $\mathscr{G}$ is a contraction mapping. As a consequence of Banach contraction principle. the problem (5.1)-(5.2) has a unique solution on J. This completes the proof.

□

Existence result via Schaefer's fixed point theorem :

Theorem 5.4. *Assume the hypothesis :*
(H2) : The function $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous.
Then, the problem (5.1)-(5.2) *has at least one solution in* J.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathscr{G}$ defined by (5.9) has a fixed point. The proof will be given in several steps.
Step 1 : $\mathscr{G}$ is continuous Let x_n be a sequence such that $x_n \to x$ in $C(\mathrm{J},\mathbb{R})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}
|(\mathscr{G}x_n)(t)-(\mathscr{G}x)(t)| \leq & I_{0^+}^{\alpha}\|f(s,x_n(s))-f(s,x(s))\|(t) \\
& +\frac{t}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}I_{0^+}^{\alpha}\|f(s,x_n(s))-f(s,x(s))\|(T)+I_{0^+}^{\alpha+\beta}\|f(s,x_n(s))-f(s,x(s))\|(\varepsilon)\right\} \\
& \leq \left\{I_{0^+}^{\alpha}(1)(T)+\frac{L\|x-y\|T}{|\Lambda|}\left\{|\delta|^{\rho}I_{0^+}^{\gamma}|(1)(T)+I_{0^+}^{\alpha+\beta}(1)(\varepsilon)\right\}\right\}\|f(s,x_n(s))-f(s,x(s))\|
\end{aligned}$$

$$\begin{aligned}
& \leq \left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\} \\
& \times\|f(s,x_n(s))-f(s,x(s))\| \\
& := M\|f(s,x_n(s))-f(s,x(s))\|
\end{aligned}$$

Since f is continuous, we have $\|(\mathscr{G}x_n)-(\mathscr{G}x)\|_\infty \to 0$ as $n\to\infty$.

Step 2 : $\mathscr{G}$ maps bounded sets into bounded sets in $C(\mathrm{J},\mathbb{R})$
Indeed, it is enough to show that for any $r>0$, we take

$$u\in B_r=\{x\in C(\mathrm{J},\mathbb{R}),\|x\|_\infty\leq r\}.$$

From (H1), Then we have

$$|f(s,x(s))|\leq|f(s,x(s))-f(t,0)|+|f(t,0)|\leq Lr+K,\quad K=\sup_{t\in\mathrm{J}}|f(t,0)|.$$

For $x\in B_r$ and for each $t\in[1,T]$, we have

$$\begin{aligned}|(\mathscr{G}x)(t)|&\leq I^{\alpha}_{0^+}|f(s,x(s))|(t)+\frac{t}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}_{0^+}I^{\alpha}_{0^+}|f(s,x(s))|(T)+I^{\alpha+\beta}_{0^+}|f(s,x(s))|(\varepsilon)\right\}\\
&\leq(Lr+K)I^{\alpha}_{0^+}(1)(T)+\frac{L_1T}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}_{0^+}I^{\alpha}_{0^+}|(1)(T)+I^{\alpha+\beta}_{0^+}(1)(\varepsilon)\right\}\\
&\leq(Lr+K)\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\}\\
&:=(Lr+K)M\end{aligned}$$

Thus,

$$\|(\mathscr{G}x)\|\leq(Lr+K)M$$

Step 3 : $\mathscr{G}$ maps bounded sets into equicontinuous sets of $C(\mathrm{J},\mathbb{R})$.
Let $t_1,t_2\in\mathrm{J},t_1<t_2$, B_r be a bounded set of $C(\mathrm{J},\mathbb{R})$ as in Step 2, and let $x\in B_r$. Then

$$\begin{aligned}|\mathscr{G}x(t_2)-\mathscr{G}x(t_1)|&\leq|I^{\alpha}_{0^+}f(s,x(s))(t_2)-I^{\alpha}_{0^+}f(s,x(s))(t_1)|\\
&\quad+\frac{|t_2-t_1|}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}_{0^+}I^{\alpha}_{0^+}|f(s,x(s))|(T)+I^{\alpha+\beta}_{0^+}|f(s,x(s))|(\varepsilon)\right\}\\
&\leq\frac{1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]f(s,x(s))ds\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\right|\\
&\quad+\frac{|t_2-t_1|}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}_{0^+}I^{\alpha}_{0^+}|f(s,x(s))|(T)+I^{\alpha+\beta}_{0^+}|f(s,x(s))|(\varepsilon)\right\}\\
&\leq\frac{(Lr+K)}{\Gamma(\alpha)}\left|\int_1^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right|+\frac{(Lr+K)}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right|\\
&\quad+\frac{(Lr+K)|t_2-t_1|}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}^{\gamma}_{0^+}I^{\alpha}_{0^+}|f(s,x(s))|(T)+I^{\alpha+\beta}_{0^+}|f(s,x(s))|(\varepsilon)\right\}\\
&\leq\frac{(Lr+K)|t_2^{\alpha}-t_1^{\alpha}|}{\Gamma(\alpha+1)}\\
&\quad+\frac{(Lr+K)|t_2-t_1|}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\end{aligned}$$

which implies $\|\mathscr{G}(t_2)-\mathscr{G}x(t_1)\|_\infty\to 0$ as $t_1\to t_2$, As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that $\mathscr{G}$ is continuous and completely

continuous.
Step 4 : A priori bounds.
Now it remains to show that the set

$$\Omega = \{x \in C(\mathrm{J}, \mathbb{R}) : x = \mu \mathscr{G}(x) \text{ for some } 0 < \rho < 1\}$$

is bounded.
For such a $x \in \Omega$. Thus, for each $t \in \mathrm{J}$, we have

$$x(t) \leq \mu \left\{ I_{0^+}^{\alpha} f(s,x(s))(t) + \frac{t}{\Lambda} \left\{ \delta^{\rho} \mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha} f(s,x(s))(T) + I_{0^+}^{\alpha+\beta} f(s,x(s))(\varepsilon) \right\} \right\}$$

For $\mu \in [0,1]$, let x be such that for each $t \in \mathrm{J}$

$$\begin{aligned}
\|\mathscr{G}x(t)\| &\leq (Lr+K) I_{0^+}^{\alpha}(1)(T) + \frac{(Lr+K)T}{|\Lambda|} \left\{ |\delta|^{\rho} \mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}(1)(T) + I_{0^+}^{\alpha+\beta}(1)(\varepsilon) \right\} \\
&\leq (Lr+K) \left\{ \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|} \left\{ |\delta| \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})} \frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}} + \frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right\} \right\} \\
&:= (Lr+K)M
\end{aligned}$$

Thus

$$\|\mathscr{G}x\| < \infty$$

This implies that the set Ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathscr{G}$ has a fixed point which is a solution on J of the problem (5.1)-(5.2).

□

Existence via the Leray-Schauder nonlinear alternative :

Theorem 5.5. *Assume the following hypotheses :*
(H4) There exist $\omega \in L^1(\mathrm{J}, \mathbb{R}^+)$ and $\Phi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,x)| \leq \omega(t)\Phi(|x|), \textit{ for a.e. } t \in \mathrm{J} \textit{ and each } x \in \mathbb{R}.$$

(H5) There exists a constant $\varepsilon > 0$ such that

$$\frac{\varepsilon}{\|\omega\|\Phi(\varepsilon)M} > 1.$$

Then the boundary value problem (5.1)-(5.2) *has at least one solution on* J.

Proof. We shall use the Leray-Schauder theorem to prove that $\mathscr{G}$ defined by (5.9) has a fixed point. As shown in Theorem 5.4, we see that the operator $\mathscr{G}$ is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem $\mathscr{G}$ is completely continuous.

Let x be such that for each $t \in \mathrm{J}$, we take the equation $x = \lambda \mathscr{G} x$ for $\lambda \in (0,1)$ and let x be a solution. After that, the following is obtained.

$$
\begin{aligned}
|x(t)| &\le I_{0^+}^{\alpha}|f(s,x(s))|(t) + \frac{t}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}|f(s,x(s))|(T) + I_{0^+}^{\alpha+\beta}|f(s,x(s))|(\varepsilon)\right\} \\
&\le \Phi(\|x\|) I_{0^+}^{\alpha}\omega(s)(T) + \frac{\Phi(\|x\|)T}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}|\omega(s)(T) + I_{0^+}^{\alpha+\beta}\omega(s)(\varepsilon)\right\} \\
&\le \|\omega\|\Phi(\|x\|)\left\{I_{0^+}^{\alpha}(1)(T) + \frac{T}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}|(1)(T) + I_{0^+}^{\alpha+\beta}(1)(\varepsilon)\right\}\right\} \\
&\le \|\omega\|\Phi(\|x\|)\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}} + \frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\} \\
&\le \|\omega\|\Phi(\|x\|)M.
\end{aligned}
$$

and consequently

$$
\frac{\|x\|_{\infty}}{\|\omega\|\psi(\|x\|)M} \le 1.
$$

Then by condition (H5), there exists ε such that $\|x\|_{\infty} \neq \varepsilon$. Let us set

$$
\kappa = \{x \in C(\mathrm{J},\mathbb{R}) : \|x\| < \varepsilon\}.
$$

Obviously, the operator $\mathscr{G} : \overline{\kappa} \to C(\mathrm{J},\mathbb{R})$ is completely continuous. From the choice of κ , there is no $x \in \partial\kappa$ such that $x = \lambda\mathscr{G}(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathscr{G}$ has a fixed point $x \in \kappa$ which is a solution of the (5.1)-(5.2). The proof is completed.

□

Existence and uniqueness result via Boyd-Wong nonlinear contraction :

Theorem 5.6. *Assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ are continuous functions and $H > 0$ satisfying the condition*

$$
|f(t,x) - f(t,y)| \le \frac{|x-y|}{H+|x-y|}, \text{ for } t \in \mathrm{J}, x,y \in \mathbb{R}. \tag{5.10}
$$

Then the fractional BVP (5.1)-(5.2) has a unique solution on J .

Proof. We define an operator $\mathscr{G} : \tau \to \tau$ as in (5.9) and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$
\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \forall\varepsilon > 0,
$$

where $M \le H$. We notice that the function ψ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x,y \in \chi$, and for each $t \in \mathrm{J}$, we obtain

$$
\begin{aligned}
|(\mathscr{G}x)(t) - (\mathscr{G}y)(t)| &\le \sup_{t\in\mathrm{J}}\{I_{0^+}^{\alpha}|f(s,x(s)) - f(s,y(s))|(t) \\
&\quad + \frac{t}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}|f(s,x(s)) - f(s,y(s))|(T) + I_{0^+}^{\alpha+\beta}|f(s,x(s)) - f(s,y(s))|(\varepsilon)\right\}\Big\} \\
&\le \frac{|x-y|}{H+|x-y|} I_{0^+}^{\alpha}(1)(T) + \frac{|x-y|}{H+|x-y|}\frac{T}{|\Lambda|}\left\{|\delta|^{\rho}\mathrm{J}_{0^+}^{\gamma} I_{0^+}^{\alpha}|(1)(T) + I_{0^+}^{\alpha+\beta}(1)(\varepsilon)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{|x-y|}{H+|x-y|}\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\} \\
&:= M\frac{|x-y|}{H+|x-y|} \\
&\leq \psi(\|x-y\|).
\end{aligned}
$$

Then, we get $\|\mathscr{G}x-\mathscr{G}y\| \leq \psi(\|x-y\|)$. Hence, $\mathscr{G}$ is a nonlinear contraction. Thus, by Theorem 1.55 the operator $\mathscr{G}$ has a unique fixed point which is the unique solution of the fractional BVP (5.1)-(5.2). The proof is completed.

$\square$

5.2.2 Example

We consider the problem for Caputo fractional differential equations of the form :

$$
\begin{cases}
{}^{C}D_{0^+}^{\frac{3}{2}}x(t) = f(t,x(t)), \quad (t,x)\in([0,\pi],\mathbb{R}), \\
x(0)=0, \quad I_{0^+}^{1/2}x(2) = \frac{1}{2}\,{}^{5}\mathrm{J}_{0^+}^{1/2}x(2)+\frac{3}{4}.
\end{cases}
\tag{5.11}
$$

Here

$$
\begin{aligned}
&\alpha=\frac{3}{2}, && \beta=\frac{1}{2}, && \gamma=\frac{1}{2}, \\
&\rho=5, && \delta=\frac{1}{2}, && \varepsilon=2, \\
&T=\pi
\end{aligned}
$$

With

$$
f(t,y(t)) = \frac{1}{(t+5)^2}(\tan^{-1}(x)+\frac{\pi}{5}), \quad t\in[0,\pi]
$$

Clearly, the function f is continuous. For each $x\in\mathbb{R}^+$ and $t\in[0,\pi]$, we have

$$
|f(t,x(t))-f(t,y(t))| \leq \frac{1}{25}|x-y|
$$

Hence, the hypothesis (H1) is satisfied with

$$
L=\frac{1}{25}
$$

. Further,

$$
M := \left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T}{|\Lambda|}\left\{|\delta|\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\alpha\rho+\rho}{\rho})}\frac{T^{\alpha+\rho\gamma}}{\Gamma(\alpha+1)\rho^{\gamma}}+\frac{\varepsilon^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\right\}
$$

and

$$
LM<1.
$$

Therefore, by the conclusion of Theorem 5.3, It follows that the problem (5.11) has a unique solution defined on $[0,\pi]$.

