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Existence de solutions pour des problème aux limites avec p-Laplacien

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List of symbols

We use the following notations throughout this thesis

Acronyms

- **FC :** Fractional calculus.
- **BVP :** Boundary value problem.
- **FBVP :** Fractional boundary value problem.
- **FIBVP :**Fractional impulsive boundary value problem.
- **FDEswP-L**: The fractional differential equations with p-Laplacien operator.

Notations

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{N}_0$: Set of positive integers n.
- $\mathbb{Z}$: Set of integer z.
- $\mathbb{Z}_0$: Set of nonegative integer z.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{R}_+$: Set of nonegative real numbers.
- $\mathbb{C}$: Set of complex numbres.
- $\in$: belongs to.
- max : Maximum.
- min : Minimum.
- $n!$: Factorial (n); $n \in \mathbb{N}$: The product of all the integers from 1 to n.

- $\mathrm{Re}(z)$: The real part of number $z \in \mathbb{C}$.
- $\mathrm{Im}(z)$: The imaginary part of number $z \in \mathbb{C}$.
- $\Gamma(.)$: Gamma function.
- $(z)_n$: Pochhammer symbol, defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$.
- $\beta(.,.)$: Beta function.
- ${}_2F_1(.,.;.;.)$: Gauss hypergemetric function.
- $I^{\alpha}_{a^+}$: The Riemann-Liouville fractional integral of order $\alpha > 0$.
- ${}^{RL}D^{\alpha}_{a^+}$: The fractional derivative of order $\alpha > 0$ in the sense of Riemann-Liouville.
- D^{α}_{a} : The fractional derivative of orde $\alpha > 0$ in the sense of Caputo.
- ${}^{\rho;\grave{a}^+}\mathbb{I}^{\sigma}_{K}$:The Katugampola fractional integral of order $\alpha > 0$, $\rho > 0$.
- ${}^{\rho;K}\mathcal{D}^{\alpha}_{\grave{a}^+}$: The Katugampola fractional derivative($K\mathbb{F}\mathbb{D}$) of order $\alpha > 0$, $\rho > 0$.
- ${}^{\rho;CK}\mathcal{D}^{\alpha}_{\grave{a}^+}$:The Caputo-Katugampola fractional derivative ($CK\mathbb{F}\mathbb{D}$) of order $\alpha > 0$, $\rho > 0$.
- ${}^{\psi}{}_{t}\mathcal{I}^{\alpha}_{a^+}$: The $\psi-$ Riemann-Liouville ($\psi-$ RL) fractional integral of order α with respect to another function ψ.
- ${}^{\psi}{}_{t}\mathcal{D}^{\alpha}_{a^+}$: The $\psi-$ Riemann-Liouville ($\psi-$ RL) fractional derivative order $\alpha \in (n-1, n)$ with respect to another function ψ.
- ${}^{\psi;C}{}_{t}\mathcal{D}^{\alpha}_{a^+}$ The $\psi-$ Caputo ($\psi-$ C) fractional derivative order $\alpha \in (n-1, n)$ with respect to another function ψ.
- $C(J)$: Space of continuous functions on J.
- $C^{+}(J)$: Space of positive continuous functions on J.
- $C^{n}(J)$: Space of n time continuously. differentiable functions on J.
- $C^{n}_{\rho}(J)$: Space of n continuously differentiable functions on J, with respect to δ_{ρ}.
- $AC(J)$: Space of absolutely continuous functions on J.
- $L(a, T)$: space of Lebesgue integrable functions on (a, T).
- $L^{p}(a, T)$: space of measurable functions u with $\mid u \mid^{p}$ belongs to $L(a, T)$, $p \in [1, +\infty)$.
- $L^{\infty}(a, T)$: Space of functions u that are essentially bounded on (a, T).

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General Introduction

The boundary value problems is considered as a subject which has a long and enriched history ranging from its theoretical (existence and uniqueness of solutions, the multiplicity of solutions, ...) development to the methods and techniques of finding or approximating their solutions. Boundary value problems often arise naturally in a variety of applied fields and can be categorized as well posed and ill posed, local and nonlocal, linear and nonlinear, singular and nonsingular, and free and fixed problems. Ordinary, partial, functional, fractional, and integrodifferential equations together with the boundary data varying from two-point and periodic to multipoint and nonlocal boundary conditions constitute interesting and important classes of boundary value problems.

Equations with a p-Laplacian operators arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws and so on. Torres [82] studied the existence of at least three positive solutions by using Leggett-Williams fixed point theorem. Liang *et al.* [53], used the fixed point theorem of Avery and Henderson to show the existence of at least two positive solutions. Zhao, Wang and Ge [86], studied the existence of at least three positive solutions by using Leggett-Williams fixed point theorem. Chai [25], obtain results for the existence of at least one nonnegative solution and two positive solutions by using fixed point theorem on cone. Su *et al.* [72], studied the existence of one and two positive solution by using the fixed point index theory. Su [71], studied the existence of one and two positive solution by using the method of defining operator by the reverse function of Green function and the fixed-point index theory. Tang *et al.* [81], studied the existence of positive solutions of fractional differential equation with p-laplacian by using the coincidence degree theory.

Fractional calculus is a generalization of ordinary differential equations and integration to arbitrary non integer orders. Recently, the number of researches dealing with differential equations with fractional derivatives has been increased so that the task of studying this type of equations is more important because of its multiple applications in various branches of science such as, physics, chemistry, chemical physics, electrical networks, control of dynamic systems, engineering, biological science, optics and signal processing, where many relations and key laws ruling between the variables appear as form of differential equations with fractional derivatives. For more explanations and examples, we refer the reader to the monographs [1, 2, 3, 44, 45, 59, 60, 62, 64, 70, 84], the papers [4, 5, 6] and the references therein.

This thesis is organized as follows:
Chabter 1 contains only, necessary notations, definitions and basic lemmas that will be used in the proofs of our main results.

In **Chapter** 2 we look at the existence and multiplicity of concave positive solutions for a boundary value problem for two-sided fractional differential equations involving the Caputo derivative:

$$D_{1^-}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha}u(t)\right)\right)+a(t)f(u(t))=0, \quad 0<t<1, \tag{0.1}$$

$$u(0)-B_0\left(D_{0^+}^{\alpha}u(0)\right)=0, \quad u''(0)=0, \quad u'(1)-\mu u'(\eta)=\lambda, \tag{0.2}$$

$$D_{0^+}^{\alpha}u(1)=0, \quad \left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]'=0, \quad \left[\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right]''=0, \tag{0.3}$$

where $f:\mathbb{R}_+\to\mathbb{R}_+,\ B_0:\mathbb{R}\to\mathbb{R}_+,\ a:(0,1)\to\mathbb{R}_+$ are given continuous functions, $D_{0^+}^{\alpha}, D_{1^-}^{\beta}$, are respectively; the left-sided and the right-sided Caputo fractional derivatives with $2<\beta,\alpha\leq 3$ and $\phi_p(s)$ is p-Laplacian operator: i.e., $\phi_p(s)=|s|^{p-2}s,\ p>1,\ \left(\phi_p\right)^{-1}=\phi_q,\ \frac{1}{p}+\frac{1}{q}=1,\ \eta\in(0,1),\ \mu\in[0,1)$ are two arbitrary constants and $\lambda\in\mathbb{R}_+:=[0,\infty)$ is a parameter.

By means of the Leggett-Williams fixed point theorem, we obtain the existence of at least three solutions. Some illustrative examples are presented in the last section of this chapter.

In **Chapter** 3 we consider the following fractional boundary value problem (FBVP):

$$\begin{cases} {}^{\rho_2;CK}\mathcal{D}_{\grave{\imath}^-}^{\sigma_2}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}\right)\right)(\tau)+\hbar(\tau)\wp(\mathrm{q}(\tau))=0, \qquad \grave{a}<\tau<\grave{\imath}, \\ \mathrm{q}(\grave{a})-F_\circ\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\grave{a})\right)=0, \\ \delta_{\rho_1}^2\mathrm{q}(\grave{a})=0, \\ \delta_{\rho_1}^1\mathrm{q}(\grave{\imath})=\mu\delta_{\rho_1}^1\mathrm{q}(\eta)+\lambda, \\ {}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\grave{\imath})=-\delta_{\rho_2}^1\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}\right)\right](\grave{a})=\delta_{\rho_2}^2\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}\right)\right](\grave{\imath})=0, \end{cases} \tag{0.4}$$

where ${}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}$ and ${}^{\rho_2;CK}\mathcal{D}_{\grave{\imath}^-}^{\sigma_2}$, $(\rho_1,\rho_2\in\mathbb{R}\setminus\{1\})$ are the right and left sided Caputo-Katugampola fractional derivatives (CKFD), $2<\sigma_1,\sigma_2\leqslant 3$, ϕ_p is the pL operator, i.e., $\phi_p(\xi)=|\xi|^{p-2}\xi$, $p>1$,

$$\delta_\rho^k=\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k,$$

$F_\circ$ is a continuous even function, $\wp$, $\hbar$ are continuous and positive. $\eta\in(\grave{a},\grave{\imath})$, $0\leq\mu<1$, and $\lambda\geq 0$.

Under some sufficient conditions, the existence of at least one, two and three positive solutions for the BVP (0.4) are ensuring.

The least **Chapter** 4 deals with the existence and uniqueness results for boundary-value problem of following nonlinear $\psi-$Caputo fractional impulsive differential equations:

$$\begin{cases} {}^{\psi_2;C}{}_t\mathcal{D}_{T^-}^{\beta}\left(\rho(t)\phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha}u\right)\right)(t)+s(t)\phi_p\left(u(t)\right)=f(t,u(t)) \quad a<t<T, \\ \Delta\left(u\left(t_k\right)\right)=I_k^1\left(u\left(t_k\right)\right), \Delta\phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha}u\right)\left(t_k\right)=I_k^2\left(u\left(t_k\right)\right), k=1,2,\ldots,m \\ u(a)=u_0+\lambda\left|{}^{\psi_3}{}_t\mathcal{I}_{a^+}^{\gamma}\eta(t)\mid u(t)|^{p-1}\right|_{t=T}^{p^\star-1}, {}^{\psi_1;C}\mathcal{D}_{a^+}^{\alpha}u(T)=u_1 \end{cases} \tag{0.5}$$

where $p, p^{\star} > 1$, $0 < \alpha, \beta, \gamma \leqq 1$, ϕ_p is a p -Laplacian operator, $s(t)$, $\rho(t)$ and $\eta(t) \in C([a,T], \mathbb{R}_{+}^{\star})$, $f \in C([a,T] \times \mathbb{R}, \mathbb{R})$, $u_0, u_1, \lambda \in \mathbb{R}$, for $k = 1, 2, \ldots, m, i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, $a = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$. $\Delta u\left(t_k\right) = u\left(t_k^{+}\right) - u\left(t_k^{-}\right)$, $u\left(t_k^{+}\right)$ and $u\left(t_k^{-}\right)$ denote the right and the left limits of $u(t)$ at $t = t_k$ $(k = 1, 2, \ldots, m)$ respectively and $\Delta \phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha} u\right)\left(t_k\right)$ has a similar meaning for $\phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha} u\right)\left(t_k\right)$.

This chapter focus to deals with the existence and uniqueness results for boundary-value problem of nonlinear $\psi-$Caputo fractional impulsive differential equations with celebrated p-Laplacian operator via of some celebrated fixed theorems, a new results are given.

Chapter 1

Preliminaries and background materials

This chapter is devoted to necessary notations, definitions and basic lemmas that will be used in the proofs of our main results. We also present fixed point theorems which are crucial in our results regarding fractional differential equations.

1.1 Functional spaces

In addition to the notations, we present some functional spaces. Let $\mathbb{R} = (-\infty, +\infty)$, $J = [a, T] \subset (0, \infty)$ and $\rho > 0$.

1. $[\alpha]$ is the largest integer less than or equal to α. Throughout the paper, we use $n = [\alpha]$ if α is an integer and $n = [\alpha] + 1$ otherwise.

2. $C^+(J) = \{y \in C(J),\ y(t) \geq 0\ \ \forall t \in J\}$.

Definition 1.1.1. [44, 45] $C(J)$ denotes the Banach space of continuous functions h on J endowed with the norm

$$\|h\|_C = \max_{x \in J} |h(x)|.$$

Analogously, $C^n(J)$ denote the spaces of n times continuously differentiable functions on J. $C^n_\rho(J)$ is the Banach space of n continuously differentiable functions on J, with respect to δ_ρ:

$$C^n_\rho(J) = \left\{ h \in C(J) : \delta^k_\rho h \in C(J), k = 0, 1, ..., n \right\},$$

endowed with the norm

$$\|h\|_{C^n_\rho} = \sum_{k=0}^{n} \|\delta^k_\rho h\|_C.$$

Denote by $L^1(a, T)$ or $(L(a, T))$ the Banach space of Lebesgue integrable functions h with the norm

$$\|h\|_{L^1} = \int_a^T |h(t)| \mathrm{d}t.$$

and we denote $L^p(a,T)$ the space of Lebesgue integrable functions on (a,T) where $|h|^p$ belongs to $L^1(a,T)$, endowed with the norm

$$\|h\|_{L^p}^p = \||h(t)|^p\|_{L^1} = \left[\int_a^T |h(t)|^p \mathrm{d}t\right].$$

.

When $p=\infty$, $L^\infty(a,T)$ is the space of all functions h that are essentially bounded on J with essential supremum

$$\|h\|_{L^\infty} = \operatorname{ess\,sup}_{t\in J} |h(t)| = \inf\{c \geq 0 : |h(t)| \leq c \text{ for a.e. } t\}.$$

Definition 1.1.2 (Function space [44, 45])**.** For $c\in\mathbb{R}$, consider the Banach space

$$\mathcal{M}_c^p(a,T) = \left\{ h : J \to \mathbb{R} : \ \|h\|_{\mathcal{M}_c^p} := \left(\int_a^T \mid t^c\, h(t) \mid^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} < +\infty \right\}.$$

In particular, if $c = 1/p$ then the space $\mathcal{M}_c^p(a,T)$ coincides with the $L^p(a,T)$-space:
$\mathcal{M}_c^p(a,T) = L^p(a,T)$.