5.3 Caputo Type Fractional Differential Dquation with Nonlocal Erdélyi-Kober Type Integral Boundary Conditions in Banach Spaces

Introduction

In this section, we give conditions for the existence of solution for a class of fractional differential equations with fractional integral boundary conditions of the type :

$$^{C}D_{0^+}^{\alpha}x(t) = f(t,x(t)), \quad t \in \mathrm{J} := [0,T], \tag{5.12}$$

associated with the following Erdélyi-Kober fractional integral boundary conditions :

$$\begin{aligned} x(T) &= \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} x(\beta_i), \quad 0 < \beta_i < T, \\ x'(T) &= \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} x'(\sigma_i), \quad 0 < \sigma_i < T, \\ x''(T) &= \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} x''(\varepsilon_i), \quad 0 < \varepsilon_i < T, \end{aligned} \tag{5.13}$$

where $^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $2 < \alpha \leq 3$ and $J_{\eta_i}^{\gamma_i,\delta_i}$ denote Erdélyi-Kober fractional integral of order $\delta_i > 0$, $\eta_i > 0$, $\gamma_i \in \mathbb{R}$ and $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a continuous function, $a_i, b_i, d_i, i = 1,2,\ldots,m$ are real constants. Recall that Erdélyi-Kober fractional integral operators play an important role especially in engineering, for more details on the Erdélyi-Kober fractional integrals, see [90, 67].

In the present paper, we initiate the study of boundary value problems like (5.12)-(5.13), in which Caputo fractional differential equations are matched to Erdélyi-Kober fractional integral boundary conditions. We will present the existence results for the problem (5.12)-(5.13) which rely on Mönchs fixed point theorem combined with the technique of Kuratowski measure of noncompactness. that technique turns out to be a very useful tool in existence for several kinds of integral equations and subsequently developed and used in many papers, see, for instance. The strong measure of noncompactness was considered first by Banaŝ et al. [33], for more details see, [10, 16, 26, 99, 125, 36, 38, 37, 34, 32].

5.3.1 Existence of solutions [2]

For the existence of solutions for the problem (5.12)-(5.13), we need the following auxiliary lemma.

Lemma 5.7. *Let $f : [0,T) \times \mathbb{E} \to \mathbb{E}$ be a continuous function. Then,for any $x \in C(\mathrm{J},\mathbb{E})$, x is a solution of the following nonlinear fractional differential equation with Erdélyi-Kober*

2. **A. Boutiara**, M. Benbachir and K. Guerbati, "Caputo type fractional differential equation with nonlocal Erdélyi-Kober type integral boundary conditions in Banach spaces," Surveys in Mathematics and its Applications, Volume 15 (2020), 399-418.

fractional integral conditions :

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}x(t) = f(t,x(t)), t \in \mathrm{J}, \\ x(T) = \sum_{i=1}^{m} a_i J^{\gamma_i,\delta_i}_{\eta_i} x(\beta_i) \\ x'(T) = \sum_{i=1}^{m} b_i J^{\gamma_i,\delta_i}_{\eta_i} x'(\sigma_i) \\ x''(T) = \sum_{i=1}^{m} d_i J^{\gamma_i,\delta_i}_{\eta_i} x''(\varepsilon_i). \end{cases} \tag{5.14}$$

if and only if

$$\begin{aligned} x(t) &= I^{\alpha}_{0^+} f(t,x(t)) + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^{m} a_i J^{\gamma_i,\delta_i}_{\eta_i} I^{\alpha}_{0^+} f(\beta_i, x(\beta_i)) - I^{\alpha}_{0^+} f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(b_i,\sigma_i)} \left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^{m} b_i J^{\gamma_i,\delta_i}_{\eta_i} I^{\alpha-1}_{0^+} f(\sigma_i, x(\sigma_i)) - I^{\alpha-1}_{0^+} f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i) v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i) v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i) t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2} \right) \\ &\times \left\{ \sum_{i=1}^{m} d_i J^{\gamma_i,\delta_i}_{\eta_i} I^{\alpha-2}_{0^+} f(\varepsilon_i, x(\varepsilon_i)) - I^{\alpha-2}_{0^+} f(T,x(T)) \right\} \end{aligned} \tag{5.15}$$

where

$$v_0(a_i,\beta_i) = \left(1 - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)} \right) \tag{5.16}$$

$$v_1(a_i,\beta_i) = \left(T - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\beta_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)} \right) \tag{5.17}$$

$$v_2(a_i,\beta_i) = \left(T^2 - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+\frac{2}{\eta_i}+1)\beta_i^2}{\Gamma(\gamma_i+\frac{2}{\eta_i}+\delta_i+1)} \right) \tag{5.18}$$

Proof. Using Lemma (1.20), the general solution of the nonlinear fractional differential equation in (5.14) can be represented as

$$x(t) = c_0 + c_1 t + c_2 t^2 + I^{\alpha}_{0^+} h(t), c_0, c_1, c_2 \in \mathbb{R}. \tag{5.19}$$

By using the first integral condition of problem (5.14) and applying the Erdélyi-Kober integral on (5.19), we get

$$\begin{aligned} c_0 + c_1 T + c_2 T^2 + I^{\alpha}_{0^+} h(T) &= \sum_{i=1}^{m} a_i J^{\gamma_i,\delta_i}_{\eta_i} I^{\alpha}_{0^+} h(\beta_i) + c_0 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)} \\ &+ c_1 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\beta_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)} + c_2 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+\frac{2}{\eta_i}+1)\beta_i^2}{\Gamma(\gamma_i+\frac{2}{\eta_i}+\delta_i+1)}. \end{aligned}$$

After collecting the similar terms in one part, we get the following equation :

$$\begin{aligned}c_0\left(1-\sum_{i=1}^{m}a_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)+c_1\left(T-\sum_{i=1}^{m}a_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\beta_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}\right)\\+c_2\left(T^2-\sum_{i=1}^{m}a_i\frac{\Gamma(\gamma_i+\frac{2}{\eta_i}+1)\beta_i^2}{\Gamma(\gamma_i+\frac{2}{\eta_i}+\delta_i+1)}\right)\\=\sum_{i=1}^{m}a_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha}h(\beta_i)-I_{0^+}^{\alpha}h(T).\end{aligned}\tag{5.20}$$

Rewriting equation (5.20) by using (5.16), (5.17), and (5.18), we obtain

$$c_0v_0(a_i,\beta_i)+c_1v_1(a_i,\beta_i)+c_2v_2(a_i,\beta_i)=\sum_{i=1}^{m}a_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha}h(\beta_i)-I_{0^+}^{\alpha}h(T).\tag{5.21}$$

Then, taking the derivative of (5.19) and using the second integral condition of (5.14), one has

$$x'(T)=c_1+c_2T+I_{0^+}^{\alpha-1}h(T).\tag{5.22}$$

Now, applying the Erdélyi-Kober integral on (5.22), we have

$$\begin{aligned}c_1+2c_2T+I_{0^+}^{\alpha-1}h(T)=\sum_{i=1}^{m}b_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha-1}h(\sigma_i)+c_1\sum_{i=1}^{m}b_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\\+2c_2\sum_{i=1}^{m}b_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\sigma_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}.\end{aligned}\tag{5.23}$$

The above equation (5.23) implies that

$$\begin{aligned}c_1\left(1-\sum_{i=1}^{m}b_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)+2c_2\left(T-\sum_{i=1}^{m}b_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\sigma_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}\right)\\=\sum_{i=1}^{m}a_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha-1}h(\sigma_i)-I_{0^+}^{\alpha-1}h(T),\end{aligned}\tag{5.24}$$

also, by using (5.16) and (5.17), equation (5.24) can be written as

$$c_1v_0(b_i,\sigma_i)+c_2v_1(b_i,\sigma_i)=\sum_{i=1}^{m}b_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha-1}h(\sigma_i)-I_{0^+}^{\alpha-1}h(T).\tag{5.25}$$

By using the last integral condition of (5.14) and applying Erdélyi-Kober integral operator on the second derivative of (5.22), we have

$$2c_2+I_{0^+}^{\alpha-2}h(T)=\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha-2}h(\varepsilon_i)+2c_2\sum_{i=1}^{m}d_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}.\tag{5.26}$$

Hence, we obtain the following equation :

$$2c_2\left(1-\sum_{i=1}^{m}d_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)=\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I_{0^+}^{\alpha-2}h(\sigma_i)-I_{0^+}^{\alpha-2}h(T),\tag{5.27}$$

by using (5.16), equation (5.27) can be written as

$$2c_2 v_0(d_i,\varepsilon_i) = \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} h(\varepsilon_i) - I_{0^+}^{\alpha-2} h(T). \tag{5.28}$$

Moreover, equation (5.28) implies that

$$c_2 = \frac{1}{2v_0(d_i,\varepsilon_i)} \left\{ \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} h(\varepsilon_i) - I_{0^+}^{\alpha-2} h(T), \right\} \tag{5.29}$$

substituting the values of (5.29) in (5.25), we obtain

$$\begin{aligned} c_1 = {} & \frac{1}{v_0(b_i,\sigma_i)} \left\{ \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} h(\sigma_i) - I_{0^+}^{\alpha-1} h(T) \right\} \\ & - \frac{v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i) v_0(d_i,\varepsilon_i)} \left\{ \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} h(\varepsilon_i) - I_{0^+}^{\alpha-2} h(T) \right\}. \end{aligned} \tag{5.30}$$

Now, substituting the values of (5.29) and (5.29) in (5.25), we have

$$\begin{aligned} c_0 = {} & \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha} h(\beta_i) - I_{0^+}^{\alpha} h(T) \right\} \\ & - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i) v_0(b_i,\sigma_i)} \left\{ \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} h(\sigma_i) - I_{0^+}^{\alpha-1} h(T) \right\} \\ & + \frac{v_1(b_i,\sigma_i) v_1(a_i,\beta_i)}{v_0(a_i,\beta_i) v_0(b_i,\sigma_i) v_0(d_i,\varepsilon_i)} \left\{ \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} h(\varepsilon_i) - I_{0^+}^{\alpha-2} h(T) \right\} \\ & - \frac{v_2(a_i,\beta_i)}{2v_0(d_i,\varepsilon_i) v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} h(\varepsilon_i) - I_{0^+}^{\alpha-2} h(T) \right\}. \end{aligned} \tag{5.31}$$

Finally, substituting the values of (5.31), (5.30), and (5.29) in equation (5.19), we obtain the general solution of problem (5.14) which is (5.15). Converse is also true by using the direct computation.

□

In the following,we prove existence results, for the boundary value problem (5.12)-(5.13) by using a Mönch fixed point theorem.
(H1) $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ satisfies the Caratheodory conditions ;
(H2) There exists $P \in C(\mathrm{J}, \mathbb{R}^+)$, such that,

$$\|f(t,x)\| \leq P(t)\|x\|, \text{ for } t \in \mathrm{J} \text{ and each } x \in \mathbb{E};$$

(H3) For each $t \in \mathrm{J}$ and each bounded set $B \subset \mathbb{E}$, we have

$$\lim_{h\to 0^+} \mu(f(\mathrm{J}_{t,h} \times B)) \leq P(t)\mu(B); \quad \textit{here} \quad \mathrm{J}_{t,h} = [t-h,t] \cap \mathrm{J}.$$

Theorem 5.8. *Assume that conditions (H1)-(H3) hold. Let* $P^* = \sup_{t\in \mathrm{J}} P(t)$. *If*

$$p^* M < 1 \tag{5.32}$$

With

$$\begin{aligned} M := &\left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i \frac{\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + 1)\beta_i^\alpha}{\Gamma(\alpha+1)\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + \delta_i + 1)} - \frac{T^\alpha}{\Gamma(\alpha+1)} \right\} \right. \\ &+ \frac{1}{v_0(b_i,\sigma_i)} \left(T - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i \frac{\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + 1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)} \right\} \\ &+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2} \right) \\ &\left. \times \left\{ \sum_{i=1}^m d_i \frac{\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + 1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \right\} \right\}, \end{aligned}$$

then the BVP (5.12)-(5.13) *has at least one solution.*

Proof. Transform the problem (5.12)-(5.13) into a fixed point problem. Consider the operator $\mathfrak{F} : C(\mathrm{J},\mathbb{E}) \to C(\mathrm{J},\mathbb{E})$ defined by

$$\begin{aligned} \mathfrak{F}x(t) = {} & I_{0^+}^\alpha f(t,x(t)) + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^\alpha f(\beta_i,x(\beta_i)) - I_{0^+}^\alpha f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(b_i,\sigma_i)} \left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} f(\sigma_i,x(\sigma_i)) - I_{0^+}^{\alpha-1} f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2} \right) \\ &\times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} f(\varepsilon_i,x(\varepsilon_i)) - I_{0^+}^{\alpha-2} f(T,x(T)) \right\}. \end{aligned} \tag{5.33}$$

Clearly, the fixed points of the operator $\mathfrak{F}$ are solutions of the problem (5.12)-(5.13). We consider

$$D = \{x \in C(\mathrm{J},\mathbb{E}) : \|x\| \le R\}.$$

where R satisfies inequality (5.32), Clearly, the subset D is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