Remark 1.1.3. If $c\in\mathbb{R}_+^\star$ and $T \leq (pc)^{\frac{1}{pc}}$ then $C(J) \hookrightarrow \mathcal{M}_c^p(J)$ and $\|h\|_{\mathcal{M}_c^p} \leq \|h\|_C$, $\forall h \in C(J)$.

Definition 1.1.4. [44, 45, 46] A function u is said absolutely continuous on J if for all $\varepsilon > 0$ there exists a number $\nu > 0$ such that; for all finite partition $[a_i, b_i]_{i=1}^n$ in J then $\sum_{k=1}^n (b_k - a_k) < \nu$ implies that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$.

We denote by $AC(J)$ (or $AC^1(J)$) the space of all absolutely continuous functions defined on J. It is well known that $AC(J)$ coincides with the space of primitives of Lebesgue summable functions:

$$h \in AC(J) \Leftrightarrow h(t) = c + \int_a^t \chi(s)\mathrm{d}s, \quad \chi \in L^1(a,T), \tag{1.1.1}$$

and so an absolutely continuous function h has a summable derivative $h'(t) = \chi(t)$ almost everywhere on J. Thus (1.1.1) yields

$$h'(t) = \chi(t) \text{ and } \ c = h(a).$$

Definition 1.1.5. [44, 45] For $n\in\mathbb{N}^*$ we denote by $AC^n(J)$ the space of functions h which have continuous derivatives up to order $n-1$ on J such that $h^{(n-1)}$ belongs to $AC(J)$:

$$AC^n(J) = \left\{h \in C^{n-1}(J) : h^{(n-1)} \in AC(J)\right\}.$$

The space $AC^n(J)$ consists of those and only those functions h which can be represented in the form

$$h(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}\chi(s)\mathrm{d}s + \sum_{k=0}^{n-1} c_k t^k, \tag{1.1.2}$$

where $\chi \in L^1(a,T), c_k\ (k=1,\ldots,n-1) \in \mathbb{R}$.
It follows from (1.1.2) that

$$\chi(t) = h^{(n)}(t) \text{ and } c_k = \frac{h^{(k)}(a)}{k!},\ (k=1,\ldots,n-1).$$

Definition 1.1.6. [44, 45, 46] Let ψ be strictly increasing and n times differentiable function on J, then $AC^n_\psi(J) = \left\{u : J \to \mathbb{R} \ \text{ and } \delta^{[n-1]}_\psi u \in AC(J), \ \delta^{[n-1]}_\psi u = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right) u\right\}$ denotes the Banach space of n times absolutely continuous with respect to the strictly increasing differentiable function ψ.

In particular, if $\psi(t) = t$ then the space $AC^n_\psi(J)$ coincides with the $AC^n(J)$-space: $AC^n_\psi(J) = AC^n(J)$.

Definition 1.1.7 (PC-Function space)**.** For $m \in \mathbb{N}^*$ we consider finite collections of intervals $J_0 = [a; t_1]$, $J_1 = (t_k, t_{k+1}]$, included in a fixed interval J, with $k = 1, 2, \ldots, m$, $a = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$. One can see that $J = \bigcup_{k=1}^{n} J_k$.

Let $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} : u \in C(J_k, \mathbb{R})$ for $k = 1, 2, \ldots, m$ and there exist $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ at $t = t_k$ with $u\left(t_k^-\right) = u\left(t_k\right)\}$.

Then $PC(J, \mathbb{R})$ is a Banach space endowed with the norm $\|u\| = \sup_{t\in J} |u(t)|$.

Definition 1.1.8 (Heaviside function H)**.** We define the Heaviside function as follows

$$H(t) = \begin{cases} 1 & \text{if } t \geqq 0 \\ 0 & \text{elseif} \end{cases} \tag{1.1.3}$$

Proposition 1.1.9. *Let u be a piecewise function ($u \in PC(J, \mathbb{R})$), $t_1, t_2, \cdots, t_k$ for $k = 1, 2, \cdots, m$ the fixed moments of impulsive effect and $\varrho^k = u(t_k^+) - u(t_k^-)$ the magnitude and direction of the impulsive effect at t_k. Then u can be written as the sum of a continuous function g and the Heaviside functions.*

$$u(t) = g(t) + \sum_{j=0}^{k} \varrho^j H(t - t_j), \tag{1.1.4}$$

where $\varrho^0 = 0$.

1.2 Some basic special functions

Here we introduce some definitions and the important of some special symbols and functions which are used in fractional calculus. The Gamma function plays a basic role of the generalized factorial and the Beta function occur when we are computing the fractional derivatives of some power functions.

1.2.1 The Gamma function

Definition 1.2.1 ([63, 70])**.** The gamma-function $\Gamma(z)$ is defined by the integral.

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t, \tag{1.2.1}$$

which converges in the right half of the complex plane, that is, $\mathrm{Re}(z) > 0$. One can see that, the gamma-function $\Gamma(z)$ is defined as the Euler integral of the second kind, the Mellin transform of the exponential function.

When we substitute e^{-t} in (1.2.1) by the well-known limit

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function (1.2.1)

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{(z)_{n+1}},$$

where $(z)_n$ is the Pochhammer symbol, defined for complex $z \in \mathbb{C}$ and nonnegative integer $n \in \mathbb{N}_0$ by this equation

$$(z)_n = z(z+1)(z+2)\cdots(z+n-1), \quad (z)_0 = 1. \tag{1.2.2}$$

The simplest properties of the gamma function is the reduction formula that is

$$\Gamma(z+1) = z\Gamma(z), \quad \text{Re}(z) > 0 \tag{1.2.3}$$

it is obtained from (1.2.1) by integration by parts. By using this simplest properties, one can see that, the Euler gamma function has an extension in the left half of the complex plane ($\text{Re}(z) < 0$) given by

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \quad (\text{Re}(z) > -n; \;\; n \in \mathbb{N}_0; \; z \notin \mathbb{Z}_0^-). \tag{1.2.4}$$

The formula (1.2.4) immediately implies that, the Euler gamma function is analytic everywhere in the complex plane $\mathbb{C}$ except negative integer numbers, wich are the simple poles of $\Gamma(z)$ in the sense of complex analysis.
For $z = n$, equations (1.2.2) and (1.2.4) yield

$$\Gamma(n+1) = (1)_n = n!, \quad n \in \mathbb{N}_0, \tag{1.2.5}$$

as usual $0! = 1$.

1.2.2 The beta function

Definition 1.2.2 ([63, 70]). The beta function $B(z, z')$ is defined by the first kind of the Euler integral:

$$B(z, z') = \int_0^1 t^z (1-t)^{z'} \mathrm{d}t, \tag{1.2.6}$$

wher $\text{Re}(z) > 0, \text{Re}(z') > 0$, while it is conditionally convergent for $\text{Re}(z) = 0$ or $\text{Re}(z') = 0$ ($z \neq 0$, $z' \neq 0$). For more explication this integral

$$B(z, z') = \int_0^{1-\epsilon} t^z (1-t)^{z'} \mathrm{d}t,$$

has a limit when $\epsilon \to 0$.

We have also that, the function beta has a relation with the gamma functions by this formula:

$$B(z, z') = \frac{\Gamma(z)\Gamma(z')}{\Gamma(z+z')} \quad (z, z' \notin \mathbb{Z}_0^-)$$

1.2.3 The Gauss hypergeometric function

Definition 1.2.3. A function is defined in the unit disk as the sum of the hypergeometric series

$$ {}_2F_1(a_1,a_2;a_3;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k}{(a_3)_k}\frac{z^k}{k!} \tag{1.2.7}$$

is called the Gauss hypergeometric function, where $\mid z \mid< 1$; $a_1, a_2 \in \mathbb{C}$, $a_3 \in \mathbb{C}\setminus \mathbb{Z}_0^-$ and $(.)_k$ is the Pochhammer symbol (1.2.2).

The series in (1.2.7) is convergent for $\mid z \mid< 1$ and for $\mid z \mid= 1$, $\text{Re}(a_3 - a_1 - a_2) > 0$, when $\mid z \mid= 1$, $-1 < \text{Re}(a_3 - a_1 - a_2) \leqq 0$ it is conditionally convergent. For other values of z, the Gauss hypergeometric function is defined as an analytic continuation of the series (1.2.7). One such analytic continuation is given by the follow Euler integral representation:

$$ {}_2F_1(a_1,a_2;a_3;z) = \frac{1}{B(a_2,a_3-a_2)}\int_0^1 \frac{t^{a_2-1}(1-t)^{a_3-a_2-1}}{(1-zt)^{a_1}}\mathrm{d}t,\ 0<\text{Re}(a_2)<\text{Re}(a_1),\ \mid \arg(1-z) \mid< \pi. \tag{1.2.8}$$

Properties 1.2.4 ([63, 70]). The Gauss hypergeometric function has some simplest properties are follow:

1. $$ {}_2F_1(a_1,a_2;a_3;z) = {}_2F_1(a_2,a_1;a_3;z).$$

2. $$ {}_2F_1(a_1,a_2;a_3;0) = {}_2F_1(0,a_2;a_3;z) = 1.$$

3. $$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right) = {}_2F_1\left(1,1;\frac{3}{2};z^2\right)\sqrt{1-z^2} = \frac{\arcsin(z)}{z}.$$

4. $$ {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right) = \frac{\arctan(z)}{z}.$$

5. $$ F_1\left(a_1,a_2;a_2;z\right) = \frac{1}{(1-z)^{a_1}}.$$

1.2.4 Mittag-Leffler Functions

In this subsection we present the definitions and some properties of two classical Mittag-Leffler functions. For more detailed information may be found in [41, 57, 63, 70, 83].

Definition 1.2.5. The Mittag-Leffler function of one parameters is an entire function introduced by Mittag-Leffler [57], is defined by the series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0) \tag{1.2.9}$$

it is a direct generalization of the exponential function $\exp(x)$ and plays a major role in fractional calculus.

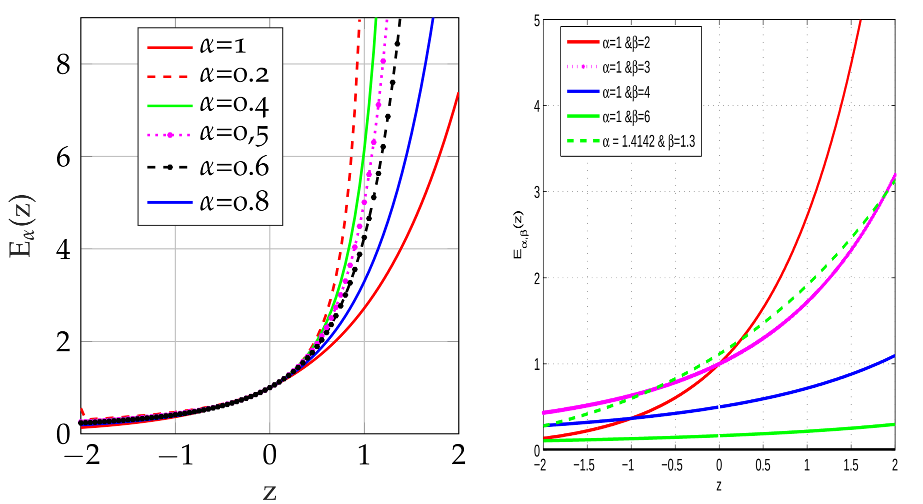
Definition 1.2.6. [70, 83] The Mittag-Leffler function $E_{\alpha,\beta}(z)$, generalizing the one in (1.2.9), is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0) \tag{1.2.10}$$

Remark 1.2.7. For $\beta = 1$ we get (1.2.9).

From (1.2.10) we have that

$$E_{\alpha,\beta}(z) = \frac{1}{\beta} + zE_{\alpha,\alpha+\beta}(z). \tag{1.2.11}$$



(a) *Mittag-Leffler function of one parameter* (b) *Mittag-Leffler function of tow parameters*

FIGURE 1.1 – *Mittag-Leffler functions of one parameter and tow parameters*

1.3 Fractional calculus

Fractional calculus (**FC**) generalizes better than integer-order integration and differentiation concepts to an arbitrary(real or complex) order[62, 70]. Fractional calculus is not a new topic; actually, it has almost the same history as that of the integer calculus. Since the occurrence of fractional or fractional-order derivative, the theories of fractional calculus fractional derivative plus fractional integral has undergone a significant and even heated development, which has been primarily contributed by pure but not applied mathematicians; the reader can refer to an encyclopedic book [44] and many references cited therein.

In the last few decades, however, applied scientists and engineers realized the need to the differential equations with fractional derivative which provided a natural framework for the discussion of the solution to various kinds of real problems modeled by the aid of fractional derivative, such as viscoelastic systems, signal processing, diffusion processes, control processing, fractional stochastic systems, allometry in biology and ecology [16, 18, 21, 42, 43, 66, 85].

Fractional calculus is considered one of the most emerging areas of investigation and has attracted the attention of many researchers over the last few decades as it is a solid and fast-growing work both in theory and in its applications [59, 62]. Different from classical or integer-order derivative, there are several kinds of definitions for fractional derivatives. These definitions are generally not equivalent to each other. In this section, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and properties can be found in the recent literature [44, 45, 59, 62, 63, 64, 70].

1.3.1 Various approaches to fractional derivation

More than 300 years after the discovery of fractional calculus, we are only beginning to overcome the difficulties.

Many mathematicians have studied the question "whether the meaning of a derivative to an integer order n could be extended to still be valid when n is not an integer" which was first raised by L'Hopital on September 30^{th} ,1695. On that day, in a letter to Leibniz, he posed a question about $D^n x/Dx^n$, Leibniz's notation for the n^{th} derivative of the linear function $f(x) = x$. L'Hopital curiously asked what the result would be if $n = \frac{1}{2}$. Leibniz responded that it would be "an apparent paradox, from which one day useful consequences will be drawn," [59], in particular, Euler (1730), Lacroix(1819), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847) and so on.