□

Step 1 : First we show that $\mathfrak{F}$ is continuous :
Let x_n be a sequence such that $x_n \to x$ in $C(\mathrm{J},\mathbb{E})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned} \|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| \;\le\; & I_{0^+}^\alpha \|f(s,x_n(s)) - f(s,x(s))\|(t) + \frac{1}{v_0(a_i,\beta_i)} \\ & \times \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^\alpha (1)(\beta_i) - I_{0^+}^\alpha (1)(T) \right\} \|f(s,x_n(s)) - f(s,x(s))\| \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{v_0(b_i,\sigma_i)}\left(t-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\\
&\times \quad \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}(1)(\sigma_i)-I_{0^+}^{\alpha-1}(1)(T)\right\}\|f(s,x_n(s))-f(s,x(s))\|\\
&+ \quad \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)}+\frac{t^2}{2}\right)\\
&\times \quad \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}(1)(\varepsilon_i)-I_{0^+}^{\alpha-2}(1)(T)\right\}\|f(s,x_n(s))-f(s,x(s))\|\\
&\leq \quad \left\{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{v_0(a_i,\beta_i)}\right.\\
&\times \quad \left\{\sum_{i=1}^{m} a_i\frac{\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+1)\beta_i^\alpha}{\Gamma(\alpha+1)\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+\delta_i+1)}-\frac{T^\alpha}{\Gamma(\alpha+1)}\right\}+\frac{1}{v_0(b_i,\sigma_i)}\\
&\times \quad \left(t-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i\frac{\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+\delta_i+1)}-\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right\}\\
&+ \quad \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)}+\frac{T^2}{2}\right)\\
&\quad\quad \left.\times\left\{\sum_{i=1}^{m} d_i\frac{\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+\delta_i+1)}-\frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\right\}\right\}\\
&\times \quad \|f(s,x_n(s))-f(s,x(s))\|
\end{aligned}
$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathfrak{F}(x_n)-\mathfrak{F}(x)\|_\infty \to 0 \text{ as } n\to\infty.$$

Step 2 : Second we show that $\mathfrak{F}$ maps D into itself :
Take $x\in D$, by $(H2)$, we have, for each $t\in$ J and assume that $\mathfrak{F}x(t)\neq 0$.

$$
\begin{aligned}
\|(\mathfrak{F}x)(t)\| \quad &\leq \quad I_{0^+}^{\alpha}\|f(s,x(s))\|(t)\\
&+ \quad \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha}\|f(s,x(s))\|(\beta_i)-I_{0^+}^{\alpha}\|f(s,x(s))\|(T)\right\}\\
&+ \quad \frac{1}{v_0(b_i,\sigma_i)}\left(t-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\\
&\times \quad \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}\|f(s,x(s))\|(\sigma_i)-I_{0^+}^{\alpha-1}\|f(s,x(s))\|(T)\right\}\\
&+ \quad \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)}+\frac{t^2}{2}\right)\\
&\times \quad \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}\|f(s,x(s))\|(\varepsilon_i)-I_{0^+}^{\alpha-2}\|f(s,x(s))\|(T)\right\}
\end{aligned}
$$

$$\begin{aligned}
&\leq I_{0^+}^{\alpha} p(s)\|x(s)\|(t)\\
&+ \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha} p(s)\|x(s)\|(\beta_i) - I_{0^+}^{\alpha} p(s)\|x(s)\|(T)\right\}\\
&+ \frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\\
&\times \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} p(s)\|x(s)\|(\sigma_i) - I_{0^+}^{\alpha-1} p(s)\|x(s)\|(T)\right\}\\
&+ \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right)\\
&\times \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} p(s)\|x(s)\|(\varepsilon_i) - I_{0^+}^{\alpha-2} p(s)\|x(s)\|(T)\right\}\\
&\leq p^* R I^{\alpha}(1)(T) + \frac{p^* R}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha}(1)(\beta_i) - I_{0^+}^{\alpha}(1)(T)\right\}\\
&+ \frac{p^* R}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}(1)(\sigma_i) - I_{0^+}^{\alpha-1}(1)(T)\right\}\\
&+ \frac{p^* R}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2}\right)\\
&\times \left\{\sum_{i=1}^{m} d_i J.{}_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}(1)(\varepsilon_i) - I_{0^+}^{\alpha-2}(1)(T)\right\}
\end{aligned}$$

Consequently

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| \leq P^* R &\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + 1)\beta_i^{\alpha}}{\Gamma(\alpha+1)\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right\}\right.\\
&+ \frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i \frac{\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + 1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right\}\\
&+ \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2}\right)\\
&\left.\times\left\{\sum_{i=1}^{m} d_i \frac{\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + 1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\right\}\right\}\\
&:= P^* R M \leq R.
\end{aligned}$$

Step 3 : we show that $\mathfrak{F}(D)$ is equicontinuous :
By Step 2, it is obvious that $\mathfrak{F}(D) \subset C(\mathrm{J},\mathbb{E})$ is bounded. For the equicontinuity of $\mathfrak{F}(D)$, let $t_1, t_2 \in \mathrm{J}$, $t_1 < t_2$ and $x \in D$, so $\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1) \neq 0$. Then

$$\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\| \leq |I_{0^+}^{\alpha} f(s,x(s))(t_2) - I_{0^+}^{\alpha} f(s,x(s))(t_1)| + \left((t_2 - t_1) - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)$$

$$
\begin{aligned}
&\times \frac{1}{v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}|f(s,x(s))|(\sigma_i) - I_{0^+}^{\alpha-1}|f(s,x(s))|(T)\right\}\\
&+\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)(t_2-t_1)}{v_0(b_i,\sigma_i)} + \frac{(t_2^2-t_1^2)}{2}\right)\\
&\times \frac{1}{v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}|f(s,x(s))|(\varepsilon_i) - I_{0^+}^{\alpha-2}|f(s,x(s))|(T)\right\}\\
&\le \frac{Rp^*}{\Gamma(\alpha+1)}\{(t_2^\alpha - t_1^\alpha) + 2(t_2-t_1)^\alpha\} + \left((t_2-t_1) - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\\
&\times \frac{1}{v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}|f(s,x(s))|(\sigma_i) - I_{0^+}^{\alpha-1}|f(s,x(s))|(T)\right\}\\
&+\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)(t_2-t_1)}{v_0(b_i,\sigma_i)} + \frac{(t_2^2-t_1^2)}{2}\right)\\
&\times \frac{1}{v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}|f(s,x(s))|(\varepsilon_i) - I_{0^+}^{\alpha-2}|f(s,x(s))|(T)\right\}.
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Hence $N(D) \subset D$.

Finally we show that the implication holds :
Let $V \subset D$ such that $V = \overline{conv}(\mathfrak{F}(V) \cup \{0\})$. Since V is bounded and equicontinuous, and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J.
By assumption $(H2)$ and the properties of the measure μ we have for each $t \in$ J.

$$
\begin{aligned}
v(t) &\le \mu(\mathfrak{F}(V)(t) \cup \{0\})) \le \mu((\mathfrak{F}V)(t))\\
&\le \mu\left\{I_{0^+}^{\alpha} f(s,x(s))(t) + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha} f(s,V(s))(\beta_i) - I_{0^+}^{\alpha} f(s,V(s))(T)\right\}\right.\\
&+\frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} f(s,V(s))(\sigma_i) - I_{0^+}^{\alpha-1} f(s,V(s))(T)\right\}\\
&+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right)\\
&\left.\times\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} f(s,V(s))(\varepsilon_i) - I_{0^+}^{\alpha-2} f(s,V(s))(T)\right\}\right\}\\
&\le I_{0^+}^{\alpha} p(s)\mu(V(s))(t) + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha} p(s)\mu(V(s))(\beta_i) - I_{0^+}^{\alpha} p(s)\mu(V(s))(T)\right\}\\
&+\frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} p(s)\mu(V(s))(\sigma_i) - I_{0^+}^{\alpha-1} p(s)\mu(V(s))(T)\right\}\\
&+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} p(s)\mu(V(s))(\varepsilon_i) - I_{0^+}^{\alpha-2} p(s)\mu(V(s))(T)\right\}\\
&\leq I_{0^+}^{\alpha} p(s)v(s)(t) + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha} p(s)v(s)(\beta_i) - I_{0^+}^{\alpha} p(s)v(s)(T)\right\}\\
&+\frac{1}{v_0(b_i,\sigma_i)}\left(t-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1} p(s)v(s)(\sigma_i) - I_{0^+}^{\alpha-1} p(s)v(s)(T)\right\}\\
&+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)}+\frac{t^2}{2}\right)\\
&\times\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2} p(s)v(s)(\varepsilon_i) - I_{0^+}^{\alpha-2} p(s)v(s)(T)\right\}\\
&\leq P^*\|v\|\left\{I_{0^+}^{\alpha}(1)(T) + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha}(1)(\beta_i) - I_{0^+}^{\alpha}(1)(T)\right\}\right.\\
&+\frac{1}{v_0(b_i,\sigma_i)}\left(T-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-1}(1)(\sigma_i) - I_{0^+}^{\alpha-1}(1)(T)\right\}\\
&+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)}+\frac{T^2}{2}\right)\\
&\left.\times\left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I_{0^+}^{\alpha-2}(1)(\varepsilon_i) - I_{0^+}^{\alpha-2}(1)(T)\right\}\right\}\\
&\leq P^*\|v\|\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+1)\beta_i^{\alpha}}{\Gamma(\alpha+1)\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+\delta_i+1)} - \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right\}\right.\\
&+\frac{1}{v_0(b_i,\sigma_i)}\left(T-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i \frac{\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+\delta_i+1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right\}\\
&+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)}+\frac{T^2}{2}\right)\\
&\left.\times\left\{\sum_{i=1}^{m} d_i \frac{\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+\delta_i+1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\right\}\right\}\\
&:= p^*\|v\|M.
\end{aligned}
$$

This means that

$$\|v\|(1-p^*M)\leq 0.$$

By (5.32) it follows that $\|v\|=0$, that is $v(t)=0$ for each $t\in$ J and then $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem (1.58), we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (5.12)-(5.13).

5.3.2 Example

Let

$$\mathbb{E} = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$$

with the norm

$$\|x\|_{\mathbb{E}} = \sum_{n=1}^{\infty} |x_n|$$

Let us consider problem (5.12)-(5.13) with specific data :

$$\begin{array}{llll} T = 1, & m = 1, & \alpha = 5/2, & \beta_1 = 1/2 \\ \sigma_1 = 3/2, & \varepsilon_1 = 5/7, & \eta_1 = 7/5, & \gamma_1 = 2/3 \\ \delta_1 = 3/2, & a_1 = 3/2, & b_1 = 1/2, & d_1 = 3/4. \end{array} \tag{5.34}$$

Using the given values of the parameters in (5.16)-(5.17) and (5.18), we find that

$$\begin{array}{lll} v_0(a_1, \beta_1) = 0.4226, & v_0(b_1, \sigma_1) = 0.8075, & v_0(d_1, \varepsilon_1) = 0.7113 \\ v_1(a_1, \beta_1) = 0.6445, & v_1(b_1, \sigma_1) = 0.8815 & \\ v_2(a_1, \beta_1) = 0.7531 & & \end{array} \tag{5.35}$$

In order to illustrate Theorem (5.8),we take

$$f(t, x(t)) = \frac{t\sqrt{\pi} - 1}{7^3} \frac{x(t)}{x(t) + 1}, t \in [0, 1]$$

Clearly, the function f is continuous, we have

$$|f(t, x(t))| \leq \frac{\sqrt{\pi}}{7^3} |x|$$

Hence, the hypothesis (H2) is satisfied with $p^* = \frac{\sqrt{\pi}}{7^3}$. We shall show that condition (5.32) holds with $T = 1$. Indeed,

$$p^* M \simeq 0.3817 < 1$$

Simple computations show that all conditions of Theorem (5.8) are satisfied. It follows that the problem (5.12)-(5.13) with data (5.34) and (5.35) has at least solution defined on $[0, 1]$.

Chapitre 6

Existence Theory for a Langevin Fractional q-Difference Equations in Banach Space

6.1 Introduction

In this chapter, we are concerned with the existence of solutions for certain classes of Langevin Fractional q-Difference Equations in Banach Space. First, we investigate the problem of Existence Theory for a Nonlinear Langevin Fractional q-Difference Equations with Dirichlet conditions on an arbitrary Banach Space. Next, we give a similar result to the coupled fractional Langevin q-difference system extends the first problem. The used approach is based on Mönch's fixed point theorem combined with the technique of measures of noncompactness. We also provide some illustrative examples in support of our existence theorems.

In particular, Fractional Langevin differential equations have been one of the important subject in physics, chemistry and electrical engineering. The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [93]. For instance, Brownian motion is well described by the Langevin equation when the random fluctuation force is assumed to be white noise. Another possible extension requires the replacement of ordinary derivative by a fractional derivative in the Langevin equation to give the fractional Langevin equation [13, 58, 69, 35, 127].

Firstly, in Section 6.3 we are interested in the existence of solutions for the following Langevin fractional q-difference equation

$$\begin{cases} D_q^{\beta}(D_q^{\alpha}+\lambda)x(t) & = f(t,x(t)), \quad t \in \mathrm{J}=[0,1], 0<\alpha,\beta \leq 1, \\ x(0)=\gamma, & x(1)=\eta, \end{cases} \tag{6.1}$$

where D_q is the fractional $q-$derivative of the Caputo type. $f : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a given function satisfying some assumptions that will be specified later and $\mathbb{E}$ is a Banach space with norm $\|x\|$, λ is any real number.

Next, In Section 6.4, we give similar result to the following coupled fractional Langevin

q-difference system

$$\begin{cases} D_q^{\beta_1}(D_q^{\alpha_1}+\lambda_1)x_1(t)=f_1(t,x_1(t),x_2(t)), \\ \\ D_q^{\beta_2}(D_q^{\alpha_2}+\lambda_2)x_2(t)=f_2(t,x_1(t),x_2(t)), \end{cases} \quad t\in \mathrm{J}, \tag{6.2}$$

with the Dirichlet boundary conditions

$$\begin{cases} x_1(0)=\gamma_1 \quad ,x_1(1)=\eta_1 \\ x_2(0)=\gamma_2 \quad ,x_2(1)=\eta_2. \end{cases} \tag{6.3}$$

where $\mathrm{J}:=[0,1]$, $0<\alpha,\beta\leq 1$, and D_q is the fractional $q-$derivative of the Caputo type. $f_i:\mathrm{J}\times\mathbb{E}^2\to\mathbb{E}$ are given functions satisfying some assumptions that will be specified later, and $\mathbb{E}$ is a Banach space with norm $\|\cdot\|$, $\lambda_i,i=1,2$, is any real number.

In the present paper, we initiate the study of boundary value problems like (6.2). We will present the existence results for the problem (6.2) which rely on Mönchs fixed point theorem combined with the technique of Kuratowski measure of noncompactness. that technique turns out to be a very useful tool in existence of solutions for several kinds of integral equations and subsequently developed and used in many papers, see, for instance. The strong measure of noncompactness was considered first by Banaṡ et al. [33], for more details are found in A. Boutiara et al. [42], Akhmerov et al. [16], Alvàrez [26], Mönch [99], Szufla [125], Benchohra et al. [36, 37]. Finally, to illustrate the theoretical results, an example is given at the end of each section.

6.2 Existence of solutions

For the existence of solutions for the problem (6.1), the following definition and Lemma will be needed.

Definition 6.1. A function $x\in C(\mathrm{J},\mathbb{E})$ is said to be a solution of the problem (6.1) if x satisfies the equation $D_q^{\beta}(D_q^{\alpha}+\lambda)x(t)=f(t,x(t))$ on J and the conditions $x(0)=\gamma$, $x(1)=\eta$.

Lemma 6.2. *Let $h:\mathrm{J}\to\mathbb{E}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t)=I_q^{\alpha+\beta}h(t)-\lambda I_q^{\alpha}h(t)+t^{(\alpha)}\left\{\eta-\gamma-I_q^{\alpha+\beta}h(1)+\lambda I_q^{\alpha}h(1)\right\}+\gamma, \tag{6.4}$$

if and only if x is a solution of the fractional boundary-value problem

$$D_q^{\beta}(D_q^{\alpha}+\lambda)x(t)=f(t,x(t)), \quad t\in\mathrm{J}, \tag{6.5}$$

$$x(0)=\gamma, \qquad x(1)=\eta. \tag{6.6}$$

Proof. Assume that x satisfies (6.5). Then by applying Lemmas 1.30, 1.31 and 1.43, we can transform the problem (6.5)-(6.6) to an equivalent integral equation

$$x(t)=I_q^{\alpha+\beta}h(t)-\lambda I_q^{\alpha}h(t)+c_0\frac{t^{\alpha}}{\Gamma_q(\alpha+1)}+c_1. \tag{6.7}$$

Applying the boundary conditions (6.6), we get

$$x(0) = c_1,$$
$$x(1) = I_q^{\alpha+\beta}h(1) - \lambda I_q^{\alpha}h(1) + \frac{c_0}{\Gamma_q(\alpha+1)} + c_1.$$

So, we have

$$c_1 = \gamma,$$
$$I_q^{\alpha+\beta}h(1) - \lambda I_q^{\alpha}h(1) + \frac{c_0}{\Gamma_q(\alpha+1)} + \gamma = \eta.$$

Consequently

$$c_1 = \gamma,$$
$$c_0 = \Gamma_q(\alpha+1)\left\{\eta - \gamma - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}h(1)\right\}.$$

Finally, we obtain

$$x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^{\alpha}h(t) + t^{(\alpha)}\left\{\eta - \gamma - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}h(1)\right\} + \gamma.$$

Which completes the proof.

□

6.3 Existence Theory for a Langevin Fractional q-Difference Equations in Banach Space.

6.3.1 First Result [1]

In the following, we prove existence results, for the boundary value problem (6.1) by using Mönch fixed point theorem, under the following hypotheses.

$(H1)$ $f : \mathrm{J}\times\mathbb{E}\to\mathbb{E}$ satisfies the Caratheodory conditions.

$(H2)$ There exists $P\in C(\mathrm{J},\mathbb{R}^+)$, such that,

$$\|f(t,x)\| \leq P(t)\|x\|, \text{ for } t\in \mathrm{J} \text{ and each } x\in\mathbb{E}.$$

$(H3)$ For each $t\in\mathrm{J}$ and each bounded set $B\subset\mathbb{E}$, we have

$$\lim_{h\to 0^+}\mu(f(\mathrm{J}_{t,h}\times B)) \leq P(t)\mu(B); \quad \textit{here} \quad \mathrm{J}_{t,h} = [t-h,t]\cap\mathrm{J}.$$

Theorem 6.3. *Assume that conditions (H1)-(H3) hold. Let $P^* = \sup_{t\in\mathrm{J}}P(t)$.*
If

$$P^*M + N < 1, \tag{6.8}$$

with

$$M := \left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\},$$

1. **A. Boutiara**, M. Benbachir, K. Guerbati, Existence Theory for a Langevin Fractional q-Difference Equations in Banach Space, (submitted).

and

$$N := |\lambda| \left\{ \frac{2}{\Gamma_q(\alpha+1)} \right\}.$$

Then the problem (6.1) *has at least one solution on* J.

Proof. Using Lemma 6.7, it is sufficient to prove existence of solutions to the integral equation (6.4). For this, we rewrite the problem (6.1) as a fixed point problem. Indeed let us consider the operator $\mathfrak{F} : C(\mathrm{J},\mathbb{E}) \to C(\mathrm{J},\mathbb{E})$ defined by

$$\mathfrak{F}x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^{\alpha}h(t) + t^{(\alpha)}\left\{\eta - \gamma - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}h(1)\right\} + \gamma. \tag{6.9}$$

It is obvious that fixed points of the operator $\mathfrak{F}$ are solutions of the problem (6.1).
Let

$$R \geq \frac{\gamma}{1-(p^*M+N)}, \tag{6.10}$$

and consider

$$D_R = \{x \in C(\mathrm{J},\mathbb{E}) : \|x\| \leq R\}.$$

We can check, without difficulty, that the subset D_R is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

□

Step 1 : First we show that $\mathfrak{F}$ is continuous :
Let x_n be a sequence such that $x_n \to x$ in $C(\mathrm{J},\mathbb{E})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| \leq{}& I_q^{\alpha+\beta}\|f(s,x_n(s)) - f(s,x(s))\|(t) + |\lambda| I_q^{\alpha}\|x_n(s) - x(s)\|(t) \\
&+ t^{(\alpha)} I_q^{\alpha+\beta}\|f(s,x_n(s)) - f(s,x(s))\|(1) + t^{(\alpha)}|\lambda| I_q^{\alpha}\|x_n(s) - x(s)\|(1), \\
\leq{}& \left\{I_q^{\alpha+\beta}(1)(t) + t^{(\alpha)} I_q^{\alpha+\beta}(1)(1)\right\}\|f(s,x_n(s)) - f(s,x(s))\| \\
&+ \left\{|\lambda| I_q^{\alpha}(1)(t) + t^{(\alpha)}|\lambda| I_q^{\alpha}(1)(1)\right\}\|x_n(s) - x(s)\|.
\end{aligned}$$

Thanks to assumption $(H1)$, the sequence $f(t,x_n(t))$ converges uniformly to $f(t,x(t))$. Lebesgue dominated convergence theorem guarantee that

$$\|\mathfrak{F}(x_n) - \mathfrak{F}(x)\|_\infty \to 0 \text{ as } n \to \infty.$$

Then $\mathfrak{F} : D_R \to D_R$ is sequentially continuous.
Step 2 : Second, we show that $\mathfrak{F}$ maps D into itself
Take $x \in D$, by (H2), we have, for each $t \in \mathrm{J}$ and assume that $\mathfrak{F}x(t) \neq 0$.

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| \leq{}& I_q^{\alpha+\beta}\|f(s,x(s))\|(t) - \lambda I_q^{\alpha}\|x\|(t) \\
&+ t^{(\alpha)}\left\{\eta - \gamma - I_q^{\alpha+\beta}\|f(s,x(s))\|(1) + \lambda I_q^{\alpha}\|x\|(1)\right\} + \gamma, \\
\leq{}& I_q^{\alpha+\beta}\|x\|P(s)(t) - \lambda I_q^{\alpha}\|x\|(t) \\
&+ t^{(\alpha)}\left\{\eta - \gamma - I_q^{\alpha+\beta}\|x\|P(s)(1) + \lambda I_q^{\alpha}\|x\|(1)\right\} + \gamma,
\end{aligned}$$

$$\begin{aligned}
&\leq P^*R\left\{I_q^{\alpha+\beta}(1)(t)+t^{(\alpha)}I_q^{\alpha+\beta}(1)(1)\right\}\\
&+R\left\{|\lambda|I_q^{\alpha}(1)(t)+t^{(\alpha)}|\lambda|I_q^{\alpha}(1)(1)\right\}+T^{(\alpha)}(\eta-\gamma)+\gamma\\
&\leq P^*R\left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\}+|\lambda|R\left\{\frac{2}{\Gamma_q(\alpha+1)}\right\}+\eta,\\
&\leq P^*RM+RN+\eta,\\
&\leq R.
\end{aligned}$$

Step 3 : we show that $\mathfrak{F}(D_R)$ is equicontinuous
By Step 2, it is obvious that $\mathfrak{F}(D_R)\subset C(\mathrm{J},\mathbb{E})$ is bounded. For the equicontinuity of $\mathfrak{F}(D_R)$, let $t_1,t_2\in\mathrm{J}$, $t_1<t_2$ and $x\in D_R$, so $\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\neq 0$. Then,

$$\begin{aligned}
\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\| \leq\; & I_q^{\alpha+\beta}|f(s,x(s))(t_2)-f(s,x(s))(t_1)|+|\lambda|I_q^{\alpha}(|x(s)|(t_2)-|x(s)|(t_2))\\
&+(t_2^{(\alpha)}-t_2^{(\alpha)})\left\{\eta-\gamma-I_q^{\alpha+\beta}|f(s,x(s))|(1)+\lambda I_q^{\alpha}|x|(1)\right\},\\
&\leq P^*R|I_q^{\alpha+\beta}(1)(t_2)-I_q^{\alpha+\beta}(1)(t_1)|+R|\lambda|(I_q^{\alpha}|(1)|(t_2)-I_q^{\alpha}|(1)|(t_1))\\
&+(t_2^{(\alpha)}-t_1^{(\alpha)})\left\{\eta-\gamma-I_q^{\alpha+\beta}|f(s,x(s))|(1)+\lambda I_q^{\alpha}|x|(1)\right\},\\
&\leq\frac{R(P^*+|\lambda|)}{\Gamma_q(\alpha+1)}\{(t_2^{\alpha}-t_1^{\alpha})+2(t_2-t_1)^{\alpha}\}\\
&+(t_2^{(\alpha)}-t_1^{(\alpha)})\left\{\eta-\gamma-I_q^{\alpha+\beta}|f(s,x(s))|(1)+\lambda I_q^{\alpha}|x|(1)\right\}.
\end{aligned}$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero.
This means that $\mathfrak{F}(D_R)\subset D_R$.

Finally we show that the implication holds
Let $V\subset D_R$ such that $V=\overline{conv}(\mathfrak{F}(V)\cup\{0\})$. Since V is bounded and equicontinuous, and therefore the function $t\to v(t)=\mu(V(t))$ is continuous on J. By assumption $(H2)$ and the properties of the measure μ we have for each $t\in\mathrm{J}$.

$$\begin{aligned}
v(t)&\leq\mu(\mathfrak{F}(V)(t)\cup\{0\}))\leq\mu((\mathfrak{F}V)(t))\\
&\leq I_q^{\alpha+\beta}P(s)\mu(V(s))(t)+|\lambda|I_q^{\alpha}\mu(V(s))(t)\\
&+t^{(\alpha)}\left\{I_q^{\alpha+\beta}P(s)\mu(V(s))(1)+|\lambda|I_q^{\alpha}\mu(V(s))(1)\right\},\\
&\leq I_q^{\alpha+\beta}P(s)\mu(V(s))(t)+|\lambda|I_q^{\alpha}\mu(V(s))(t)\\
&+t^{(\alpha)}\left\{I_q^{\alpha+\beta}P(s)\mu(V(s))(1)+|\lambda|I_q^{\alpha}\mu(V(s))(1)\right\},\\
&\leq P^*\|v\|\left\{I_q^{\alpha+\beta}(1)(t)+t^{(\alpha)}I_q^{\alpha+\beta}(1)(1)\right\}\\
&+\|v\|\left\{|\lambda|I_q^{\alpha}(1)(t)+t^{(\alpha)}|\lambda|I_q^{\alpha}(1)(1)\right\},\\
&\leq P^*\|v\|\left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\}+|\lambda|\|v\|\left\{\frac{2}{\Gamma_q(\alpha+1)}\right\},\\
&\leq P^*\|v\|M+\|v\|N.
\end{aligned}$$

This means that

$$\|v\|(1-p^*M-N)\leq 0.$$

By (6.8) it follows that $\|v\|=0$, that is $v(t)=0$ for each $t\in \mathrm{J}$ and then $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 1.58, we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (6.1).