Different approaches have been used to generalize the notion of fractional derivation.

1.3.1.1 Riemann-Liouville and Caputo fractional derivatives approaches

Here, we focus on two of which are the most popular ones, the Riemann-Liouville and the Caputo fractional derivatives since they are the most used ones in applications. We will formulate the conditions of their equivalence and derive the most important properties.

Definition 1.3.1. [44, 45] The Riemann-Liouville fractional integral $I^{\alpha}_{a^+}$ and $I^{\alpha}_{T^-}$ of order $\alpha > 0$ of a function $u \in L^1([a,T])$ are defined by:

$$I^{\alpha}_{a^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \tag{1.3.1}$$

and

$$I^{\alpha}_{T^-}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}u(s)ds,$$

provided the integral exists (respectively). Here $\Gamma(\alpha)$ is the Gamma function (1.2.1). These integrals are called the left-sided and the right-sided fractional integrals.

Definition (1.3.1) can be obtained in several ways. We shall consider one approach that uses the convolution kernel of order α (1.3.2).

H^{α}, denots the convolution kernel of order $\alpha \in \mathbb{R}^+$ for fractional integrals, is defined by

$$H^{\alpha}(t) = \frac{t^{\alpha-1}_{a^+}}{\Gamma(\alpha)} \in L^1_{\text{loc}}\left(\mathbb{R}^+\right) \tag{1.3.2}$$

where the suffix $+$ is just denoting that the function is vanishing for $t < a$ that is

$$t^{\alpha-1}_{a^+} = \begin{cases} (t-a)^{\alpha-1}, & t > a \\ 0, & t \leq a \end{cases}.$$

We agree to mention this function as Gel'fand-Shilov function of order α who have treated it in their book [Gel'fand and Shilov (1964)].

The fractional integral (1.3.1) (or the left-sided Riemann-Liouville integral) with fractional order $\alpha \in R^+$of function $u(t)$ is defined as

$$I^{\alpha}_{a^+}u(t) = H^{\alpha} * u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau.$$

H^{α} has an important convolution property (or semigroup property), that is, $H^{\alpha} * H^{\beta} = H^{a+\beta}$ for arbitrary $\alpha > 0$ and $\beta > 0$.

Examples 1.3.2. We consider the power functions $u_1 = (t-a)^{\beta-1}$, $u_2 = (T-t)^{\beta-1}$, $u_3 = (t-a)^{\beta-1}(T-t)^{\gamma-1}$ and $u_4 = (t-a)^{\beta-1}(t-c)^{\gamma-1}$, where $\beta > 0, \gamma \in \mathbb{R}$ and $c < a$. Then we have respectively

$$I^{\alpha}_{a^+}u_1(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\alpha+\beta-1} \tag{1.3.3}$$

$$I^{\alpha}_{T^-}u_2(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(T-t)^{\alpha+\beta-1} \tag{1.3.4}$$

$$I^{\alpha}_{a^+}u_3(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\alpha+\beta-1}_2F_1\left(1-\gamma,\beta;\alpha+\beta;\frac{x-a}{T-a}\right) \tag{1.3.5}$$

$$I^{\alpha}_{a^+}u_4(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\alpha+\beta-1}_2F_1\left(1-\gamma,\beta;\alpha+\beta;-\frac{x-a}{a-c}\right) \tag{1.3.6}$$

The formulae (1.3.3)and (1.3.4) can be proved by direct evaluation, the formula (1.3.5)and (1.3.6) can be proved by simple transformations of the Euler representation (1.2.8).

Definition 1.3.3. [44, 45] The fractional derivatives of a function $u \in C([a,T],\mathbb{R}), n \in \mathbb{N}_0$ in the sense of Riemann-Liouville $\left({}^{RL}D^{\alpha}_{a^+}u\right)(t)$, $\left({}^{RL}D^{\alpha}_{T^-}u\right)(t)$, of order α, $0 < n-1 < \alpha \leq n$ are defined by:

$$\left({}^{RL}D^{\alpha}_{a^+}u\right)(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{u(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

and

$$\left({}^{RL}D^{\alpha}_{T^-}u\right)(t) = (-1)^n \cdot \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{T^-}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^T \frac{u(s)}{(s-t)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of real number α.

Examples 1.3.4. The fractional derivatives in the sense of Riemann-Liouville for the power functions u_1 and u_2 is given respectively by:

$$\left({}^{RL}D^{\alpha}_{a^+}u_1\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1},$$
$$\left({}^{RL}D^{\alpha}_{T^-}u_2\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(T-t)^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$ and $\alpha \geqq 0$, then the Riemann-Liouville fractional derivatives of a constant are, in general, not equal to zero:

$$\left({}^{RL}D^{\alpha}_{a^+}1\right)(t) = \frac{\Gamma(1)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \ \left({}^{RL}D^{\alpha}_{T^-}1\right)(t) = \frac{\Gamma(1)}{\Gamma(1-\alpha)}(T-t)^{-\alpha} \tag{1.3.7}$$

For $\beta = \alpha - i$, $i = 1, 2, \cdots, n$ we have that

$$\left({}^{RL}D^{\alpha}_{a^+}(t-a)^{\alpha-i}\right)(t) = 0, \ \left({}^{RL}D^{\alpha}_{T^-}(t-a)^{\alpha-i}\right)(t) = 0 \tag{1.3.8}$$

Properties 1.3.5 ([44, 62]). Let $n-1 < \alpha < n$

$$\left({}^{RL}D^{\alpha}_{a^+}u\right)(t) = 0 \Rightarrow u(t) = \sum_{i=1}^{n} c_i(t-a)^{\alpha-i}, \ c_i \in \mathbb{R}, \tag{1.3.9}$$

$$\left({}^{RL}D^{\alpha}_{T^-}u\right)(t) = 0 \Rightarrow u(t) = \sum_{i=1}^{n} d_i(T-t)^{\alpha-i}, \ d_i \in \mathbb{R}. \tag{1.3.10}$$

Lemma 1.3.6 ([44, 45, 62, 70]). *Let $\alpha > \beta > 0$, $u \in L^p([a,T])$ $(p \in [1,\infty))$ then :*

(1) The Riemann-Liouville fractional integration operators and the derivatives are linear.

(2) The fractional integration operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{T^-}$ are bounded in $L^p([a,T])$.

(3) $I^{\alpha}_{a^+}I^{\beta}_{a^+}u(t) = I^{\alpha+\beta}_{a^+}u(t)$ and $I^{\alpha}_{T^-}I^{\beta}_{T^-}u(t) = I^{\alpha+\beta}_{T^-}u(t)$.

(4) ${}^{RL}D^{\alpha}_{a^+}I^{\alpha}_{a^+}u(t) = u(t)$ and $D^{\alpha}_{T^-}I^{\alpha}_{T^-}u(t) = u(t)$.

(5) ${}^{RL}D^{\beta}_{a^+} I^{\alpha}_{a^+} u(t) = I^{\alpha-\beta}_{a^+} u(t)$ *and* ${}^{RL}D^{\beta}_{T^-} I^{\alpha}_{T^-} u(t) = I^{\alpha-\beta}_{T^-} u(t)$.

Proof. See [70] □

Lemma 1.3.7 ([44, 45, 62, 70]). *Let $\alpha > 0$, $n = [\alpha] + 1$. If $u \in L^1([a,T])$ and $Lu_{n-\alpha}, Ru_{n-\alpha} \in AC^n([a,T])$, then*

$$I^{\alpha}_{a^+}\left({}^{RL}D^{\alpha}_{a^+} u(t)\right) = u(t) + \sum_{k=1}^{n} \frac{Lu^{(n-k)}_{n-\alpha}(a)(t-a)^{\alpha-k}}{\Gamma(\alpha-k-1)}. \tag{1.3.11}$$

$$I^{\alpha}_{T^-}\left({}^{RL}D^{\alpha}_{T^-} u(t)\right) = u(t) + \sum_{k=1}^{n} \frac{Ru^{(n-k)}_{n-\alpha}(T)(T-t-)^{\alpha-k}}{\Gamma(\alpha-k-1)}. \tag{1.3.12}$$

hold almost everywhere on $[a,T]$, where $Lu_{n-\alpha}$ and $Ru_{n-\alpha}$ are the Reimann-Liouville left- sided and right-sided fractional integrals of order $n-\alpha > 0$ of the function u.

Definition 1.3.8 ([44, 70]). The fractional derivatives of a function $u \in C^n([a,T],\mathbb{R})$ (or $AC^n([a,T],\mathbb{R}), n \in N_0$ in the sense of Caputo $\left(D^{\alpha}_{a^+} u\right)(t)$, $\left(D^{\alpha}_{T^-} u\right)(t)$, of order α, $0 < n-1 < \alpha \leq n$ are defined by:

$$\left(D^{\alpha}_{a^+} u\right)(t) = I^{n-\alpha}_{a^+} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

and

$$\left(D^{\alpha}_{T^-} u\right)(t) = (-1)^n \cdot I^{n-\alpha}_{T^-} u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T \frac{u^{(n)}(s)}{(s-t)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$.

Remark 1.3.9 ([44, 45, 62, 70]). Let $u \in AC^n([a,T],\mathbb{R})$, then the Caputo fractional derivatives $\left(D^{\alpha}_{a^+} u\right)(t)$, $\left(D^{\alpha}_{T^-} u\right)(t)$, of order α, $0 < n-1 < \alpha \leq n$ exist almost everywhere on $[a,T]$

Remark 1.3.10. By Definition 1.3.8, under natural conditions on the function u, for $\alpha \to n$, the Caputo derivative becomes a conventional n-th derivative of the function u.

Remark 1.3.11 ([44, 45, 62, 70]). Let $u \in C^n([a,T],\mathbb{R})$, then the Caputo fractional derivatives $\left(D^{\alpha}_{a^+} u\right)(t)$, $\left(D^{\alpha}_{T^-} u\right)(t)$, of order α, $0 < n-1 < \alpha \leq n$ are continuous. Moreover,

$$\left(D^{\alpha}_{a^+} u\right)(a) = \left(D^{\alpha}_{T^-} u\right)(T) = 0. \tag{1.3.13}$$

Remark 1.3.12 ([44, 45, 62, 70]). By Definition 1.3.8, under natural conditions on the function u, the Caputo derivatives $\left(D^{\alpha}_{a^+} u\right)(t)$, $\left(D^{\alpha}_{T^-} u\right)(t)$, of order α, have properties similar to those of the Riemann-Liouville fractional derivatives $\left({}^{RL}D^{\alpha}_{a^+} u\right)(t)$, $\left({}^{RL}D^{\alpha}_{T^-} u\right)(t)$, given in Lemma 1.3.7 , but different from those in (1.3.7), (1.3.8)

Remark 1.3.13 ([44, 45, 62, 70]). As a basic example, let $\alpha, \beta > 0$. Then the following relations hold:

$$\left(D^{\alpha}_{a^+}u_1\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad \beta > n$$
$$\left(D^{\alpha}_{T^-}u_2\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(T-t)^{\beta-\alpha-1}, \quad \beta > n.$$

In particular $D^{\alpha}_{a^+}(t-a)^{\mu} = 0$ and $D^{\alpha}_{T^-}(T-t)^{\mu} = 0, \mu = 0, 1, \ldots, n-1$, then the Caputo fractional derivatives of a constant are, in general, equal to zero:

$$\left(D^{\alpha}_{a^+}1\right)(t) = \left(D^{\alpha}_{T^-}1\right)(t) = 0. \tag{1.3.14}$$

Lemma 1.3.14 ([44, 45, 62, 70]). *Let $\alpha > \beta > 0$, then*

(1) $\left(D^{\alpha}_{a^+}I^{\alpha}_{a^+}u\right)(t) = u(t)$ *and* $\left(D^{\alpha}_{T^-}I^{\alpha}_{T^-}u\right)(t) = u(t), \quad u \in L^{\infty}\big([a,T]\big)$ *or* $u \in C(J)$.

(2) $D^{\beta}_{0^+}I^{\alpha}_{0^+}u(t) = I^{\alpha-\beta}_{0^+}u(t), \quad u \in L^1([a,T])$.

From the definition of the Caputo derivative and Remark 1.3.13 we can obtain the following statement.

Lemma 1.3.15 ([44, 45, 62, 70]). *Let $\alpha > 0$. If we assume $u \in C^n(a,T) \cap L(a,T)$, then the fractional differential equation*

$$D^{\alpha}_{a^+}u(t) = 0$$

has at least one solution

$$u(t) = C_0 + C_1(t-a) + C_2(t-a)^2 + \cdots + C_{n-1}(t-a)^{n-1}, \quad C_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1,$$

where n is the smallest integer greater than or equal to α.

Lemma 1.3.16 ([44, 45, 62, 70]). *Let $\alpha > 0$, $n = [\alpha] + 1$. If $u \in C^n(0,1)$, then*

$$I^{\alpha}_{0^+}\left(D^{\alpha}_{0^+}u(t)\right) = u(t) + \sum_{k=0}^{n-1} C_k t^k, \quad C_k \in \mathbb{R}. \tag{1.3.15}$$
$$I^{\alpha}_{1^-}\left(D^{\alpha}_{1^-}u(t)\right) = u(t) + \sum_{k=0}^{n-1} (-1)^k D_k (1-t)^k, \quad D_k \in \mathbb{R}. \tag{1.3.16}$$

Lemma 1.3.17 ([44, 45, 62, 70]). *Let $\alpha > 0$, and $n = [\alpha] + 1$. If ${}^{C}D^{\alpha}_{0^+}u(t) \in C[0,1]$, then $u(t) \in C^{n-1}([0,1])$.*

Proof. Let $h(t) \in C[0,1]$, such that ${}^{C}D^{\alpha}_{0^+}u(t) = h(t)$. From Lemma 1.3.16, we have

$$u(t) = I^{\alpha}_{0^+}h(t) + \sum_{k=0}^{n-1} C_k t^k, \quad C_k \in \mathbb{R}.$$

It is easy to check that $u(t) \in C^{n-1}([0,1])$. □

Next, we recall the Katugampola (K) and Caputo-Katugampola (CK) fractional integrals and derivatives [50].