6.3.2 Example

In this section, we present an example to illustrate the main result.
Let $\mathbb{E}=l^1=\{x=(x_1,x_2,\ldots,x_n,\ldots):\sum_{n=1}^{\infty}|x_n|<\infty\}$ with the norm

$$\|x\|_{\mathbb{E}}=\sum_{n=1}^{\infty}|x_n|$$

Consider the following nonlinear Langevin $\frac{1}{4}$-fractional equation :

$$\begin{cases} D^{1/2}_{1/4}\left(D^{1/3}_{1/4}-\frac{5}{27}\right)x(t) & =\frac{(\sin t+1)e^{-t}}{24}\left(\frac{x^2(t)}{1+|x(t)|}\right), \qquad t\in \mathrm{J}=[0,1], \\ x(0)=\gamma & ,x(1)=\eta. \end{cases} \tag{6.11}$$

Here

$$\begin{array}{lll} \alpha=1/2, & \beta=1/3, & q=1/4, \\ \gamma=3/4, & \eta=1/4, & \lambda=5/27, \end{array}$$

with

$$f(t,x)=(((\sin t+1)e^{-t})/24)(x^2/(1+|x|)).$$

Clearly, the function f is continuous. For each $x\in\mathbb{E}$ and $t\in[0,1]$, we have

$$|f(t,x)|\leq\frac{1}{12}|x|,$$

and

$$p^*=\frac{1}{12}.$$

Hence, the hypothesis (H2) is satisfied with $p^*=\frac{1}{12}$. We shall show that condition (6.8) holds with $\mathrm{J}=[0,1]$. Indeed,

$$p^*M+N\simeq 0.6785<1.$$

Therefore, we deduce from the conclusion of Theorem 1.58 that the problem (6.11) has a solution on $[0,1]$.

6.4 Existence Theory for a Nonlinear Langevin Fractional q-Difference System in Banach Space

6.4.1 Second Result [2]

Let $\mathscr{U} = C([0,1],\mathbb{E})$ denotes the Banach space of all continuous functions from $u : \mathrm{J} \to \mathbb{E}$ with

$$\|u\|_\infty = \sup\{|u(t)| : \quad t \in \mathrm{J}\}.$$

Then the product space $\mathscr{C} := \mathscr{U} \times \mathscr{V}$ defined by $\mathscr{C} = \{(u,v) : u \in \mathscr{U}, v \in \mathscr{V}\}$ is Banach space under the norm

$$\|(u,v)\|_{\mathscr{C}} = \|u\|_\infty + \|v\|_\infty.$$

We further will use the following hypotheses.
(H1) For any $i = 1,2$, $f_i : \mathrm{J} \times \mathbb{E}^2 \to \mathbb{E}$ satisfie the Caratheodory conditions.
(H2) There exist $p_i, q_i \in C(\mathrm{J},\mathbb{R}^+)$, such that,

$$\|f(t,x)\| \leq p_i(t)\|x_1\| + q_i(t)\|x_1\|, \text{ for } t \in \mathrm{J} \text{ and each } x_i \in \mathbb{E}, i = 1,2.$$

(H3) For any $t \in \mathrm{J}$ and each bounded measurable sets $B_i \subset \mathbb{E}$, i=1,2, we have

$$\lim_{h \to 0^+} \mu(f(\mathrm{J}_{t,h} \times B_1, B_2), 0) \leq p_1(t)\mu(B_1) + q_1(t)\mu(B_2)$$

and

$$\lim_{h \to 0^+} \mu(0, f(\mathrm{J}_{t,h} \times B_1, B_2)) \leq p_2(t)\mu(B_1) + q_2(t)\mu(B_2)$$

where μ is the Kuratowski measure of compactness and $\mathrm{J}_{t,h} = [t-h,t] \cap \mathrm{J}$.
Set

$$p_i^* = \sup_{t \in \mathrm{J}} p_i(t) \quad \textit{and} \quad q_i^* = \sup_{t \in \mathrm{J}} q_i(t), i = 1,2.$$

Theorem 6.4. *Assume that conditions (H1)-(H3) hold. If*

$$M < 1 \tag{6.12}$$

With

$$\begin{aligned} M : \quad &= \textstyle\sum_{i=1}^2 (M_i) \\ &= \textstyle\sum_{i=1}^2 \left\{ \frac{2(p_i^* + q_i^*)R}{\Gamma_q(\alpha_i + \beta_i + 1)} + \frac{2|\lambda_i|R}{\Gamma_q(\alpha_i + 1)} \right\} \end{aligned}$$

then the problem (6.2) *has at least one solution on* J.

Proof. Transform the problem (6.2) into a fixed point problem. Consider the operator $\mathfrak{F} : \mathscr{C} \to \mathscr{C}$ defined by the formula

$$\mathfrak{F}x_i(t) = I_q^{\alpha_i+\beta_i}h(t) - \lambda_i I_q^{\alpha_i}h(t) + t^{(\alpha_i)}\left\{\eta_i - \gamma_i - I_q^{\alpha_i+\beta_i}h(1) + \lambda_i I_q^{\alpha_i}h(1)\right\} + \gamma_i \tag{6.13}$$

2. **A. Boutiara**, M. Benbachir, K. Guerbati, Existence Theory for a Langevin Fractional q-Difference System in Banach Space, (submitted).

Clearly, the fixed points of the operator $\mathfrak{F}$ are solutions of the problem (6.2). Let

$$R \geq \frac{\eta_i}{1-M}, \quad i=1,2. \tag{6.14}$$

and consider

$$D_R = \{x_i \in \mathscr{C}, i=1,2 : \|(x_1,x_2)\| \leq R\}.$$

Clearly, the subset D_R is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

□

Step 1 : First we show that $\mathfrak{F}_i$ is sequentially continuous :
Let $\{x_{1,n},x_{2,n}\}_n$ be a sequence such that $(x_{1,n},x_{2,n}) \to (x_1,x_2)$ in $\mathscr{C}$. Then for any $t \in$ J ,

$$\begin{aligned}
\|(\mathfrak{F}_i x_{i,n})(t)-(\mathfrak{F}_i x_i)(t)\| \leq\; & I_q^{\alpha_i+\beta_i}\|f(s,x_{1,n}(s),x_{2,n}(s))-f(s,x_1(s),x_2(s))\|(t)\\
&+|\lambda_i|I_q^{\alpha_i}\|x_{i,n}(s)-x_i(s)\|(t)\\
&+t^{(\alpha_i)}I_q^{\alpha_i+\beta_i}\|f(s,x_{1,n}(s),x_{2,n}(s))-f(s,x_1(s),x_2(s))\|(1)\\
&+t^{(\alpha_i)}|\lambda_i|I_q^{\alpha_i}\|x_{i,n}(s)-x_i(s)\|(1)\\
\leq\; & \left\{2I_q^{\alpha_i+\beta_i}(1)(t)\right\}\|f(s,x_{1,n}(s),x_{2,n}(s))-f(s,x_1(s),x_2(s))\|\\
&+\left\{2|\lambda_i|I_q^{\alpha_i}(1)(t)\right\}\|x_{i,n}(s)-x_i(s)\|.
\end{aligned}$$

Since for any $i=1,2$, the function f_i satisfies assumptions (H1), we have $f_i(t,x_{1,n}(t),x_{2,n}(t))$ converge uniformly to $f_i(t,x_1(t),x_2(t))$.
Hence, the Lebesgue dominated convergence theorem implies that $(\mathfrak{F}_i(x_{1,n},x_{2,n}))(t)$ converges uniformly to $(\mathfrak{F}_i(x_1,x_{2,}))(t)$ Thus $(\mathfrak{F}(x_{1,n},x_{2,n})) \to (\mathfrak{F}(x_1,x_{2,}))$. Hence $\mathfrak{F}: D_R \to D_R$ is sequentially continuous.
Step 2 : Second we show that $\mathfrak{F}_i$ maps D_R into itself :
Take $x_i \in D_R$, i=1,2, by (H2), we have, for each $t \in$ J and assume that $(\mathfrak{F}_i(x_i))(t) \neq 0$, i=1,2.

$$\begin{aligned}
\|(\mathfrak{F}_i x_i)(t)\| \leq\; & I_q^{\alpha_i+\beta_i}\|f(s,x_1(s),x_2(s))\|(t)-\lambda I_q^{\alpha}\|x_i\|(t)\\
&+t^{(\alpha_i)}\left\{\eta_i-\gamma_i-I_q^{\alpha_i+\beta_i}\|f(s,x_1(s),x_2(s))\|(1)+\lambda_i I_q^{\alpha}\|x_i\|(1)\right\}+\gamma_i\\
\leq\; & I_q^{\alpha_i+\beta_i}\left[\|x_1\|p_i(s)+\|x_2\|q_i(s)\right](t)-\lambda_i I_q^{\alpha_i}\|x_i\|(t)\\
&+t^{(\alpha_i)}\left\{\eta_i-\gamma_i-I_q^{\alpha_i+\beta_i}\left[\|x_1\|p_i(s)+\|x_2\|q_i(s)\right](1)+\lambda_i I_q^{\alpha_i}\|x_i\|(1)\right\}+\gamma_i\\
\leq\; & (p_i^*+q_i^*)R\left\{I_q^{\alpha_i+\beta_i}(1)(t)+t^{(\alpha_i)}I_q^{\alpha_i+\beta_i}(1)(1)\right\}\\
&+R\left\{|\lambda_i|I_q^{\alpha_i}(1)(t)+t^{(\alpha_i)}|\lambda_i|I_q^{\alpha_i}(1)(1)\right\}+t^{(\alpha_i)}(\eta_i-\gamma_i)+\gamma_i\\
\leq\; & \left\{\frac{2(p_i^*+q_i^*)R}{\Gamma_q(\alpha_i+\beta_i+1)}\right\}+\left\{\frac{2|\lambda_i|R}{\Gamma_q(\alpha_i+1)}\right\}+\eta_i\\
=\; & RM_i+\eta_i.
\end{aligned}$$

Hence we get

$$\|(\mathfrak{F}(x_1,x_2))\|_{\mathscr{C}} \leq \sum_{i=1}^{2}(RM_i+\eta_i) \leq R. \tag{6.15}$$

Step 3 : we show that $\mathfrak{F}_i(D_R)$ is equicontinuous :
By Step 2, it is obvious that $\mathfrak{F}(D_R) \subset \mathscr{C}$ is bounded. For the equicontinuity of $\mathfrak{F}(D_R)$, let $t_1, t_2 \in$ J , $t_1 < t_2$ and $x \in D_R$, so $\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1) \neq 0$. Then

$$\begin{aligned}
\|\mathfrak{F}x_i(t_2) - \mathfrak{F}x_i(t_1)\| &\leq |I_q^{\alpha_i+\beta_i} f(s,x_1(s),x_2(s))(t_2) - I_q^{\alpha_i+\beta_i} f(s,x_1(s),x_2(s))(t_1)| \\
&+ |\lambda_i|(I_q^{\alpha_i}|x_i(s)|(t_2) - I_q^{\alpha_i}|x_i(s)|(t_2)) \\
&+ (t_2^{(\alpha_i)} - t_2^{(\alpha_i)})\left\{\eta_i - \gamma_i - I_q^{\alpha_i+\beta_i}|f(s,x_1(s),x_2(s))|(1) + \lambda_i I_q^{\alpha_i}|x_i|(1)\right\} \\
&\leq (p_i^* + q_i^*)R|I_q^{\alpha_i+\beta_i}(1)(t_2) - I_q^{\alpha_i+\beta_i}(1)(t_1)| + R|\lambda_i|(I_q^{\alpha}|(1)|(t_2) - I_q^{\alpha}|(1)|(t_1)) \\
&+ (t_2^{(\alpha_i)} - t_1^{(\alpha_i)})\left\{\eta_i - \gamma_i - I_q^{\alpha_i+\beta_i}|f(s,x_1(s),x_2(s))|(1) + \lambda_i I_q^{\alpha_i}|x_i|(1)\right\} \\
&\leq \frac{R(p_i^* + q_i^*)}{\Gamma_q(\alpha_i+\beta_i+1)}\left\{(t_2^{\alpha_i+\beta_i} - t_1^{\alpha_i v}) + 2(t_2 - t_1)^{\alpha_i+\beta_i}\right\} \\
&+ \frac{R|\lambda_i|}{\Gamma_q(\alpha_i+1)}\left\{(t_2^{\alpha_i} - t_1^{\alpha_i}) + 2(t_2 - t_1)^{\alpha_i}\right\} \\
&+ (t_2^{(\alpha_i)} - t_1^{(\alpha_i)})\left\{\eta_i - \gamma_i - I_q^{\alpha_i+\beta_i}|f(s,x_1(s),x_2(s))|(1) + \lambda_i I_q^{\alpha_i}|x_i|(1)\right\}
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
This means that $\mathfrak{F}(D_R) \subset D_R$. Finally we show that the implication (1.12) holds :
Let $V \subset D_R$ such that $V = \overline{conv}(\mathfrak{F}(V) \cup \{(0,0)\})$. Since V is bounded and equicontinuous, and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J. By assumption (H2), and the properties of the measure μ, for any $t \in$ J we get.

$$\begin{aligned}
v(t) &\leq \mu(\mathfrak{F}(V)(t) \cup \{(0,0)\})) \leq \mu((\mathfrak{F}V)(t)) \\
&\leq \mu\left(\{((N_1v_1)(t), (N_2v_2)(t) : (v_1,v_2) \in V\}\right) \\
&\leq 2I^{\alpha_1+\beta_1}\mu\left(\{(\{(f_1(s,v_1(s),v_2(s))(t));0) : (v_1,v_2) \in V\}\right) \\
&+2|\lambda_1|I^{\alpha_1}\mu\left(\{(v_1(s),0) : (v_1,0) \in V\}\right) \\
&+2I^{\alpha_2+\beta_2}\mu\left(\{(0,f_2(s,v_1(s),v_2(s))) : (v_1,v_2) \in V\}\right) \\
&+2|\lambda_2|I^{\alpha_2}\mu\left(\{(0,v_2(s)) : (0,v_2) \in V\}\right) \\
&\leq 2I^{\alpha_1+\beta_1}[p_1(s)\mu\left(\{(v_1(s),0) : (v_1,0) \in V\}\right) \\
&+q_1(s)\mu\left(\{(0,v_2(s)) : (0,v_2) \in V\}\right)] \\
&\quad +2|\lambda_1|I^{\alpha_1}\mu\left(\{(v_1(s),0) : (v_1,0) \in V\}\right) \\
&+2I^{\alpha_2+\beta_2}[p_2(s)\mu\left(\{(v_1(s),0) : (v_1,0) \in V\}\right) \\
&+q_2(s)\mu\left(\{(0,v_2(s)) : (0,v_2) \in V\}\right)] \\
&+2|\lambda_2|I^{\alpha_2}\mu\left(\{(0,v_2(s)) : (0,v_2) \in V\}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\mu\left(V(t)\right) \leq & 2I^{\alpha_1+\beta_1}\left(p_1(s)+q_1(s)\right) \times \mu\left(V(s)\right) \\
& 2|\lambda_1|I^{\alpha_1}\left((1)(s)\right) \times \mu\left(V(s)\right) \\
& + 2I^{\alpha_2+\beta_2}\left(p_2(s)+q_2(s)\right) \times \mu\left(V(s)\right) \\
& 2|\lambda_2|I^{\alpha_2}\left((1)(s)\right) \times \mu\left(V(s)\right)
\end{aligned}$$

Hence

$$\mu(V(t)) \le \left\{ \frac{2(p_1^*+q_2^*)}{\Gamma_q(\alpha_1+\beta_1+1)} + \frac{2|\lambda_1|}{\Gamma_q(\alpha_1+1)} \right\} \sup_{t\in I} \mu(V(t)) + \left\{ \frac{2(p_2^*+q_2^*)}{\Gamma_q(\alpha_2+\beta_2+1)} + \frac{2|\lambda_2|}{\Gamma_q(\alpha_2+1)} \right\} \sup_{t\in I} \mu(V(t))$$

This means that

$$\sup_{t\in I} \mu(V(t)) \le M \sup_{t\in I} \mu(V(t))$$

By (6.12) it follows that $\sup_{t\in \mathrm{J}} \mu((V(t)) = 0$, that is $\mu(V(t)) = 0$ for each $t \in \mathrm{J}$, and then $V(t)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 1.58, we conclude that $\mathfrak{F}$ has a fixed point, which is a solution of the problem (6.2)-(6.3).

6.4.2 Example

In this section, we present some examples to illustrate our results.
Let $\mathbb{E} = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with the norm

$$\|x\|_{\mathbb{E}} = \sum_{n=1}^{\infty} |x_n|$$

Consider the following nonlinear Langevin $\frac{1}{4}$-fractional equation :

$$\begin{cases} D_{1/4}^{1/4}\left(D_{1/4}^{1/3} - \frac{1}{27}\right)x(t) = \frac{\sqrt{3}|x|\cos^2(2\pi t)}{3(27-t)} + \frac{\sqrt{2}\pi|y|}{(7\pi-t)^2}\left(\frac{|y|}{|y|+3}+1\right) \\ t \in \mathrm{J} = [0,1], \\ D_{1/4}^{1/2}\left(D_{1/4}^{2/3} - \frac{2}{37}\right)x(t) = \frac{\sqrt{2}\pi|x|}{4(4\pi-t)^2}\left(\frac{|x|}{|x|+3}+1\right) + \frac{|y|\sin^2(2\pi t)}{(10-t)^2} \\ t \in \mathrm{J} = [0,1], \\ x(0) = \frac{3}{4}, \quad x(1) = \frac{1}{4}, \quad x(0) = \frac{2}{3}, \quad x(1) = \frac{5}{2}. \end{cases} \tag{6.16}$$

Here

$\alpha_1 = 1/4,$	$\alpha_2 = 1/2,$	$\beta_1 = 1/3,$	$\beta_2 = 2/3$
$\gamma_1 = 3/4,$	$\gamma_2 = 1/4,$	$\lambda_1 = 1/27,$	$\lambda_2 = 2/37,$
$\eta_1 = 2/3,$	$\eta_2 = 5/2,$	$q = 1/4.$	

with

$$f(t,x) = (((\sin t + 1)e^{-t})/24)(x^2/(1+|x|))$$

Clearly, the function f is continuous. For each $x \in \mathbb{E}$ and $t \in [0,1]$, we have

$$|f(t,x_1,x_2)| \le \frac{\sqrt{3}}{81}|x_1| + \frac{\sqrt{2}}{49\pi}|x_2|$$

and

$$|g(t,x_1,x_2)| \leq \frac{\sqrt{2}}{64\pi}|x_1| + \frac{1}{100}|x_2|$$

Hence, the hypothesis (H2) is satisfied with $p_1^* = \frac{\sqrt{3}}{81}$, $q_1^* = \frac{\sqrt{2}}{49\pi}$, $p_2^* = \frac{\sqrt{2}}{64\pi}$ and $q_2^* = \frac{1}{100}$. We shall show that condition (6.12) holds with $\mathrm{J} = [0,1]$. Indeed,

$$\left\{\frac{2(p_1^*+q_1^*)}{\Gamma_q(\alpha_1+\beta_1+1)} + \frac{2|\lambda_1|}{\Gamma_q(\alpha_1+1)}\right\} + \left\{\frac{2(p_2^*+q_2^*)}{\Gamma_q(\alpha_2+\beta_2+1)} + \frac{2|\lambda_2|}{\Gamma_q(\alpha_2+1)}\right\} \simeq 0.6758 < 1$$

Simple computations show that all conditions of Theorem 6.4 are satisfied. It follows that the coupled system (6.16) has at least one weak solution defined on J.

6.5 Conclusion

We have provided sufficient conditions for the existence of the solutions of a new class of nonlinear Langevin fractional q-difference system with Dirichlet boundary conditions in Banach space. by using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting an example.

Chapitre 7

Existence and Uniqueness Results to a Fractional q-Difference Coupled System with Integral Boundary Conditions via Topological Degree Theory

7.1 Introduction

At the present day, there are many results on the existence of solutions for fractional differential equations. In this study, we focus on which that uses the topological degree. This method is a powerful tool for the existence of solutions to BVPs of many mathematical models that arise in applied nonlinear analysis. Very recently F. Isaia [81] proved a new fixed theorem that was obtained via coincidence degree theory for condensing maps. To see more applications about the usefulness of coincidence degree theory approach for condensing maps in the study for the existence of solutions of certain integral equations, the reader can be referred to [18, 19, 132, 81, 87, 120, 121, 122, 123, 124, 130].

Let $\mathscr{U} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $u : \mathrm{J} \to \mathbb{R}$ with

$$\|u\|_\infty = \sup\{|u(t)| : \quad t \in \mathrm{J}\}.$$

Then the product space $\mathscr{C} := \mathscr{U} \times \mathscr{V}$ defined by $\mathscr{C} = \{(u,v) : u \in \mathscr{U}, v \in \mathscr{V}\}$ is Banach space under the norm

$$\|(u,v)\|_{\mathscr{C}} = \|u\|_\infty + \|v\|_\infty.$$

This section is mainly concerned with the existence results for the following fractional q-difference system of the form

$$\begin{cases} D_q^{q_1} u_1(\tau) = \mathscr{F}_1(\tau, u_1(\tau), u_2(\tau)), \\ \qquad\qquad\qquad\qquad\qquad\qquad , \tau \in \mathrm{J} := [0,1], \\ D_q^{q_2} u_2(\tau) = \mathscr{F}_2(\tau, u_1(\tau), u_2(\tau)), \end{cases} \tag{7.1}$$

with the fractional boundary conditions

$$\begin{cases} u_1(0) = a_1\, I_q^{\beta_1} u(\eta_1), 0 < \eta_1 < 1,\ \beta_1 > 0, \\ u_1(1) = b_1\, I_q^{\alpha_2} u(\sigma_1), 0 < \sigma_1 < 1,\ \alpha_1 > 0, \\ u_2(0) = a_2\, I_q^{\beta_2} u(\eta_2), 0 < \eta_2 < 1,\ \beta_2 > 0, \\ u_2(1) = b_2\, I_q^{\alpha_2} u(\sigma_2), 0 < \sigma_2 < 1,\ \alpha_2 > 0. \end{cases} \tag{7.2}$$

For all $i = 1,2$, $D_q^{q_i}$ is the fractional q-derivative of the Caputo type of order $1 < q_i \leq 2$, and $\mathscr{F} : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given continuous function, $a_i, b_i, i = 1,2$ are suitably chosen real constants.

7.2 Existance Result [1]

For the existence of solutions for the problem (7.1)-(7.15), we need the following auxiliary lemmas.

Lemma 7.1. *Let $\mathscr{F}_i : \mathrm{J} \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function for each $i = 1,2$. Then problem (7.1)-(7.15) is equivalent to the problem of obtaining the solutions of the integral equation*

$$u_i(\tau) = I_q^{q_i} \mathscr{F}_{u_i}(\tau) + (\Lambda_{1,i} - \Lambda_{4,i}\tau)\, I_q^{q_i+\beta_i} \mathscr{F}_{u_i}(\eta_i) + (\Lambda_{2,i} + \Lambda_{3i,}\tau)\left(b_i I_q^{q_i+\alpha_i} \mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i} \mathscr{F}_{u_i}(1)\right) \tag{7.3}$$

if and only if u_i, $i = 1,2$ is a solution of the fractional boundary-value problem

$$\begin{cases} D_q^{q_1} u_1(\tau) = \mathscr{F}_{u_1}, \\ \\ D_q^{q_2} u_2(\tau) = \mathscr{F}_{u_2}, \end{cases} \quad , \tau \in J := [0,1], \tag{7.4}$$

subject with

$$\begin{cases} u_1(0) = a_1\, I_q^{\beta_1} u(\eta_1), 0 < \eta_1 < 1,\ \beta_1 > 0, \\ u_1(1) = b_1\, I_q^{\alpha_2} u(\sigma_1), 0 < \sigma_1 < 1,\ \alpha_1 > 0, \\ u_2(0) = a_2\, I_q^{\beta_2} u(\eta_2), 0 < \eta_2 < 1,\ \beta_2 > 0, \\ u_2(1) = b_2\, I_q^{\alpha_2} u(\sigma_2), 0 < \sigma_2 < 1,\ \alpha_2 > 0, \end{cases} \tag{7.5}$$

where

$$\begin{aligned} \Lambda_{1,i} &= \frac{a_i}{\Lambda_i}\left(1 - \frac{b_i \sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right), \quad \Lambda_{2,i} = \frac{a_i \eta_i^{\beta_i+1}}{\Lambda_i \Gamma(\beta_i+2)}, \\ \Lambda_{3,i} &= \frac{1}{\Lambda_i}\left(1 - \frac{a_i \eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right), \quad \Lambda_{4,i} = \frac{a_i}{\Lambda_i}\left(1 - \frac{b_i \sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right), \\ \Lambda_i &= \left(1 - \frac{a_i \eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right)\left(1 - \frac{b_i \sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right) + \frac{a_i \eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)}\left(1 - \frac{b_i \sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right). \end{aligned} \tag{7.6}$$

1. **A. Boutiara**, M. Benbachir, K. Guerbati, Existence and Uniqueness Results to a Fractional q-Difference Coupled System with integral Boundary conditions via Topological Degree Theory, (submitted).

Proof. For some constants $c_{0,i}$, $c_{1,i} \in \mathbb{R}$ and $1 < q_i \leq 2$, the general solution of $\mathscr{D}_q^{q_i} u_i(\tau) = \mathscr{F}_{u_i}(\tau)$ can be written as

$$u_i(\tau) = I_q^{q_i} \mathscr{F}_{u_i}(\tau) + c_{0,i} + c_{1,i}\, \tau. \tag{7.7}$$

Using the boundary conditions (7.5) in (7.7) we may obtain

$$\begin{aligned} \left(1 - \frac{a_i \eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right) c_{0,i} - \frac{a_i \eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)} c_{1,i} &= a_i I_q^{q_i+\beta_i} \mathscr{F}_{u_i}(\eta_i), \\ \left(1 - \frac{b_i \sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)}\right) c_{0,i} + \left(1 - \frac{b_i \sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right) c_{1,i} &= b_i I_q^{q_i+\alpha_i} \mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i} \mathscr{F}_{u_i}(1). \end{aligned} \tag{7.8}$$

which, on solving, yields

$$c_{0,i} = \frac{1}{\Lambda_i} \left\{ a_i \left(1 - \frac{b_i \sigma_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}\right) I_q^{q_i+\beta_i} \mathscr{F}_{u_i}(\eta_i) + \frac{a_i \eta_i^{\beta_i+1}}{\Gamma(\beta_i+2)} \left(b_i I_q^{q_i+\alpha_i} \mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i} \mathscr{F}_{u_i}(1)\right) \right\},$$

and

$$c_{1,i} = \frac{1}{\Lambda_i} \left\{ a_i \left(\frac{b_i \sigma_i^{\alpha_i}}{\Gamma(\alpha_i+1)} - 1\right) I_q^{q_i+\beta_i} \mathscr{F}_{u_i}(\eta_i) + \left(1 - \frac{a_i \eta_i^{\beta_i}}{\Gamma(\beta_i+1)}\right) \left(b_i I_q^{q_i+\alpha_i} \mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i} \mathscr{F}_{u_i}(1)\right) \right\}.$$

Substituting the value of $c_{0,i}, c_{1,i}$ in (7.7) we get (7.3), which completes the proof.

□

We use the following sufficient assumptions in the proofs of our main results.