1.3.1.2 Katugampola and Caputo-Katugampola fractional derivatives approaches

We present here basic definitions and lemmas from fractional calculus theory (See [44, 50, 59, 62, 64, 70, 75] and references therein. Now, we recall the Katugampola and Caputo-Katugampola fractional integrals and derivatives [50].

Definition 1.3.18. The Katugampola left-sided ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma}$ and right-sided ${}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{\sigma}$ fractional integrals of non integer order $\alpha > 0$ of a function $\mathrm{q} \in \mathcal{M}_r^p(\grave{a},\grave{\imath})$ are defined by

$$ {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha}\mathrm{q}(\tau) = \frac{\rho^{1-\tau}}{\Gamma(\tau)}\int_{\grave{a}}^{\tau}(\tau^{\rho}-\xi^{\rho})^{\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \qquad \tau > \grave{a}, $$

$$ {}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{\alpha}\mathrm{q}(\tau) = \frac{\rho^{1-\sigma}}{\Gamma(\sigma)}\int_{\tau}^{\grave{\imath}}(\xi^{\rho}-\tau^{\rho})^{\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \qquad \tau < \grave{\imath}. $$

The Katugampola fractional derivatives (K𝔽𝔻) of q are defined by

$$ \begin{aligned} {}^{\rho;K}\mathcal{D}_{\grave{a}^+}^{\alpha}\mathrm{q}(\tau) &= \delta_{\rho}^{n}\left({}^{\rho;\grave{a}^+}\mathbb{I}_K^{n-\sigma}\mathrm{q}\right)(\tau) \\ &= \frac{\rho^{1-n+\sigma}}{\Gamma(n-\sigma)}\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\int_{\grave{a}}^{\tau}(\tau^{\rho}-\xi^{\rho})^{n-\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \\ {}^{\rho;K}\mathcal{D}_{\grave{\imath}^-}^{\alpha}\mathrm{q}(\tau) &= (-1)^{n}\delta_{\rho}^{n}\left({}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{n-\sigma}\right)\mathrm{q}(\tau) \\ &= \frac{(-1)^{n}\rho^{1-n+\sigma}}{\Gamma(n-\sigma)}\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\int_{\tau}^{\grave{\imath}}(\xi^{\rho}-\tau^{\rho})^{n-\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi. \end{aligned} $$

When α is integer, we consider the ordinary definition.

In the following, we present some properties for left-sided integrals and derivatives. But, the same properties are also true for the right-sided ones.

Lemma 1.3.19. *Let $r \in \mathbb{R}$, $\alpha_1, \alpha_2, \rho > 0, n \in N_0$, and $1 \le p \le \infty$. Then, on $\mathcal{M}_r^p(\grave{a},\grave{\imath})$, we have the following. i) ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1} : \mathcal{M}_r^p(\grave{a},\grave{\imath}) \to \mathcal{M}_r^p(\grave{a},\grave{\imath})$; ii) ${}^{\rho;K}\mathcal{D}_{\grave{a}^+}^{\alpha_1}$ and ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_1}$ are linear; iii) ${}^{\rho;K}\mathcal{D}_{\grave{a}^+}^{\alpha_1}, \circ {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_1} = \mathrm{I}_d$, ${}^{\rho;K}\mathcal{D}_{\grave{a}^+}^{\alpha_1}({}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_2}\mathrm{q})(\tau) = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_2-\alpha_1}\mathrm{q}(\tau)$ when $\alpha_2 \ge \alpha_1$; iv) ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_1} \circ {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2} = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\alpha_1+\alpha_2}$.*

Definition 1.3.20. Let $n \in N_0$. The Caputo-Katugampola fractional derivatives (CK𝔽𝔻) of a function $\mathrm{q} \in C_{\delta}^{n}([\grave{a},\grave{\imath}])$ (or $\in AC_{\delta}^{n}([\grave{a},\grave{\imath}])$) are defined by ${}^{\rho;CK}\mathcal{D}_{\grave{a}^+}^{\alpha}\mathrm{q}(\tau) = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{n-\sigma}\delta_{\rho_1}^{n}\mathrm{q}(\tau)$ and

$$ {}^{\rho;CK}\mathcal{D}_{\grave{\imath}^-}^{\alpha}\mathrm{q}(\tau) = (-1)^{n}\left({}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{n-\sigma}\delta_{\rho_1}^{n}\mathrm{q}(\tau)\right). $$

Lemma 1.3.21. *The CKFD of a function* $q \in C^n_\delta(J)$ *(or* $\in AC^n_\delta(J)$*) can also be written as*

$$ {}^{\rho;CK}\mathcal{D}^{\alpha}_{\grave{a}^+} q(\tau) = \left({}^{\rho;K}\mathcal{D}^{\alpha}_{\grave{a}^+} \right) \left[q(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k_\rho q(\tau)}{k!} \Big|_{\grave{a}} \left(\frac{\tau^\rho - \grave{a}^\rho}{\rho} \right)^k \right], \tag{1.3.17} $$

$$ {}^{\rho;CK}\mathcal{D}^{\alpha}_{\grave{\imath}^-} q(\tau) = \left({}^{\rho;K}\mathcal{D}^{\alpha}_{\grave{\imath}^-} \right) \left[q(\tau) - \sum_{k=0}^{n-1} (-1)^k \frac{\delta^k_\rho q(\tau)}{k!} \Big|_{\grave{\imath}} \left(\frac{\grave{\imath}^\rho - \tau^\rho}{\rho} \right)^k \right]. \tag{1.3.18} $$

Lemma 1.3.22. *Let* $\alpha_2 > \alpha_1 > 0$, $q \in \mathcal{M}^p_r(\grave{a}, \grave{\imath})$, $q \in AC^n_\delta(J)$ *or* $C^n_\delta(J)$*. Then we have*

$$ {}^{\rho;CK}\mathcal{D}^{\alpha_1}_{\grave{a}^+} \left({}^{\rho;\grave{a}^+}\mathbb{I}^{\alpha_2}_K q(\tau) \right) = {}^{\rho;\grave{a}^+}\mathbb{I}^{\alpha_2-\alpha_1}_K q(\tau), $$

and for some real constants N_k *and* M_k*,*

$$ {}^{\rho;\grave{a}^+}\mathbb{I}^{\alpha_1}_K \left({}^{\rho;CK}\mathcal{D}^{\alpha_1}_{\grave{a}^+} q \right)(\tau) = q(\tau) - \sum_{k=0}^{n-1} N_k \left(\frac{\tau^\rho - \grave{a}^\rho}{\rho} \right)^k, \tag{1.3.19} $$

$$ {}^{\rho;\grave{\imath}^-}\mathbb{I}^{\alpha_1}_K \left({}^{\rho;CK}\mathcal{D}^{\alpha_1}_{\grave{\imath}^-} q \right)(\tau) = q(\tau) - \sum_{k=0}^{n-1} M_k \left(\frac{\grave{\imath}^\rho - \tau^\rho}{\rho} \right)^k. \tag{1.3.20} $$

Proposition 1.3.23. *We have the following properties for Caputo-Katugampola fractional derivative approach. When* $\rho = 1$*, the Caputo-Katugampola fractional derivative approach coincides with Caputo fractional derivative approach.*
When $\rho \to 0^+$*, the Caputo-Katugampola fractional derivative approach coincides with Caputo-Hadamard fractional derivative approach.*

Lemma 1.3.24. *[59] If* ${}^{\rho;CK}\mathcal{D}^{\alpha_1}_{\grave{a}^+} q \in C(J)$*, then* $q \in C^{n-1}_\rho(J)$*.*

1.3.1.3 The Riemann-Liouville and Caputo fractional derivatives approaches that involve a function ψ

We present here basic, a general definitions and lemmas form of fractional calculus theory that involves a function ψ. We can easily see that under suitable choices of ψ, we obtain some well-known fractional operators, like the Riemann-Liouville, the Caputo, the Hadamard, the Katugampola, or the Erdelyi-Kober fractional derivatives. (See [11, 14, 48, 50, 54, 59, 62, 64, 70] and references therein).

There are some definitions in fractional calculus which are very widely used and have importance in proving various results of fractional calculus. In this section, We will see a new class of integrals and fractional derivatives. Due to the huge amount of definitions, i.e., fractional operators, the following definition is a special approach when the kernel is unknown, involving a function ψ. That generalized all definitions of fractional integral and fractional differential operators that we use throughout this thesis.

Now, we recall the definitions and some properties of the ψ-Riemann-Liouville (ψ-RL) and the ψ-Caputo (ψ-C) fractional fractional derivative of a function. Some of these definitions and results were given in Samko et al [70], Almeida [11].

Definition 1.3.25. For $\alpha > 0$, we define the left-sided and right-sided $\psi-$ Riemann-Liouville ($\psi-$ RL) fractional integral respective of order α for an integrable function $u : J \longrightarrow \mathbb{R}$ with respect to another function ψ as follows

$$
\begin{aligned}
{}^{\psi}_{t}\mathcal{I}^{\alpha}_{a^+}u(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)\mathrm{d}s \\
{}^{\psi}_{t}\mathcal{I}^{\alpha}_{T^-}u(t) &= \frac{1}{\Gamma(\alpha)}\int_t^T \psi'(s)(\psi(s)-\psi(t))^{\alpha-1}u(s)\mathrm{d}s
\end{aligned}
$$

where Γ is the special Euler's function and $\psi : J \longrightarrow \mathbb{R}$ is strictly increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in$ J.

Definition 1.3.26. Let $\psi \in C^n(\,\mathrm{J}, \mathbb{R})$ be a function such that ψ is strictly increasing and $\psi'(t) \neq 0$, for all $t \in$ J. The left-sided and right-sided $\psi-$ Riemann-Liouville ($\psi-$ RL) fractional derivative respective of a function $u : J \to \mathbb{R}$ of order $\alpha \in (n-1, n)$ are defined by

$$
\begin{aligned}
{}^{\psi}_{t}\mathcal{D}^{\alpha}_{a^+}u(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n {}^{\psi}_{t}\mathcal{I}^{n-\alpha}_{a^+}u(t), t > a, \\
{}^{\psi}_{t}\mathcal{D}^{\alpha}_{T^-}u(t) &= \left(\frac{-1}{\psi'(t)}\frac{d}{dt}\right)^n {}^{\psi}_{t}\mathcal{I}^{n-\alpha}_{T^-}u(t), t < T,
\end{aligned}
$$

provided the right- side and left- side integrals are pointwise defined on J .

Definition 1.3.27. Let $n \in N_0$ and let $\psi, u \in C^n(J, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided and right-sided ψ -Caputo ($\psi - C$) fractional derivatives respective of u of order α are defined by

$$
\begin{aligned}
{}^{\psi;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u(t) &= {}^{\psi}_{t}\mathcal{I}^{n-\alpha}_{a^+}\delta^{[n]}_{\psi}u(t) \\
{}^{\psi;C}_{\ \ t}\mathcal{D}^{\alpha}_{T^-}u(t) &= {}^{\psi}_{t}\mathcal{I}^{n-\alpha}_{T^-}(-1)^n\delta^{[n]}_{\psi}u(t)
\end{aligned}
$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}, n = \alpha$ for $\alpha \in \mathbb{N}$. and $\delta^{[n]}_{\psi}u(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t)$.

Lemma 1.3.28. *The left-sided and right-sided ψ -Caputo ($\psi - C$) fractional derivatives respective of order α of $u \in C^n(J)$ can also be determined as:*

$$
{}^{\psi;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u(t) = {}^{\psi}_{t}\mathcal{D}^{\alpha}_{a^+}\left[u(t) - \sum_{k=0}^{n-1}\frac{\delta^{[k]}_{\psi}u(t)}{k!}|_a\,(\psi(t)-\psi(a))^k\right], \tag{1.3.21}
$$

$$
{}^{\psi;C}_{\ \ t}\mathcal{D}^{\alpha}_{T^-}u(t) = {}^{\psi}_{t}\mathcal{D}^{\alpha}_{T^-}\left[u(t) - \sum_{k=0}^{n-1}(-1)^k\frac{\delta^{[k]}_{\psi}u(t)}{k!}|_a\,(\psi(T)-\psi(t))^k\right]. \tag{1.3.22}
$$

In the following, we present some properties for left-sided integrals and derivatives. But, the same properties are also true for the right-sided ones.

Lemma 1.3.29. *Let $\alpha, \beta > 0$.*

- *If $u \in L^1(J, \mathbb{R})$, then*

$$ {}^{\psi}_{t}\mathcal{I}^{\alpha}_{a^+} {}^{\psi}_{t}\mathcal{I}^{\beta}_{a^+} u(t) = {}^{\psi}_{t}\mathcal{I}^{\alpha+\beta}_{a^+} u(t), \text{ a.e. } t \in \mathrm{J} \tag{1.3.23} $$

- *If $u \in \mathcal{C}(J, \mathbb{R})$, then*

$$ {}^{\psi}_{t}\mathcal{I}^{\alpha}_{a^+} {}^{\psi}_{t}\mathcal{I}^{\beta}_{a^+} u(t) = {}^{\psi}_{t}\mathcal{I}^{\alpha+\beta}_{a^+} u(t) \text{ and } {}^{\psi;C}_{\quad t}\mathcal{D}^{\alpha}_{a^+} {}^{\psi}_{t}\mathcal{I}^{\alpha}_{a^+} u(t) = u(t), t \in \mathrm{J} \tag{1.3.24} $$

- *If $u \in C^n(J, \mathbb{R}), n-1 < \alpha < n$. Then*

$$ {}^{\psi}_{t}\mathcal{I}^{\alpha}_{a^+} {}^{\psi;C}_{\quad t}\mathcal{D}^{\alpha}_{a^+} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{[k]}_{\psi}(a)}{k!} [\psi(t) - \psi(a)]^k, \quad t \in \mathrm{J} \tag{1.3.25} $$

1.4 Some motivations for using fractional calculus

The fractional calculus adds powerful information to the classical calculus, with a more accurate description of certain natural phenomena, nevertheless it does not reflect any physical process. Unclear physical meaning has been a big obstacle that keeps fractional derivatives lagging far behind the integer-order calculus. It can be applied to several areas of knowledge, such as physics, chemistry, engineering, technology...