(H1) There exist constants $\mathscr{L}_i > 0$, $i = 1,2$ such that for $\tau \in J$ and each $u_i, v_i \in \mathscr{C}$, $i = 1,2$.

$$\begin{aligned} |\mathscr{F}_1(\tau, u_1, u_2) - \mathscr{F}(\tau, v_1, v_2)| &\leq \mathscr{L}_1 \sum_{i=1}^{2} (|u_i - v_i|), \\ |\mathscr{F}_2(\tau, u_1, u_2) - \mathscr{F}(\tau, v_1, v_2)| &\leq \mathscr{L}_2 \sum_{i=1}^{2} (|u_i - v_i|). \end{aligned} \tag{7.9}$$

(H2) For arbitrary $\tau \in J$ and each $u_1, u_2 \in \mathscr{C}$ there exist constants $K_i, M_i, N_i > 0, i = 1,2$, and $p \in (0,1)$ such that

$$\begin{aligned} |\mathscr{F}_1(\tau, u_1(s), u_2(s))| &\leq K_1 \|u_1\|^p + M_1 \|u_2\|^p + N_1, \\ |\mathscr{F}_2(\tau, u_1(s), u_2(s))| &\leq K_2 \|u_1\|^p + M_2 \|u_2\|^p + N_2. \end{aligned} \tag{7.10}$$

In the following, we set an abbreviated notation for the fractional q-integral of the Caputo type of order $q_i > 0$, for a function with two variables as

$$I_q^{q_i} \mathscr{F}_{u_i}(\tau) = \frac{1}{\Gamma(q_i)} \int_0^\tau (\tau - qs)^{\alpha-1} \mathscr{F}(s, u_1(s), u_2(s)) ds.$$

Moreover, for computational convenience we put

$$\omega_i = \left\{ (|\Lambda_{1,i}| + |\Lambda_{4,i}|) \frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)} + (|\Lambda_{2,i}| + |\Lambda_{3,i}|) \left(\frac{|b_i| \sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)} \right) \right\}, \tag{7.11}$$

and

$$\bar{\omega}_i = \left\{ |\Lambda_{4,i}| \frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)} + |\Lambda_{3,i}| \left(\frac{|b_i|\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)} \right) \right\}. \tag{7.12}$$

By Lemma 7.1, we consider two operators $\mathscr{T}, \mathscr{S} : \mathscr{C} \longrightarrow \mathscr{C}$ as follows :

$$\mathscr{T} u_i(\tau) = I_q^{q_i} \mathscr{F}_{u_i}(\tau), \ \tau \in J,$$

and

$$\mathscr{S} u_i(\tau) = (\Lambda_{1,i} - \Lambda_{4,i}\ \tau) I_q^{q_i+\beta_i} \mathscr{F}_{u_i}(\eta_i) + (\Lambda_{2,i} + \Lambda_{3,i}\ \tau) \left(b_i I_q^{q_i+\alpha_i} \mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i} \mathscr{F}_{u_i}(1) \right), \ \tau \in J.$$

Then the integral equation (7.3) in Lemma 7.1 can be written as an operator equation

$$\mathscr{K} u_i(\tau) = \mathscr{T} u_i(\tau) + \mathscr{S} u_i(\tau), \ \ \tau \in J.$$

The continuity of $\mathscr{F}_i$, i=1,2, shows that the operator $\mathscr{K} : \mathscr{C} \to \mathscr{C}$ is well define and fixed points of the operator equation are solutions of the integral equations (7.3) in Lemma 7.1.

Lemma 7.2. *The operator $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is Lipschitz with constant $\sum_{i=1}^{2} \ell_{\mathscr{F}_i} = \sum_{i=1}^{2} \frac{\mathscr{L}_i}{\Gamma(q_i+1)}$. Moreover, $\mathscr{T}$ satisfies the growth condition given below*

$$\|\mathscr{T}(u_1,u_2)\| \le \sum_{i=1}^{2} \frac{1}{\Gamma(\alpha+1)} (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),$$

for every $u_i \in \mathscr{C}$.

Proof.

To show that the operator $\mathscr{T}$ is Lipschitz. Let $u_i, v_i \in \mathscr{C}$,i=1,2, then we have

$$\begin{aligned} |\mathscr{T} u_i(\tau) - \mathscr{T} v_i(\tau)| &= \left| I_q^{q_i} \mathscr{F}_{i,u_i} - I_q^{q_i} \mathscr{F}_{i,v_i} \right| \\ &\le I_q^{q_i} |\mathscr{F}_{i,u_i} - \mathscr{F}_{i,v_i}|(\tau) \\ &\le I_q^{q_i}(1)\ \mathscr{L}_i \sum_{i=1}^{2} (\|u_i - v_i\|) \\ &= \frac{\mathscr{L}_i}{\Gamma(q_i+1)} \sum_{i=1}^{2} (\|u_i - v_i\|). \end{aligned}$$

For all $\tau \in J$, we obtain

$$\|\mathscr{T} u_i - \mathscr{T} v_i\| \le \frac{\mathscr{L}_i}{\Gamma(q_i+1)} \sum_{i=1}^{2} (\|u_i - v_i\|).$$

Hence, $\mathscr{T} : \mathscr{C} \longrightarrow \mathscr{C}$ is a Lipschitzian on $\mathscr{C}$ with Lipschitz constant $\ell_{\mathscr{F}_i} = \frac{\mathscr{L}_i}{\Gamma(q_i+1)}$. By Proposition 1.67, $\mathscr{T}$ is κ–Lipschitz with constant $\ell_{\mathscr{F}_i}$. Moreover, for growth condition, we have

$$\begin{aligned} |\mathscr{T} u_i(\tau)| &\le I_q^{q_i} |\mathscr{F}_{u_i}|(\tau) \\ &\le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\ I_q^{\alpha}(1) \\ &= \frac{1}{\Gamma(q_i+1)} (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i). \end{aligned}$$

Hence it follows that

$$\|\mathscr{T}u_i\| \le \frac{1}{\Gamma(q_i+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),$$

which implies that

$$\|\mathscr{T}(u_1,u_2)\| \le \sum_{i=1}^{2} \frac{1}{\Gamma(\alpha+1)}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i).$$

□

Lemma 7.3. *$\mathscr{S}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathscr{S}u_i\| \le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i, \textit{ for every } u_i \in \mathscr{C},$$

where ω_i is given by (7.11).

Proof. Choose a bounded subset $D_r = \{(u_1,u_2) \in \mathscr{C} : \|(u_1,u_2)\| \le r\} \subset \mathscr{C}$ and consider a sequence $\{z_n = (u_{1,n}, u_{2,n})\} \in D_r$ such that $z_n \to z = (u_1,u_2)$ as $n \to \infty$ in D_r. We need to show that $\|\mathscr{S}z_n - \mathscr{S}z| \to 0, n \to \infty$. From the continuity of $\mathscr{F}_{i,u}$, it follows that $\mathscr{F}_{i,u_n} \to \mathscr{F}_{i,u}$, as $n \to \infty$. In view of (H_2), we obtain the following relations :

$$\begin{aligned}
&(\tau - sq)^{q_i-1}\|\mathscr{F}_{i,u_n} - \mathscr{F}_{i,u}\| \le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\tau - sq)^{q_i-1}, i = 1,2.\\
&(\eta_i - sq)^{q_i+\beta_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\eta_i - sq)^{q_i+\beta_i-1}, i = 1,2,\\
&(\sigma_i - sq)^{q_i+\alpha_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(\sigma_i - sq)^{q_i+\alpha_i-1}, i = 1,2,\\
&(1 - sq)^{q_i-1} \mapsto (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(1 - sq)^{q_i-1}, i = 1,2,
\end{aligned}$$

which implies that each term on the left is integrable. By Lebesgue Dominated convergent theorem, we obtain

$$\begin{aligned}
&I_q^{q_i+\beta_i}|\mathscr{F}_{i,u_n} - \mathscr{F}_{i,u}|(\eta_i) \to 0 \text{ as } n \to +\infty,\\
&I_q^{q_i+\alpha_i}|\mathscr{F}_{i,u_n} - \mathscr{F}_{i,u}|(\sigma_i) \to 0 \text{ as } n \to +\infty,\\
&I_q^{q_i}|\mathscr{F}_{i,u_n} - \mathscr{F}_{i,u}|(1) \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

It follows that $\|\mathscr{S}z_n - \mathscr{S}z\| \to 0$ as $n \to +\infty$. Which implies the continuity of the operator $\mathscr{S}$.

For the growth condition, using the assumption (H2) we have

$$\begin{aligned}
|\mathscr{S}u_i(\tau)| &\le (|\Lambda_{1,i}| + |\Lambda_{4_i}|)I_q^{q_i+\beta_i}\mathscr{F}_{u_i}(\eta_i) + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b|I_q^{q_i+\alpha_i}\mathscr{F}_{u_i}(\sigma_i) + I_q^{q_i}\mathscr{F}_{u_i}(1)\right),\\
&\le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(|\Lambda_{1,i}| + |\Lambda_{4_i}|)I_q^{q_i+\beta_i}(1)(\eta_i)\\
&+ (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)(|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b_i|I_q^{q_i+\alpha_i}(1)(\sigma_i) + I_q^{q_i}(1)\right)\\
&\le (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\Bigg\{(|\Lambda_{1,i}| + |\Lambda_{4_i}|)\frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}\\
&+ (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b_i|\frac{\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)}\right)\Bigg\}\\
&= (K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i.
\end{aligned}$$

Which implies that,

$$\|\mathscr{S}(u_1,u_2)\| \leq \sum_{i=1}^{2}(K_i\|u_1\|^p + M_i\|u_2\|^p + N_i)\omega_i, i = 1,2. \tag{7.13}$$

where ω_i, i=1,2 is given by (7.11). This completes the proof of Lemma 7.3.

□

Lemma 7.4. *The operator $\mathscr{S} : \mathscr{C} \longrightarrow \mathscr{C}$ is compact. Consequently, $\mathscr{S}$ is κ-Lipschitz with zero constant.*

Proof. In order to show that $\mathscr{S}$ is compact. Let us take a bounded set $\Omega \subset \mathscr{B}_r$, i=1,2. We are required to show that $\mathscr{S}(\Omega)$ is relatively compact in $\mathscr{C}$. For arbitrary $u_i \in \Omega \subset \mathscr{B}_r$, then with the help of the estimates (7.13) we can obtain

$$\|\mathscr{S}u\| \leq (K_i r^p + M_i r^p + N_i)\,\omega_i,$$

where ω_i is given by (7.11), which shows that $\mathscr{S}(\Omega)$ is uniformly bounded.
Now, for equi-continuity of $\mathscr{S}$ take $\tau_1,\tau_2 \in J$ with $\tau_1 < \tau_2$, and let $u_i \in \Omega$. Thus, we get

$$\begin{aligned}|\mathscr{S}u_i(\tau_2) - \mathscr{S}u_i(\tau_1)| &\leq |\Lambda_{4,i}|\,(\tau_2-\tau_1)\,I_q^{q_i+\beta_i}\mathscr{F}_{u_i}(\eta_i)\\ &\quad + |\Lambda_{3,i}|\,(\tau_2-\tau_1)\left(b_i I_q^{q_i+\alpha_i}\mathscr{F}_{u_i}(\sigma_i) - I_q^{q_i}\mathscr{F}_{u_i}(1)\right)\\ &\leq \bar{\omega}_i\,(K_i\|u_i\|^p + M_i\|v_i\|^p + N_i)\,(\tau_2-\tau_1).\end{aligned}$$

Which implies that,

$$|\mathscr{S}(u_1,u_2)(\tau_2) - \mathscr{S}(u_1,u_2)(\tau_1)| \leq \sum_{i=1}^{2}\bar{\omega}_i\,(K_i\|u_i\|^p + M_i\|v_i\|^p + N_i)\,(\tau_2-\tau_1).$$

where $\bar{\omega}_i$ is given by (7.12). From the last estimate, we deduce that

$$\|\mathscr{S}(u_1,u_2)(\tau_2) - \mathscr{S}(u_1,u_2)(\tau_1)\| \to 0 \quad \text{when} \quad \tau_2 \to \tau_1.$$

Therefore, $\mathscr{S}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathscr{S}$ is compact and hence by Proposition 1.66. $\mathscr{S}$ is κ–Lipschitz with zero constant.

□

Theorem 7.5. *Suppose that (H1)–(H2) are satisfied, then the BVP (7.1) has at least one solution $(u_1,u_2) \in \mathscr{C}$, provided that $\sum_{i=1}^{2}\ell_{\mathscr{F}_i} < 1$, $i = 1,2$, and the set of the solutions is bounded in $\mathscr{C}$.*

Proof. Let $\mathscr{T},\mathscr{S},\mathscr{K}$ are the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 7.2, $\mathscr{T}$ is κ–Lipschitz and by Lemma 7.4, $\mathscr{S}$ is κ–Lipschitz with constant 0. Thus, $\mathscr{K}$ is κ–Lipschitz with constant $\ell_{\mathscr{F}_i}$. Hence $\mathscr{K}$ is strict κ–contraction with constant $\ell_{\mathscr{F}_i}$. Since $\sum_{i=1}^{2}\ell_{\mathscr{F}_i} < 1$, so $\mathscr{K}$ is κ-condensing.

Now consider the following set

$$\Theta = \{(u_1,u_2) \in \mathscr{C} : \text{ there exist } \xi \in [0,1] \text{ such that } u_i = \xi\mathscr{K}u_i, i = 1,2\}.$$

We will show that the set Θ is bounded. For $u_i \in \Theta$, we have $u_i = \xi \mathscr{K} u_i = \xi(\mathscr{T}(u_i) + S(u_i))$, which implies that

$$\begin{aligned}\|u_i\| &\leq \xi(\|\mathscr{T} u_i\| + \|\mathscr{S} u_i\|) \\ &\leq \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right](K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),\end{aligned}$$

hence we get

$$\begin{aligned}\|(u_1,u_2)\| &\leq \xi(\|\mathscr{T}(u_1,u_2)\| + \|\mathscr{S}(u_1,u_2)\|) \\ &\leq \sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right](K_i\|u_1\|^p + M_i\|u_2\|^p + N_i),\end{aligned}$$

where ω_i is given by (7.11). From the above inequalities, we conclude that Θ is bounded in $\mathscr{C}$. If it is not bounded, then dividing the above inequality by $a := \|u_i\|$ and letting $a \to \infty$, we arrive at

$$1 \leq \sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right]\lim_{a\to\infty}\frac{K_i a^p + M_i a^p + N_i}{a} = 0,$$

which is a contradiction. Thus the set Θ is bounded in $\mathscr{C}$ and the operator $\mathscr{K}$ has at least one fixed point which represent the solution of BVP (7.1).

□

To end this section, we give an existence and uniqueness result.

Theorem 7.6. *Under assumption (H1) the BVP (7.1) has a unique solution if*

$$\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right]\mathscr{L}_i < 1. \tag{7.14}$$

Proof. Let $u_i, v_i \in \mathscr{C}$ and $\tau \in J$, then we have

$$\begin{aligned}|\mathscr{K} u_i(\tau) - \mathscr{K} v_i(\tau)| &\leq {}^{H}\mathscr{I}_q^{q_i}|\mathscr{F}_{u_i} - \mathscr{F}_{v_i}|(\tau) + (|\Lambda_{1,i}| + |\Lambda_{4,i}|)\, I_q^{q_i+\beta_i}|\mathscr{F}_{u_i} - \mathscr{F}_{v_i}|(\eta_i) \\ &\quad + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left\{|b_i| I_q^{q_i+\alpha_i}|\mathscr{F}_{u_i} - \mathscr{F}_{v_i}|(\sigma_i) + I_q^{q_i}|\mathscr{F}_{u_i} - \mathscr{F}_{v_i}|(1)\right\} \\ &\leq \mathscr{L}_i\sum_{i=1}^{2}(\|u_i - v_i\|)\left\{I_q^{q_i}(1)(1) + (|\Lambda_{1,i}| + |\Lambda_{4,i}|)\, I_q^{q_i+\beta_i}(1)(\eta_i)\right. \\ &\quad \left. + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\,|b_i| I_q^{q_i+\alpha_i}(1)(\sigma_i) + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\, I_q^{q_i}(1)\right\} \\ &\leq \mathscr{L}_i\sum_{i=1}^{2}(\|u_i - v_i\|)\left(\frac{1}{\Gamma(q_i+1)} + \left\{(|\Lambda_{1,i}| + |\Lambda_{4,i}|)\frac{\eta_i^{q_i+\beta_i}}{\Gamma(q_i+\beta_i+1)}\right.\right. \\ &\quad \left.\left. + (|\Lambda_{2,i}| + |\Lambda_{3,i}|)\left(|b_i|\frac{\sigma_i^{q_i+\alpha_i}}{\Gamma(q_i+\alpha_i+1)} + \frac{1}{\Gamma(q_i+1)}\right)\right\}\right) \\ &= \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right]\mathscr{L}_i\sum_{i=1}^{2}(\|u_i - v_i\|).\end{aligned}$$

Hence $\mathscr{K}$ is contraction as $\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right]\mathscr{L}_i<1$ and by Banach contraction principle $\mathscr{K}$ has a unique fixed point which is a unique solution of problem (7.1). This completes the proof.

$\square$

Remark 7.7. If the growth condition (H2) is formulated for $p=1$, then the conclusions of Theorem 7.5 remain valid provided that

$$\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right](K_i+M_i)<1.$$

7.3 Examples

In this section, in order to illustrate the main result, we consider two examples.

Example 7.8. Consider the following boundary value problem of a fractional differential equation :

$$\begin{cases} D_{\frac{1}{4}}^{\frac{4}{3}}u_1(\tau)=\frac{1}{e^{(\tau)}+9}\left(\frac{|u_1(\tau)|}{1+|u_1(\tau)|}\right)+\frac{\sqrt{3+\tau^2}|u_1(\tau)|}{20}+\tau, \quad \tau\in J:=[0,1], \\ D_{\frac{1}{4}}^{q_2}u_2(\tau)=\frac{\sin\left(\sqrt{|u_2(\tau)|}\right)}{16}+\left(\frac{e^{-\pi\tau}|u_2(\tau)|}{16+|u_2(\tau)|}\right)+(1+\tau^2), \\ u_1(0)=a_1\, I_{\frac{1}{4}}^{\frac{5}{2}}u(\tfrac{1}{4}), \quad u_1(1)=b_1\, I_{\frac{1}{4}}^{\frac{1}{4}}u(\tfrac{1}{5}), \\ u_2(0)=a_2\, I_{\frac{1}{4}}^{\frac{4}{5}}u(\tfrac{2}{5}), \quad u_2(1)=b_2\, I_{\frac{1}{4}}^{\frac{4}{5}}u(\tfrac{2}{5}). \end{cases} \tag{7.15}$$

Note that, this problem is a particular case of BVP (7.1), where

$$\begin{aligned} &q_1=\frac{4}{3}, q_2=\frac{7}{5}, \mathbf{q}=\frac{\mathbf{1}}{\mathbf{4}}, \sigma_1=\frac{1}{5}, \\ &a_1=b_2=\frac{1}{2}; a_2=b_1=\frac{1}{5}; \eta_2=\sigma_2=\frac{2}{5}, \beta_1=\frac{5}{2}, \\ &\alpha_1=\eta_1=\frac{1}{4}, \beta_2=\frac{1}{3}, \alpha_2=\frac{4}{5}. \end{aligned} \tag{7.16}$$

Using the given values of the parameters in (7.6) and (7.11), by the Matlab program, we find that

$$\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right]=2.332,$$

In order to illustrate Theorem 7.5, we take

$$\begin{aligned} \mathscr{F}_1(\tau,u_1(\tau),u_2(\tau))&=\frac{1}{e^{(\tau-1)}+9}\left(\frac{|u_1(\tau)|}{1+|u_1(\tau)|}\right)+\frac{\sqrt{3+\tau^2}|u_2(\tau)|}{20}+\tau, \\ \mathscr{F}_2(\tau,u_1(\tau),u_2(\tau))&=\frac{\sin\left(\sqrt{|u_1(\tau)|}\right)}{16}+\left(\frac{e^{-\pi\tau}|u_2(\tau)|}{16+|u_2(\tau)|}\right)+(1+\tau^2). \end{aligned} \tag{7.17}$$

We can easily show that

$$\begin{aligned}
|\mathscr{F}_1(\tau,u_1,u_2)-\mathscr{F}(\tau,v_1,v_2)| &\le \frac{1}{10}\sum_{i=1}^{2}[|u_i-v_i|],\\
|\mathscr{F}_2(\tau,u_1,u_2)-g(\tau,v_1,v_2)| &\le \frac{1}{16}\sum_{i=1}^{2}[|u_i-v_i|].
\end{aligned} \tag{7.18}$$

Hence the condition (H1) holds with $\mathscr{L}_1=\frac{1}{10}$, $\mathscr{L}_2=\frac{1}{16}$. Further from the above given data it is easy to calculate

$$\sum_{i=1}^{2}\ell_{\mathscr{F}_i}=\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}\right]\mathscr{L}_i=1.8703,$$

On the other hand, for any $\tau\in \mathrm{J}, u\in\mathbb{R}$ we have

$$|\mathscr{F}(\tau,u_1,u_2)|\le\frac{1}{10}|u_1|+\frac{1}{10}|u_2|+1,$$

$$|\mathscr{F}(\tau,u_1,u_2)|\le\frac{1}{16}|u_1|+\frac{1}{16}|u_2|+2,$$

Hence condition (H2) holds with $M_1=K_1=\frac{1}{10}$, $M_2=K_2=\frac{1}{16}$, $p=N_1=1$ and $N_2=2$. In view of Theorem 7.5

$$\Theta=\{(u_1,u_2)\in\mathscr{C}:\ \text{there exist } \xi\in[0,1] \text{ such that } u_i=\xi\mathscr{K}u_i, i=1,2\},$$

is the solution set ; then

$$\begin{aligned}
\|(u_1,u_2)\| &\le \xi(\|\mathscr{T}(u_1,u_2)\|+\|\mathscr{S}(u_1,u_2)\|)\\
&\le \sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right]((K_i+M_i)(\|u_1\|+\|u_2\|)+N_i).
\end{aligned}$$

From which, we have

$$\|(u_1,u_2)\|\le\frac{\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right]N_i}{1-\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right](M_i+K_i)}=19.8124.$$

By Theorem 7.5, the BVP (7.1) with the data (7.19) and (7.17) has at least a solution u in $C(\mathrm{J}\times\mathbb{R},\mathbb{R})$. Furthermore $\sum_{i=1}^{2}\left[\frac{1}{\Gamma(q_i+1)}+\omega_i\right]\mathscr{L}_i=0.1854,<1$. Hence by Theorem 7.6 the boundary value problem (7.1) with the data (7.19) and (7.17) has a unique solution.

Example 7.9. Let us consider coupled system (7.1) with specific data :

$$\begin{aligned}
&q_1=\frac{3}{2}, q_2=\frac{5}{4}, \mathrm{q}=\frac{1}{2}, \sigma_1=\frac{1}{3},\\
&a_1=b_1=a_2=b_2=1; \beta_1=\eta_2=\sigma_2=\frac{1}{2},\\
&\alpha_1=\eta_1=\frac{3}{4}, \beta_2=\frac{2}{3}, \alpha_2=\frac{2}{5}.
\end{aligned} \tag{7.19}$$

In order to illustrate Theorem 7.5, we take

$$\begin{aligned} \mathscr{F}_1(\tau,u_1,u_2) &= \tfrac{1}{4} + \tfrac{e^{-\pi\tau}\sqrt{|u_1(\tau)|}}{16+\sqrt{|u_1(\tau)|}} + \tfrac{\cos\sqrt{|u_2(\tau)|}}{16} \\ \mathscr{F}_2(\tau,u_1,u_2) &= \tfrac{1}{8} + \tfrac{\sin\sqrt{|u_1(\tau)|}}{24} + \tfrac{\sqrt{|u_2(\tau)|}}{24} \end{aligned} \tag{7.20}$$

One has

$$\begin{aligned} |\mathscr{F}_1(\tau,u_1,u_2) - \mathscr{F}(\tau,v_1,v_2)| &\le \frac{1}{16}\sum_{i=1}^{2}[|u_i - v_i|], \\ |\mathscr{F}_2(\tau,u_1,u_2) - g(\tau,v_1,v_2)| &\le \frac{1}{24}\sum_{i=1}^{2}[|u_i - v_i|]. \end{aligned} \tag{7.21}$$

Hence the condition (H1) holds with $\mathscr{L}_1 = \frac{1}{16}$ and $\mathscr{L}_1 = \frac{1}{24}$. Further from the above given data it is easy to calculate

$$\sum_{i=1}^{2} \ell_{\mathscr{F}_i} = \sum_{i=1}^{2} \frac{\mathscr{L}_i}{\Gamma(\alpha+1)} = 0.1446.$$

Using the given values of the parameters in (7.6) and (7.11), by the Matlab program, we find that

$$\sum_{i=1}^{2} \frac{1}{\Gamma(q_i+1)} + \omega_i = 4.6588. \tag{7.22}$$

Hence condition (H1) holds with $\mathscr{L}_1 = \frac{1}{16}$, $\mathscr{L}_2 = \frac{1}{24}$. We shall check that condition (7.14) is satisfied. Indeed using the Matlab program, we can find

$$\sum_{i=1}^{2} \left[\frac{1}{\Gamma(q_i+1)} + \omega_i\right] \mathscr{L}_i = 0.2332 < 1.$$

Hence by Theorem 7.6 the boundary value problem (7.1)-(7.15) has a unique solution.

Conclusion and Perspective

In this thesis research, our main scientific contributions focused on the existence and uniqueness of solutions for various classes of initial value problem and boundary value problem for nonlinear fractional differential equations involving different types of fractional derivatives and integrals. As well as we studied different classes of fractional differential equations. We have shown the interest of a new fractional derivative with respect to another function ψ, in the sense of the Hilfer fractional derivative so-called ψ-Hilfer for which, it can be considered as an interpolant between the derivatives of ψ-Riemann-Liouville and of ψ-Caputo because it is a generalization of all fractional derivatives and also fractional integrals used in this thesis. The results are based on the argument of the fixed points theorems Some appropriate fixed point theorems have been used, in particular ; Banach contraction, Schaefer's fixed point theorem, Boyd and Wong fixed point theorem, Leray-Schauder nonlinear alternative fixed point theorem, Dhage fixed point theorem and Mönch's fixed points combined with the technique of measures of noncompactness. Also using Isaia topological degree theory.

For the perspective and the possible generalization, it would be interesting to extend the results of the present thesis by considering differential inclusions and extend the problems studied on Banach and Fréchet spaces with another technique, other fixed point theorem and determine the conditions that befit closer to obtain the best results. As another proposal, considering some type of fractional derivatives and integrals have been presented recently with respect to another function (namely, ψ-Hilfer). We will use the numerical method to solve these problems. Also, we will study the problem of stability for a class of boundary value problem for nonlinear fractional differential equations. These suggestions will be treated in the future.

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