In this section, we present here some motivations for the use of fractional calculus for example calculate a heat flow by using Fourier's law, measuring memory with the order of fractional derivative the effect of memory, and we give a statistical approach to some fractional operators.

1.4.1 Calculate a heat flow by using Fourier's law

The concept of the fractional derivative of $\frac{1}{2}$ order appears occurs naturally in [34] when they were tried to calculate a heat flow by using Fourier's law.

They researched to give the solution of the heat equation

$$ \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial y^2} = f(t), \quad t > 0, \quad y \geq 0 \tag{1.4.1} $$

the unknown function $u(y, t)$ satisfies a nullity constraint at $y = 0$ (the temperature is imposed) and its gradient $\frac{\partial u}{\partial y}$ tends to 0 if y tends to $+\infty$ (the temperature becomes uniform at large distances):

$$ u(0, t) = 0 \tag{1.4.2} $$

$$ \frac{\partial u}{\partial y}(y, t) \longrightarrow 0 \quad \text{if} y \longrightarrow +\infty. \tag{1.4.3} $$

Engineers are directly interested in the heat flow $\Phi(t)$ to the wall, given by Fourier's law:

$$\Phi(t) = -\iota \frac{\partial u}{\partial y}(0,t), \quad t > 0. \tag{1.4.4}$$

They solved the previous problem (1.4.1)-(1.4.3) by using the Fourier transform in time.

Then, they obtained this expression

$$\widehat{u}(y,\omega) = \frac{1}{i\omega}\widehat{f}\left[1 - \exp\left(-\sqrt{\frac{i\omega}{\iota}}y\right)\right] \tag{1.4.5}$$

According to Fourier's reciprocity formula, we get the function u

We can then derive this expression with respect to y:

$$\frac{\partial \widehat{u}}{\partial y}(y,\omega) = \frac{1}{\sqrt{i\iota\omega}}\widehat{f}\exp\left(-\sqrt{\frac{i\omega}{\iota}}y\right) \tag{1.4.6}$$

and substitute this expression at $y = 0$, they have:

$$\widehat{\Phi}(\omega) = -\sqrt{\frac{\iota}{i\omega}}\widehat{f} \tag{1.4.7}$$

As results, the quantity $\sqrt{\frac{1}{i\omega}}\widehat{f}(\omega)$in (1.4.7) seems to be able to be interpreted as the transform of Fourier of an integral of the order of one-half of the function f, why? because Fourier's transform of the function derivative of f is obtained by the multiplication of the Fourier's transform of the function of f and $i\omega$.

More precisely, if we pose

$$\nu(t) \equiv \frac{1}{\sqrt{t}}H(t) \tag{1.4.8}$$

where $H(t)$ Heaviside's function (1.1.3), we see that

$$\widehat{\nu} = \sqrt{\frac{\pi}{i\omega}} \tag{1.4.9}$$

The expression above permits to write more precisely the expression of the heat flux under the following form:

$$\widehat{\Phi} = -\sqrt{\frac{\iota}{\pi}}\widehat{\nu}\widehat{f} \tag{1.4.10}$$

According to the convolution, $\nu * f$ of the functions f and ν which defines as follow

$$(\nu * f)(x) \equiv \int_{-\infty}^{\infty} \nu(y) f(x-y)\mathrm{d}y$$

and the classical relation

$$\widehat{\nu * f} = \widehat{\nu}\widehat{f} \tag{1.4.11}$$

They have that

$$\Phi(t) = -\sqrt{\frac{\iota}{\pi}}\nu * f = -\sqrt{\frac{\iota}{\pi}}\int_0^t f(\theta)\frac{\mathrm{d}\theta}{\sqrt{t-\theta}} \tag{1.4.12}$$

We have just shown the half-order integrator of the function f.

1.4.2 Order of fractional derivative as tool for measuring memory

As we know the fractional derivative has rich history as long as that of classical calculus and is a promising tool for describing memory phenomena. This is back to the kernel function of fractional derivative, it is called memory function, but it is much less popular than of the calculus of integer order . Why ? Because we need the physical meaning of fractional derivative, which still an open problem. In modeling various memory phenomena. In [33] they observed that a memory process usually consists of two stages. The first stage is short (the fresh stage) with permanent retention, and the other (the working stage) is governed by a simple model of fractional derivative. With the numerical least square method, they emphasized that the fractional model perfectly fits the test data of memory phenomena in different disciplines, not only in mechanics, but also in biology and psychology. Based on this model (Scott-Blair's model [64])

$$\left({}^{RL}D^{\alpha}_{0^+}\varepsilon\right)(t) = d\Xi(t) \tag{1.4.13}$$

where $\left({}^{RL}D^{\alpha}_{0^+}\varepsilon\right)(t)$ is the fractional derivative which depends on the strain history from 0 to t, and d is a positive, they found that Scott-Blair's model originally a material model, can be a formula for memory phenomena in various disciplines. This leads to to say that a physical meaning of the fractional order is an index of memory.

Compared to the results obtained by others, we can see that their results may be among the best partial answers to the question "what are the physical interpretations of fractional calculus" which was put forward as an open problem in 1974 [64], for example in 2002, a physical Explanation was proposed in terms of inhomogeneous and changing time scale by analogy reasoning, but the new time scale has not been validated by any experiment [64].

1.4.3 Traditional approach of memory effect

As stated in previous subsection 1.4.1, Fractional calculus is a great tool that can be employed to describe real-life phenomena with the so-called memory effect and is considered for measuring memory with the order of fractional derivative. Classic models of autonomous differential equations have no memory, because their solution is independent of the previous instant. In general, this assertion is not true for fractional differential equations. Among some ways to introduce and detect the memory effect into a mathematical model is by changing the order of the derivative of a classical model so that it is non-integer [33]. Thus, fractional calculus has been shown to be efficient in mathematical modeling. for that we mention the following. Let u be function defined on $[a, T]$ and $t_1,\ t_2 \in [a, T]$ such that $a < t_1 < t_2 < T$, $P_1 = \left(I^{\alpha}_{a^+}u\right)(t_2) - \left(I^{\alpha}_{a^+}u\right)(t_1)$ and $P_2 = \left(I^{\alpha}_{T^-}u\right)(t_2) - \left(I^{\alpha}_{T^-}u\right)(t_1)$ for $\alpha \in \mathbb{R}^+$. From equalities below, one can observe that the value of P_1 and P_2 depend on the entire range of u over $[a, t_2]$ and $[t_1, T]$ respectively if $\alpha \neq 1$, whereas P_1 and P_2 depend only on the range of u over $[t_1, t_2]$ if $\alpha = 1$:

$$\begin{aligned} P_1 &= \left(I^{\alpha}_{a^+}u\right)(t_2) - \left(I^{\alpha}_{a^+}u\right)(t_1) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{t_2}(t_2-s)^{\alpha-1}u(s)ds - \frac{1}{\Gamma(\alpha)}\int_a^{t_1}(t_1-s)^{\alpha-1}u(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}u(s)ds + \int_a^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]u(s)ds\right] \end{aligned} \tag{1.4.14}$$

and

$$\begin{aligned} P_2 &= \left(I^{\alpha}_{T^-}u\right)(t_1) - \left(I^{\alpha}_{T^-}u\right)(t_2) \\ &= \frac{1}{\Gamma(\alpha)}\int_{t_1}^{T}(s-t_1)^{\alpha-1}u(s)ds - \frac{1}{\Gamma(\alpha)}\int_{t_2}^{T}(s-t_2)^{\alpha-1}u(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(s-t_1)^{\alpha-1}u(s)ds + \int_{t_2}^{T}\left[(s-t_1)^{\alpha-1}-(s-t_2)^{\alpha-1}\right]u(s)ds\right]. \end{aligned} \tag{1.4.15}$$

Note that if $\alpha = 1$, then the second integral in (1.4.14) and (1.4.15) is canceled:

$$P_1 = P_2 = \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}u(s)ds.$$

1.4.4 Memory effect in fractional calculus based on statistical expectation

Here, we focus to present the so-called memory effect in the Riemann-Liouville and Caputo fractional derivative approaches on $[a, T]$ based on statistical expectation, to the best of our knowledge, which has been presented in the literature [22, 58].

Proposition 1.4.1 ([22]). *Let $\alpha \in \mathbb{R}^+$ and $u \in AC[a, T]$. Under these conditions, we have*

$$\left(I^{\alpha}_{a^+}u\right)(t) = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}E\left(u\left((t-a)X_1+a\right)\right); \tag{1.4.16}$$

$$\left(I^{\alpha}_{T^-}u\right)(t) = \frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)}E\left(u\left(T-(T-t)X_1\right)\right); \tag{1.4.17}$$

$$\left({}^{RL}D^{\alpha}_{a^+}u\right)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}E\left(u\left((t-a)X_2+a\right)\right) + \frac{(t-a)^{1-\alpha}}{\Gamma(3-\alpha)}E\left(u'\left((t-a)X_3+a\right)\right), \quad 0<\alpha<1; \tag{1.4.18}$$

$$\left({}^{RL}D^{\alpha}_{T^-}u\right)(t) = \frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)}E\left(u\left(T-(T-t)X_2\right)\right) + \frac{(T-t)^{1-\alpha}}{\Gamma(3-\alpha)}E\left(-u'\left(T-(T-t)X_3\right)\right), \quad 0<\alpha<1; \tag{1.4.19}$$

$$\left(D^{\alpha}_{a^+}u\right)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(3-\alpha)}E\left(u'\left((t-a)X_2+a\right)\right), \quad 0<\alpha<1; \tag{1.4.20}$$

$$\left(D^{\alpha}_{T^-}u\right)(t) = \frac{(T-t)^{-\alpha}}{\Gamma(3-\alpha)}E\left(-u'\left(T-(T-t)X_2\right)\right), \quad 0<\alpha<1, \tag{1.4.21}$$

where X_1, X_2 and X_3 are random variables follow the beta distributions $B(1,\alpha)$, $B(2,1-\alpha)$, $B(1,1-\alpha)$ respectively and $E\left(u(.)\right)$ is the expectation or expected value of $u(.)$[58]

Proof. First, we note that $B(1;\alpha) = \frac{1}{\alpha}$. From (1.3.1), we have

$$\left(I^{\alpha}_{a^+}u\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\mathrm{d}s = \frac{1}{\Gamma(\alpha)}\int_a^t (t-a)^{\alpha-1}\left(1-\frac{s-a}{t-a}\right)^{\alpha-1}u(s)\mathrm{d}s$$

Put $\tau = \frac{s-a}{t-a}$, we obtain

$$\begin{aligned}\left(I^{\alpha}_{a^+}u\right)(t) &= \frac{(t-a)^{\alpha}}{\Gamma(\alpha)}\int_0^1 (1-x)^{\alpha-1}u((t-a)x+a)\mathrm{d}x \\ &= \frac{(t-a)^{\alpha}B(1,\alpha)}{\Gamma(\alpha)}\int_0^1 \frac{(1-x)^{\alpha-1}}{B(1,\alpha)}u((t-a)x+a)\mathrm{d}x \\ &= \frac{(t-a)^{\alpha}}{\alpha\Gamma(\alpha)}\int_0^1 \frac{(1-x)^{\alpha-1}}{B(1,\alpha)}u((t-a)x+a)\mathrm{d}x.\end{aligned}$$

We use the fact of $\dfrac{(1-x)^{\alpha-1}}{B(1,\alpha)}$ is the density function of a beta distribution, we have that (1.4.18) holds true for $X_1 \sim B(1,\alpha)$.

Similarly by the saime technique of above discussions we find that (1.4.17)-(1.4.21) hold true. □

Remark 1.4.2. Note that, by Lemma 1.3.28, exchanging $u(t)$ by $u(t)-u(a)$ and $u(t)-u(T)$ or, equivalently $u\left((t-a)X_2+a\right)$ by $u\left((t-a)X_2+a\right)-u\left(a\right)$ and $u\left((T-(T-t)X_2\right)$ by $u\left(T-(T-t)X_2\right)-u(T)$ in (1.4.18) and (1.4.19) respectively, for $0<\alpha<1$, we can rewrite Caputo's derivatives as follows:

$$\left(D^{\alpha}_{a^+}u\right)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}E\left(u\left((t-a)X_2+a\right)-u\left(a\right)\right) + \frac{(t-a)^{1-\alpha}}{\Gamma(3-\alpha)}E\left(u'\left((t-a)X_3+a\right)\right); \tag{1.4.22}$$

$$\left(D^{\alpha}_{T^-}u\right)(t) = \frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)}E\left(u\left(T-(T-t)X_2\right)-u(T)\right) + \frac{(T-t)^{1-\alpha}}{\Gamma(3-\alpha)}E\left(-u'\left(T-(T-t)X_3\right)\right). \tag{1.4.23}$$

We can see that (1.4.22) and (1.4.23) coincid with (1.4.18) and (1.4.19)respectively. This follows from the proof of Proposition 1.4.1 and Definitions 1.3.8 and Lemma 1.3.28 .

Next, we provide some examples of Riemann-Liouville and Caputo fractional derivatives using the formulas (1.4.18)-(1.4.21).

Examples 1.4.3. 1. If $u(t)=c$, where c is constant, we have $E[u((t-a)X_1+a))] = E[c] = c$, $E[u(T-(T-t)X_1))] = E[c] = c$ and $E[u'((t-a)X_2+a))] = E[u'(T-(T-t)X_2))] = 0$. Thus, from (1.4.18)-(1.4.21) we have $\left({}^{RL}D^{\alpha}_{a^+}u\right)(t) = \frac{c(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$, $\left({}^{RL}D^{\alpha}_{T^-}u\right)(t) = \frac{c(T-t)^{-\alpha}}{\Gamma(1-\alpha)}$, and $\left(D^{\alpha}_{a^+}u\right)(t) = \left(D^{\alpha}_{T^-}u\right)(t) = 0$.

2. For the power functions u_1 and u_2, we find $\left({}^{RL}D^{\alpha}_{a^+}u_1\right)(t)$, $\left({}^{RL}D^{\alpha}_{T^-}u_2\right)(t)$, $\left(D^{\alpha}_{a^+}u_1\right)(t)$, and $\left({}^{RL}D^{\alpha}_{T^-}u_2\right)(t)$, for $0<\alpha<1$. Note that

$$\begin{aligned}
E[u_1((t-a)X_1+a))] &= \int_0^1 \frac{(1-x)^{-\alpha}}{B(1,\alpha)} u_1((t-a)x+a)\mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^{-\alpha}}{B(1,1-\alpha)} ((t-a)x)^{\beta-1}\mathrm{d}x \\
&= (t-a)^{\beta-1}\int_0^1 \frac{(1-x)^{-\alpha}}{B(1,1-\alpha)} (x)^{\beta-1}\mathrm{d}x \\
&= (t-a)^{\beta-1}(1-\alpha)B(\beta,1-\alpha)
\end{aligned}$$

and

$$\begin{aligned}
E[u_1'((t-a)X_2+a))] &= \int_0^1 \frac{(1-x)^{-\alpha}\,x}{B(2,\alpha)} u_1'((t-a)x+a)\mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^{-\alpha}\,x}{B(2,1-\alpha)} (\beta-1)((t-a)x)^{\beta-2}\mathrm{d}x \\
&= (\beta-1)(t-a)^{\beta-2}\int_0^1 \frac{(1-x)^{-\alpha}\,x^2}{B(2,1-\alpha)} (x)^{\beta-2}\mathrm{d}x \\
&= \frac{(\beta-1)(t-a)^{\beta-2}}{B(2,1-\alpha)}B(\beta,1-\alpha).
\end{aligned}$$

By (1.4.18) and the fact $u_1(a)=u_2(T)=0$, we have

$$\begin{aligned}
\left({}^{RL}D^{\alpha}_{a^+}u_1\right)(t) &= \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(t-a)^{\beta-1}(1-\alpha)B(\beta,1-\alpha) \\
&\quad + \frac{(t-a)^{1-\alpha}}{\Gamma(3-\alpha)}\frac{(\beta-1)(t-a)^{\beta-2}}{B(2,1-\alpha)}B(\beta,1-\alpha) \\
&= \frac{(1-\alpha)(t-a)^{\beta-\alpha-1}}{\Gamma(1-\alpha)}B(\beta,1-\alpha) + \frac{(\beta-1)(t-a)^{\beta-\alpha-1}}{\Gamma(3-\alpha)B(2,1-\alpha)}B(\beta,1-\alpha) \\
&= (t-a)^{\beta-\alpha-1}B(\beta,1-\alpha)\left[\frac{(1-\alpha)}{\Gamma(1-\alpha)} + \frac{(\beta-1)}{\Gamma(3-\alpha)B(2,1-\alpha)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= (t-a)^{\beta-\alpha-1}\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)}\left[\frac{(1-\alpha)}{\Gamma(1-\alpha)}+\frac{(\beta-1)\Gamma(3-\alpha)}{\Gamma(3-\alpha)\Gamma(2)\Gamma(1-\alpha)}\right] \\
&= (t-a)^{\beta-\alpha-1}\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)}\left[\frac{(1-\alpha)}{\Gamma(1-\alpha)}+\frac{(\beta-1)}{\Gamma(1-\alpha)}\right] \\
&= (t-a)^{\beta-\alpha-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\left[\beta-\alpha\right] \\
&= (t-a)^{\beta-\alpha-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)},
\end{aligned} \tag{1.4.24--1.4.25}
$$

and

$$\left(D^{\alpha}_{a^+}u_1\right)(t) = \left({}^{RL}D^{\alpha}_{a^+}\right)\left(u_1(t)-u_1(a)\right) = (t-a)^{\beta-\alpha-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}$$

Furthermore,

$$\left({}^{RL}D^{\alpha}_{T^-}u_2\right)(t) = \left(D^{\alpha}_{T^-}u_2\right)(t) = (T-t)^{\beta-\alpha-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \tag{1.4.26}$$

3. For the two functions $u_5(t) = e^{c(t-a)}$ and $u_6(t) = e^{c(T-a)}$, we have

$$\left({}^{RL}D^{\alpha}_{a^+}u_5\right)(t) = (t-a)^{-\alpha}E_{1,1-\alpha}(c(t-a)) \tag{1.4.27}$$

$$\left({}^{RL}D^{\alpha}_{T^-}u_6\right)(t) = (T-t)^{-\alpha}E_{1,1-\alpha}(c(T-t)) \tag{1.4.28}$$

$$\left(D^{\alpha}_{a^+}u_5\right)(t) = c(t-a)^{-\alpha}E_{1,2-\alpha}(c(t-a)) \tag{1.4.29}$$

$$\left(D^{\alpha}_{T^-}u_6\right)(t) = c(T-t)^{-\alpha}E_{1,2-\alpha}(c(T-t)). \tag{1.4.30}$$

The results (1.4.25)-(1.4.30) of Examples 1.4.3 coincide with those presented in the literature, we can see that the equations (1.4.16)–(1.4.21) can be used instead of the traditional approach to fractional calculus.

Next we present some results based on statistical expectation

Proposition 1.4.4. *The weight in each weighted average:*

1. $\left(I^{\alpha}_{a^+}u\right)(t)$ *is affected by historical values of* $u((t-a)X_1+a)$ *according to density function* $u_{X_1}(x) = \frac{(1-x)^{\alpha-1}}{B(1,\alpha)}$, *which is increases,if* $0<\alpha<1$, *decreases, for* $\alpha>1$;
2. $\left(I^{\alpha}_{T^-}u\right)(t)$ *is affected by future values of* $u(T-(T-t)X_1)$ *according to density function* $u_{X_1}(x)$;

3. $\left({}^{RL}D^{\alpha}_{a^+}u\right)(t)$ *is affected by historical values of* $u((t-a)X_2+a)$ *and* $u'((t-a)X_3+a)$*, according to the two densities* $u_{X_2}(x) = \frac{(1-x)^{-\alpha}}{B(1,1-\alpha)}$*and* $u_{X_3}(x) = \frac{x(1-x)^{-\alpha}}{B(2,1-\alpha)}$ *which are both increasing for* $\alpha \in (0,1)$*;*

4. $\left({}^{RL}D^{\alpha}_{T^-}u\right)(t)$ *is affected by future values of* $u(T-(T-t)X_2)$ *and* $-u'(T-(T-t)X_3)$*, according to the two densities* $u_{X_2}(x)$ *and* $u_{X_3}(x)$*;*

5. $\left(D^{\alpha}_{a^+}u\right)(t)$ *is affected by historical values of* $u'((t-a)X_2+a)$*, according to the density* $u_{X_2}(x)$*;*

6. $\left(D^{\alpha}_{T^-}u\right)(t)$ *is affected by future values of* $-u'(T-(T-t)X_3)$*, according to the two density* $u_{X_2}(x)$*.*

Items 1, 3 and 5 indicate that the values of the fractional integral, as well as the fractional derivatives in t, are affected by the historical values of u (at every time before t). Thus, the fractional calculus seems to be a more adequate mathematical tool for modeling phenomena with hysteresis than the classical calculus.

It is said that such a system presents the phenomenon of hysteresis, when the current state of a system is influenced by the dynamics of its historical past. More general, hysteresis is a type of non-limited window memory (that is, limited from the origin), so it can be formulated mathematically with a convolution kernel from the origin. This is a typical kernel used to define integral and differential fractional operators, such as The Riemann-Liouville and Caputo Differintegral approach.

Items 2, 4 and 6 indicate that the values of the fractional integral, as well as the fractional derivatives in t, are affected by the future values of u (at every time after t). Thus, the fractional calculus seems to be a more adequate mathematical tool for modeling phenomena which has this properties than the classical calculus.

1.5 Some topics from functional analysis

In this section, we present some notations and terminologies of functional analysis that will be used in this thesis.

1.5.1 Cone in Banach Space

Let $\mathbb{E}$ be a real Banach function space, endowed with the infinity norm.

Definition 1.5.1. A nonempty closed convex set $K \subset \mathbb{E}$ is called cone if the following properties are satisfied.

1. $\forall u \in K, \forall \lambda > 0: \ \lambda u \in K.$

2. $\forall u \in K: -u \in K \implies u = 0.$

Definition 1.5.2. An operator $\mathcal{L}: \mathbb{E} \to \mathbb{E}$ is said to be Lipschitzian if there exists a positive real constant k such that for all x and y in $\mathbb{E}$,

$$\| \mathcal{L}x - \mathcal{L}y \| \leq k \| x - y \| . \tag{1.5.1}$$

that for all x and y in

Remark 1.5.3. The smallest k for which (1.5.1) holds is said to be the Lipschitz constant for $\mathcal{L}$ and is denoted by L. If $L < 1$ we say that $\mathcal{L}$ is a contraction, whereas if $L = 1$, we say that $\mathcal{L}$ is nonexpansive.

1.5.2 Fixed point theorems

The theory of fixed point is one of the most powerful tools of modern mathematics. The theorems which are concerning with the existence of solutions for differential equations.

Definition 1.5.4. [69] A point $x \in \mathbb{E}$ is called a fixed point of an operator $\mathcal{L} : \mathbb{E} \to \mathbb{E}$, if

$$\mathcal{L}(x) = x, x \in \mathbb{E}.$$

Banach[12] proved that a contraction mapping in the field of a complete normed space possesses a unique fixed point. This theorem is probably the most well-known fixed-point theorem.
This theorem is outstanding among fixed point theorems, because it is not only guarantees existence of a fixed point, but also its uniqueness, an approximation method actually to find the fixed point, a priori and a posteriori estimates for the rate of convergence.

Definition 1.5.5 (Banach Fixed Point Theorem). [12] Let $\mathbb{E}$ be a Banach space and $\mathcal{L}$ be a contraction mapping with Lipschitz constant k. Then $\mathcal{L}$ has an unique fixed point.

Definition 1.5.6. [69] A continuous operator is called completely continuous if it maps bounded sets into precompact sets.

Lemma 1.5.7 ([38], p. 219, PC-type Ascoli-Arzelá Theorem)**.** *Let $\Omega \subset PC(J, \mathbb{R})$. Suppose the following conditions are satisfied:*

1. *Ω is uniformly bounded subset of $PC(J, \mathbb{R})$;*
2. *Ω is equicontinuous in $J_k, k = 0, 1, 2, \ldots, m$. Then Ω is a relatively compact subset of $PC(J, \mathbb{R})$.*

We recall some fundamental results of the fixed-point theory:

Theorem 1.5.8. *Let $\mathcal{X}$ be a Banach space and Ω is an open bounded subset of $\mathcal{X}$ with $\theta \in \Omega$. Assume that $\mathcal{L} : \Omega \to \mathcal{X}$ be a completely continuous operator such that $\| \mathcal{L}u \| \leqslant \| u \|$, $\forall u \in \partial\Omega$. Then $\mathcal{L}$ has a fixed point in Ω*

Theorem 1.5.9 (Schaefer's Fixed Point Theorem)**.** *Let $\mathcal{X}$ be a Banach space and $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ be a completely continuous operator. If the set*

$$E(\mathcal{L}) = \{u \in \mathcal{X} : u = \sigma \mathcal{L} u \text{ for some } \sigma \in [0, 1]\}$$

is bounded, then $\mathcal{L}$ has at least a fixed point.

In many problems that arise from models of neutron transport, chemical reactors,population biology, infectious diseases, and other systems, we need to didcuss the existence of nonnegative solutions with certain qualitative and quantitative properties to the considered problem. What we normally understand by nonnegativily can be developed by arbitrary cones. The proof of existence of solution in a cone is based upon an applications of the following theorems.

Let $K \subset \mathbb{E}$ be a cone, $r > 0$, $\Omega_r = \{u \in K : \|u\| < r\}$, and $\mathbf{i}$ is the fixed point index function .

Theorem 1.5.10 ([36, 37]). *Let $\mathcal{L} : K \cap \overline{\Omega}_r \to K$ be a completely continuous operator such that $\mathcal{L}u \neq u$ for all $u \in \partial\Omega_r$.*

1. *If $\|\mathcal{L}u\| \leqslant \|u\|$ for all $u \in \partial\Omega_r$, then $\mathbf{i}\,(\mathcal{L}, \Omega_r, K) = 1$.*
2. *If $\|\mathcal{L}u\| \geqslant \|u\|$ for all $u \in \partial\Omega_r$, then $\mathbf{i}\,(\mathcal{L}, \Omega_r, K) = 0$.*

Theorem 1.5.11 (Guo-Krasnoselskii [44]). *Assume that Ω_1 and Ω_2 are open subsets of $\mathbb{E}$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $\mathcal{L} : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be completely continuous operator. Consider*

(D_1) $\|\mathcal{L}u\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{L}u\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2$,

(D_2) $\|\mathcal{L}u\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2$ *and* $\|\mathcal{L}u\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$.

If (D_1) or (D_2) holds, then $\mathcal{L}$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let $f : J \to (0, \infty)$ be continuous.

Definition 1.5.12. ([11, 87]) We say that f is $\rho-$ convex if

$$f\left([(1-t)x^\rho + ty^\rho]^{\frac{1}{\rho}}\right) \leq (1-t)f(x) + tf(y) \qquad \text{for all } x, y \in J \text{ and } t \in [0,1].$$

f is called $\rho-$concave if $(-f)$ is $\rho-$convex.

Remark 1.5.13. ([11, 87])

1. f is $\rho-$convex (concave) if and only if $f(\varphi^{-1})$ is convex (concave), where $\varphi(t) = \frac{t^\rho}{\rho}$.
2. f is $\rho-$convex (concave) if and only if $\delta_\rho f(x)$ is increasing (decreasing).

In order to show existence of multiple solutions we will use the Leggett-Williams fixed point theorem [52]. For this we define the following subsets of a cone K,

$$\Omega_c = \Big\{\mathrm{q} \in K : \|\mathrm{q}\| < c\Big\},$$
$$\Omega_\varphi(b, d) = \Big\{\mathrm{q} \in K : b \leq \varphi(\mathrm{q}),\ \|\mathrm{q}\| \leq d\Big\}.$$

Definition 1.5.14. A map $\Pi : K \to [0, \infty)$ is said to be a nonnegative continuous concave functional on a cone K of a real Banach space $\mathbb{E}$, if it's continuous and

$$\Pi(\bar{\lambda}\mathrm{q} + (1-\bar{\lambda})\acute{\mathrm{q}}) \geq \bar{\lambda}\Pi(\mathrm{q}) + (1-\bar{\lambda})\Pi(\acute{\mathrm{q}}),$$

for all $\mathrm{q}, \acute{\mathrm{q}} \in K$ and $\bar{\lambda} \in [0, 1]$.

Theorem 1.5.15. *[52] Let $\mathcal{T} : \overline{\Omega_c} \to \overline{\Omega_c}$ be a completely continuous operator and φ a nonnegative continuous concave functional on K such that $\varphi(\mathrm{q}) \leq \|\mathrm{q}\|$ for all $\mathrm{q} \in \overline{\Omega_c}$. Suppose that there exist constants $0 < å < b < d \leq c$ such that*

*(**D1**)* $\{\mathrm{q} \in \Omega_\varphi(b, d) : \varphi(\mathrm{q}) > b\} \neq \emptyset$ *and* $\varphi(\mathcal{T}\mathrm{q}) > b$ *if* $\mathrm{q} \in K_\varphi(b, d)$;

*(**D2**)* $\|\mathcal{T}\mathrm{q}\| < å$ *if* $\mathrm{q} \in \Omega_{å}$;

*(**D3**)* $\varphi(\mathcal{T}\mathrm{q}) > b$ *for* $\mathrm{q} \in \Omega_\varphi(b, c)$ *with* $\|\mathcal{T}\mathrm{q}\| > d$.

Then, $\mathcal{T}$ has at least three fixed points $\mathrm{q}_1, \mathrm{q}_2$ and q_3 such that $\|\mathrm{q}_1\| < å$, $b < \varphi(\mathrm{q}_2)$ and $\|\mathrm{q}_3\| > å$ with $\varphi(\mathrm{q}_3) < b$.

Chapter 2

Existence of Concave Positive Solutions for Nonlinear Fractional Differential Equation with p-Laplacian Operator

2.1 Introduction

This chapter essentially contains the intitled worck"Existence of Concave Positive Solutions for Nonlinear Fractional Differential Equation with p-Laplacian Operator [26] ".

The present work investigates the existence of multiple concave positive solutions of the following nonlinear mixed-orders three points boundary value problem for p-Laplacian

$$D_{1^-}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha}u(t)\right)\right)+a(t)f(u(t))=0,\quad 0<t<1, \tag{2.1.1}$$

$$u(0)-B_0\left(D_{0^+}^{\alpha}u(0)\right)=0,\quad u''(0)=0,\quad u'(1)-\mu u'(\eta)=\lambda, \tag{2.1.2}$$

$$D_{0^+}^{\alpha}u(1)=0,\quad \left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]'=0,\quad \left[\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right]''=0, \tag{2.1.3}$$

where, $f:\mathbb{R}_+\to\mathbb{R}_+,\ B_0:\mathbb{R}\to\mathbb{R}_+,\ a:(0,1)\to\mathbb{R}_+$ are given continuous functions, $D_{0^+}^{\alpha}, D_{1^-}^{\beta}$, are respectively; the left-sided and the right-sided Caputo fractional derivatives with $2<\beta,\alpha\leq 3$ and $\phi_p(s)$ is p-Laplacian operator: i.e., $\phi_p(s)=|s|^{p-2}s,\ p>1,\ \left(\phi_p\right)^{-1}=\phi_q,\ \frac{1}{p}+\frac{1}{q}=1,\ \eta\in(0,1),\ \mu\in[0,1)$ are two arbitrary constants and $\lambda\in\mathbb{R}_+:=[0,\infty)$ is a parameter.

The following hypotheses will be used in the sequel.

(H_1) The function a does not vanish identically on any closed subinterval of $(0,1)$.

(H_2) The function B_0 is odd on $\mathbb{R}$, and there exist $A,\ B>0$ such that

$$Bv\leqslant B_0(v)\leqslant Av;\quad \text{for all } v\geqslant 0.$$

Motivated by the above works, under some sufficient conditions we obtain existence of at least one, two and three positive solutions for (2.1.1)–(2.1.3). The organization of this chapter is as follows: First of all, we present some necessary definitions and preliminary results that will be used to prove our results. Secondly, we discuss the existence of at least one positive solution for (2.1.1)– (2.1.3). Next, we discuss the existence of multiple positive solutions for (2.1.1)–(2.1.3). Finally, we conclude by giving an example.

2.2 Preliminary results

Consider the boundary- value problem

$$D^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leqslant 3, \tag{2.2.1}$$

$$u(0) - B_0\left({}^cD^{\alpha}_{0+}u(0)\right) = 0, u''(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda. \tag{2.2.2}$$

Lemma 2.2.1. *Let $y \in C^+[0,1] = \{y \in C[0,1] : y(t) \geq 0;\ t \in [0,1]\}$, then the (BVP) (2.2.1)-(2.2.2) has a unique solution, defined by*

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)y(s)ds + \frac{\lambda t}{1-\mu} + B_0\left(y(0)\right),$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}; & s \leq t \\ (\alpha-1)t(1-s)^{\alpha-2}; & t \leq s \end{cases} \tag{2.2.3}$$

and

$$G_1(\eta,s) = \frac{1}{\Gamma(\alpha-1)}\begin{cases} \left((1-s)^{\alpha-2} - (\eta-s)^{\alpha-2}\right); & s \leq \eta \\ (1-s)^{\alpha-2}; & \eta \leq s. \end{cases} \tag{2.2.4}$$

Proof. By applying Lemma 1.3.15 and (1.3.15) in Lemma 1.3.16, the equation (2.2.1) is equivalent to the following integral equation,

$$u(t) = -C_0 - C_1 t - C_2 t^2 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \tag{2.2.5}$$

for some arbitrary constants $C_0, C_1, C_2 \in \mathbb{R}$. Boundary conditions (2.2.2) permit us to deduce the values,

$$-C_0 = -B_0\left(y(0)\right),\ C_2 = 0$$

$$C_1 = \frac{1}{(1-\mu)\,\Gamma(\alpha-1)}\left[\int_0^1 (1-s)^{\alpha-2}y(s)ds - \mu\int_0^{\eta}(\eta-s)^{\alpha-2}y(s)ds\right] + \frac{\lambda}{(1-\mu)},$$

then, the unique solution of (2.2.1)-(2.2.2) is given by the formula,

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t}{(1-\mu)\,\Gamma(\alpha-1)}\left[\int_0^1 (1-s)^{\alpha-2}y(s)ds\right.\\
&\left. -\mu\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds\right] + \frac{t\lambda}{(1-\mu)} + B_0\,(y(0))\\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s\\
&+ \frac{\mu t}{(1-\mu)\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}y(s)ds - \mu\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds + \frac{t\lambda}{(1-\mu)} + B_0\,(y(0))\\
&= \frac{1}{\Gamma(\alpha)}\left[-\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s + (\alpha-1)t\int_0^t (1-s)^{\alpha-2}y(s)ds\right.\\
&\qquad\left. +(\alpha-1)t\int_t^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s\right]\\
&+ \frac{\mu t}{(1-\mu)\,\Gamma(\alpha-1)}\left[\int_0^\eta (1-s)^{\alpha-2}y(s)ds + \int_\eta^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s - \int_0^\eta (\eta-s)^{\alpha-2}y(s)\mathrm{d}s\right]\\
&+ \frac{t\lambda}{(1-\mu)} + B_0\,(y(0))\\
&= \int_0^1 G(t,s)y(s)\mathrm{d}s + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)y(s)\mathrm{d}s + \frac{\lambda t}{(1-\mu)} + B_0\,(y(0))\,.
\end{aligned}
$$

□

Lemma 2.2.2. *Assume that $h(t) \in C([0,1])$ and $2 < \beta, \alpha \leq 3$, $\eta \in (0,1)$, $\mu \in [0,1)$, $\lambda \in \mathbb{R}_+$. Then, the following differential equation*

$$D^{\alpha}_{1^-}\left(\phi_p\left(D^{\alpha}_{0^+}u(t)\right)\right) = h(t), \quad 0 < t < 1, \tag{2.2.6}$$

satisfying the boundary conditions

$$u(0) - B_0\left(D^{\alpha}_{0^+}u(0)\right) = 0, u''(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda, \tag{2.2.7}$$

$$D^{\alpha}_{0^+}u(1) = 0, \left[\phi_p\left(D^{\alpha}_{0^+}u(0)\right)\right]' = 0, \left[\phi_p\left(D^{\alpha}_{0^+}u(1)\right)\right]'' = 0 \tag{2.2.8}$$

has a unique solution

$$
\begin{aligned}
u(t) = &\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\\
&+ \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\\
&+ \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right),
\end{aligned}
\tag{2.2.9}
$$

where

$$H(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}(\beta-1)(1-t)s^{\beta-2}-(s-t)^{\beta-1}; & t\leq s,\\ (\beta-1)(1-t)s^{\beta-2}; & s\leq t,\end{cases} \tag{2.2.10}$$

and $G(t,s)$ is defined in Lemma 2.2.1.

Proof. From Lemma 1.3.16 the boundary value problem (2.2.6)-(2.2.8) is equivalent to the integral equation

$$\phi_p\left(D_{0^+}^{\alpha}u(t)\right)=I_{1^-}^{\beta}u(t)-D_0+D_1(1-t)-D_2(1-t)^2$$

for some $D_0, D_1, D_2\in\mathbb{R}$; that is,

$$\phi_p\left(D_{0^+}^{\alpha}u(t)\right)=-D_0+D_1(1-t)-D_2(1-t)^2+\frac{1}{\Gamma(\beta)}\int_t^1(\tau-t)^{\beta-1}h(\tau)\mathrm{d}\tau.$$

By the boundary conditions $D_{0^+}^{\alpha}u(1)=0$, $\left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]'=0$, $\left[\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right]''=0$.
We have

$$D_0=D_2=0 \quad ,D_1=-\frac{1}{\Gamma(\beta-1)}\int_0^1\tau^{\beta-2}h(\tau)\mathrm{d}\tau.$$

Therefore, the solution u of fractional differential equation boundary value problem (2.2.6) − (2.2.8) satisfies

$$\begin{aligned}\phi_p\left(D_{0^+}^{\alpha}u(t)\right)&=I_{1^-}^{\beta}u(t)-(1-t)\frac{1}{\Gamma(\beta-1)}\int_0^1\tau^{\beta-2}h(\tau)\mathrm{d}\tau\\ &=-\int_0^1H(t,\tau)h(\tau)\mathrm{d}\tau.\end{aligned}$$

Consequently, $D_{0^+}^{\alpha}u(t)=-\phi_q\left(\int_0^1H(t,\tau)h(\tau)\mathrm{d}\tau\right)$. Thus, the boundary value problem (2.2.6) − (2.2.8) is equivalent to the problem

$$(P)\begin{cases}D_{0^+}^{\alpha}u(t)+\phi_q\left(\int_0^1H(t,\tau)h(\tau)\mathrm{d}\tau\right)=0, & t\in(0,1), \quad 2<\alpha\leqslant 3\\ u(0)-B_0\left(D_{0^+}^{\alpha}u(0)\right)=0, u''(0)=0, & u'(1)-\mu u'(\eta)=\lambda.\end{cases}$$

Lemma 2.2.1 implies that the boundary value problem (2.2.6) − (2.2.8) has a unique solution,

$$\begin{aligned}u(t)=&\int_0^1G(t,s)\phi_q\left(\int_0^1H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\\ &+\frac{\mu t}{(1-\mu)}\int_0^1G_1(\eta,s)\phi_q\left(\int_0^1H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\\ &+\frac{\lambda t}{(1-\mu)}+B_0\left(\phi_q\left(\int_0^1H(0,\tau)h(\tau)d\tau\right)\right).\end{aligned} \tag{2.2.11}$$

The proof is complete.

□

Lemma 2.2.3. *Assume that $\mu \in [0,1)$ holds, then the functions $G(t,s)$, $G_1(t,s)$ and $H(t,s)$ defined by* (2.2.3), (2.2.4) *and* (2.2.10) *respectively such that*

(i) $G(t,s)$, $G_1(t,s)$ *and* $H(t,s)$ *are continuous functions on* $[0,1]\times[0,1]$,

(ii) $G(t,s) \leqslant \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ *for* $(t,s)\in[0,1]\times[0,1]$, $G_1(\eta,s) \leqslant \frac{\eta(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$,
$H(t,s) \leqslant \frac{(1-t)s^{\beta-2}}{\Gamma(\beta-1)}$ *for* $(t,s)\in[0,1]\times[0,1]$, $\int_0^1 G_1(\eta,s)\mathrm{d}s = \frac{(1-\eta)\eta}{\Gamma(\alpha)}$,
$\int_0^1 H(s,\tau)\mathrm{d}\tau = \frac{(1-s)}{\Gamma(\beta)} - \frac{(1-s)^\beta}{\Gamma(\beta+1)}$,

(iii) $G(t,s), G_1(t,s), H(t,s) \geqslant 0$, *for all* $(t,s)\in[0,1]$,

(iv) $t^{\alpha-1}G(1,s) \leqslant G(t,s) \leqslant G(1,s);\quad (1-t)^{\beta-1}H(0,s) \leqslant H(t,s) \leqslant H(0,s)$ *for all* $0 \leqslant t,s \leqslant 1$,

(v) $(1-t^{\alpha-2})G(1,s) \leqslant G'_t(t,s) \leqslant \frac{\alpha-1}{\alpha-2}G(1,s)$ *for all* $0 \leqslant t,s \leqslant 1$.

Proof. From the definition of $G(t,s)$, $G_1(n,s)$ and $H(t,s)$, it easy to check that (i) and (ii) are both satisfied. We shall prove that (iii) holds, we set

$$g_1(t,s) = (\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \qquad \text{for} \qquad 0 \le s \le t \le 1$$
$$g_2(t,s) = (\alpha-1)t(1-s)^{\alpha-2} \qquad \text{for} \qquad 0 \le t \le s \le 1.$$

To prove that (iii) is true, we need to show that $g_i \geqslant 0$ for $i=1,2$
(1) if $0 \le s \le t \le 1$, we have

$$\begin{aligned} g_1(t,s) &:= (\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \\ &\geqslant (\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \\ &= t^{\alpha-1}\left[(\alpha-1)(1-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\right]. \end{aligned}$$

Notice that

$$\left(1-\frac{s}{t}\right)^r \leqslant (1-s)^r, \forall t,s \in (0,1);\ r>0. \tag{2.2.12}$$

Thus according to (2.2.12), we have

$$t^{\alpha-1}\left[(\alpha-1)(1-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\right] \geqslant 0$$

(2) if $s \le t$, then it is clearly $g_2 \geqslant 0$.
Similarly, $G_1(t,s)$ and $H(t,s)$ for $t,s \in (0,1)$. From above discussions, we conclude that $G(t,s) \geqslant 0$, $G_1(t,s) \geqslant 0$ and $H(t,s) \geqslant 0$ for any $t,s \in (0,1)$. So property (iii) holds.
Now we prove that (iv) holds. Firstly we check that $g_1(t,s)$ and $g_2(t,s)$ are nondecreasing with

respect to $t \in [0,1]$.
From (2.2.3) and (2.2.4) we have,

$$G'_t(t,s) = G_1(t,s) = \begin{cases} \frac{(\alpha-1)(1-s)^{\alpha-2}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1 \\ \frac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.2.13}$$

Clearly, $G_1(t,s) \geqslant 0$, for $0 \leqslant t,s \leqslant 1$, so $G(t,s)$ is increasing with respect to $t \in [0,1]$ and therefore, $G(t,s) \leqslant G(1,s)$, for $0 \leqslant t,s \leqslant 1$. This to say that (1) of Lemma 2.2.3 is satisfied. Secondly setting

$$h_1(t,s) = (\beta-1)(1-t)s^{\beta-2} - (s-t)^{\beta-1} \qquad 0 \leqslant t \leqslant s \leqslant 1$$

$$h_2(t,s) = (\beta-1)(1-t)s^{\beta-2} \qquad 0 \leqslant s \leqslant t \leqslant 1,$$

we have

$$\begin{aligned} \frac{\partial h_1(t,s)}{\partial t} &= -(\beta-1)s^{\beta-2} + (\beta-1)(s-t)^{\beta-2} \\ &= (\beta-1)[(s-t)^{\beta-2} - s^{\beta-2}] \\ &= (\beta-1)s^{\beta-2}[(1-\frac{t}{s})^{\beta-2} - s^{\beta-2}] \leqslant 0, \end{aligned}$$

which means that $h_1(t,s)$ is nonincreasing witch respect to t for $0 \leqslant t \leqslant s \leqslant 1$. It easily to see that $h_2(t,s)$ is nonincreasing witch respect to t for $0 \leqslant s \leqslant t \leqslant 1$.
Thus

$$H(t,s) \leqslant H(0,s) \quad \text{for} \quad 0 \leqslant t \leqslant s \leqslant 1,$$

and

$$H(t,s) \leqslant H(0,s) \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant 1.$$

From the above discussion, we have

$$H(t,s) \leqslant H(0,s) \quad \text{for} \quad (t,s) \in [0,1] \times [0,1].$$

On the other hand, if $t \geqslant s$, then,

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\ &\geqslant \frac{t^{\alpha-1}\left[(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\right]}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\ &= t^{\alpha-1}. \end{aligned}$$

If $t \leqslant s$, then $G(t,s)/G(1,s) = t^{\alpha-1}$; therefore, $G(t,s) \geqslant t^{\alpha-1}G(1,s)$, for $0 \leqslant s,t \leqslant 1$, which means that (iii) of Lemma 2.2.3 holds.

Similarly by the same technique of above discussions we find

$$H(t,s) \geqslant (1-t)^{\beta-1} H(0,s), \text{ for } 0 \leqslant s,t \leqslant 1.$$

So the property (iv) of Lemma 2.2.3 holds.
Finally we shall prove that (v) hold. There are two cases to consider.

Case 1: $0 \leq s \leq t \leq 1$. In this case, by (2.2.13), (iii) and (iv) of Lemma 2.2.3, we obtain:

$$\begin{aligned}
0 \leqslant \frac{G'_t(t,s)}{G(1,s)} :=& \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\
\leqslant& \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-2}} \\
=& \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{(\alpha-2)(1-s)^{\alpha-2}} \\
\leqslant& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-2)(1-s)^{\alpha-2}} - \frac{(\alpha-1)(t-s)^{\alpha-2}}{(\alpha-2)(1-s)^{\alpha-2}} \\
=& \frac{(\alpha-1)}{(\alpha-2)} - \frac{(\alpha-1)t^{\alpha-2}(1-\frac{s}{t})^{\alpha-2}}{(\alpha-2)(1-s)^{\alpha-2}} \\
\leqslant& \frac{\alpha-1}{\alpha-2}.
\end{aligned} \tag{2.2.14}$$

Also, we have

$$\begin{aligned}
\frac{G'_t(t,s)}{G(1,s)} :=& \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\
\geqslant& \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}} \\
\geqslant& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}} - \frac{(\alpha-1)(t-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}} \\
\geqslant& 1 - \frac{t^{\alpha-2}(1-\frac{s}{t})^{\alpha-2}}{(1-s)^{\alpha-2}}.
\end{aligned} \tag{2.2.15}$$

Therefore, from (2.2.12), we get

$$1 - \frac{t^{\alpha-2}(1-\frac{s}{t})^{\alpha-2}}{(1-s)^{\alpha-2}} \geq 1 - t^{\alpha-2} \geqslant 0.$$

Case 2: $0 \le t \le s \le 1$. In this case, by (2.2.13), (iii) and (iv) of Lemma 2.2.3, we can write:

$$\begin{aligned} 0 \leqslant \frac{G_t'(t,s)}{G(1,s)} :=& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}} \\ \leqslant& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-2}} \\ \leqslant& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-2)(1-s)^{\alpha-2}} \\ =& \frac{(\alpha-1)}{(\alpha-2)} \end{aligned} \tag{2.2.16}$$

Then by similar arguments to (2.2.15), one has

$$\begin{aligned} \frac{G_t'(t,s)}{G(1,s)} :=& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}} \\ \geqslant& \frac{(\alpha-1)(1-s)^{\alpha-2}}{(\alpha-1)(1-s)^{\alpha-2}} \\ =&1 \\ \geqslant& 1-t^{\alpha-2} \geqslant 0. \end{aligned} \tag{2.2.17}$$

Hence, from (2.2.14), (2.2.15), (2.2.16) and (2.2.17), we deduce

$$\left(1-t^{\alpha-2}\right)G(1,s) \leqslant G_t'(t,s) \leqslant \frac{\alpha-1}{\alpha-2}G(1,s); \quad \text{for all} \quad 0 \le t,s \le 1, \tag{2.2.18}$$

the proof is complete. □

We shall consider the Banach space $E = C^3([0,1])$ equipped with standard norm

$$\|u\| = \max\left\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|, \max_{0\le t\le 1}|u''(t)|, \max_{0\le t\le 1}|u'''(t)|\right\}.$$

Set the cone on E,

$$P = \{u \in E : u \text{ is a nonnegative, monotone increasing and concave function on } [0,1]\}.$$

Lemma 2.2.4. *Let $h(t) \in C^+[0,1]$. If $u \in P$ is a solution of BVP (2.2.6)- (2.2.8), then $u(t)$ satisfies*

1.
$$\min_{\rho\leqslant t\leqslant 1} u(t) \geqslant \rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u(t)|,$$

2.
$$\min_{\rho\leqslant t\leqslant 1} u(t) \geqslant \frac{\alpha-2}{\alpha-1}\rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u'(t)|,$$

3.
$$\min_{\rho\leqslant t\leqslant 1} u(t) \geqslant B\Gamma(\alpha-1)\rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u''(t)|,$$

4.
$$\min_{\rho\leqslant t\leqslant 1} u(t) \geqslant B\rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u'''(t)|.$$

Proof. (1) From the Definition 1.3.1, 1.3.8 and Remark 1.3.10, we have $D^{\alpha}_{0^+}u(t) = I^{3-\alpha}_{0^+}u'''(t), 2 < \alpha < 3$, $D^{\alpha}_{0^+}u(t) = u'''(t), \alpha = 3$. So by Lemma 1.3.14 and 2.2.6, we deduce that $D^{\alpha}_{0^+}u(t)$ is continuous for all $u(t) \in E$.
Hence, from the definition, u is nonnegative, continuous and increasing function on [0,1]. From Lemma 2.2.2 and 2.2.3, we get

$$\begin{aligned}
u(t) :=& \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&+ \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right) \\
\leqslant& \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\max_{0\leqslant t\leqslant 1}|u(t)| \leqslant& \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right).
\end{aligned}$$

On the other hand, Lemmas 2.2.2 and 2.2.3 imply that, for any $t \in [\rho, 1]$,

$$\begin{aligned}
u(t) &\geqslant \int_0^1 \rho^{\alpha-1} G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&+ \frac{\mu\rho^{\alpha-1}}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&\quad + \frac{\lambda\rho^{\alpha-1}}{(1-\mu)} + \rho^{\alpha-1}B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right) \\
&\geqslant \rho^{\alpha-1}\left[\int_0^1 \rho^{\alpha-1} G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s\right. \\
&+ \frac{\mu\rho^{\alpha-1}}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&\quad \left. + \frac{\lambda\rho^{\alpha-1}}{(1-\mu)} + \rho^{\alpha-1}B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right)\right] \\
&\geqslant \rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u(t)|.
\end{aligned}$$

(2) From Lemmas 2.2.2 and 2.2.3, clearly u' is nonnegative, decreasing on $[0,1]$ and therefore,

$$\begin{aligned}
u'(t) &:= \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s + \frac{\lambda}{(1-\mu)} \\
&\leqslant \int_0^1 \frac{\alpha-1}{\alpha-2} G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s + \frac{\lambda}{(1-\mu)}.
\end{aligned}$$

Since $2 < \alpha \leq 3$, we have $\frac{\alpha-1}{\alpha-2} \geq 1$. Then,

$$\begin{aligned}
u'(t) &\leqslant \frac{\alpha-1}{\alpha-2}\left(\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s\right. \\
&\left. + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s + \frac{\lambda}{(1-\mu)}\right), \\
&\leqslant \frac{\alpha-1}{\alpha-2}\left[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s\right. \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right) \mathrm{d}s \\
&\quad \left. + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right)\right].
\end{aligned}$$

Thus

$$\max_{0\leqslant t\leqslant 1}|u'(t)| \leqslant \frac{\alpha-1}{\alpha-2}\left[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right)\right]. \tag{2.2.19}$$

From (2.2.19), we have

$$\begin{aligned} u(t) \geqslant \rho^{\alpha-1}&\left[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\right. \\ &+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\ &\left.+\frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right)\right], \\ \geqslant &\frac{\alpha-2}{\alpha-1}\rho^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u'(t)|. \end{aligned}$$

Thus, it follows that (2) holds.

(3) By Lemmas 2.2.3 and 2.2.12, one has

$$D_{0^+}^{\alpha}(u)(t) = -\left(\phi_q\left(\int_0^1 H(t,\tau)h(\tau)d\tau\right)\right), \quad 2<\alpha\leq 3. \tag{2.2.20}$$

From above discussions, we conclude that $u^{(3)}$ is negative and continuous function on $[0,1]$. Also

$$\begin{aligned} D_{0^+}^{2}(u(t)) &= D_{0^+}^{2-\alpha}D_{0^+}^{\alpha}(u(t)) \\ &= -D_{0^+}^{2-\alpha}\left(\phi_q\left(\int_0^1 H(t,\tau)h(\tau)d\tau\right)\right) \\ &= -I_{0^+}^{\alpha-2}\left(\phi_q\left(\int_0^1 H(t,\tau)h(\tau)d\tau\right)\right) \\ &= -\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\left(\phi_q\left(\int_0^1 H(s,\tau)h(\tau)\mathrm{d}\tau\right)\right)\mathrm{d}s, \end{aligned}$$

so, u'' is negative, decreasing on $[0,1]$. Hence from Lemma 2.2.3 and Condition (H_2), we obtain

$$\begin{aligned}
|u''(t)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\left(\phi_q\left(\int_0^1 H(s,\tau)h(\tau)\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&\le \phi_q\left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau\right)\times\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\mathrm{d}s \\
&\le \phi_q\left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau\right)\times\frac{1}{\Gamma(\alpha-1)}t^{\alpha-2} \\
&\leqslant \phi_q\left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau\right)\times\frac{1}{\Gamma(\alpha-1)} \\
&\leqslant \frac{1}{B\Gamma(\alpha-1)}\times B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau\right)\right) \\
&\leqslant \frac{1}{B\Gamma(\alpha-1)}\times\left[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s\right. \\
&+\frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)d\tau\right)\mathrm{d}s \\
&\quad\left.+\frac{\lambda}{(1-\mu)}+B_0\left(\phi_q\left(\int_0^1 H(0,\tau)h(\tau)d\tau\right)\right)\right] \\
&\le \frac{1}{B\Gamma(\alpha-1)}\times\max_{0\leqslant t\leqslant 1}|u(t)|.
\end{aligned}$$

Thus,

$$\max_{0\leqslant t\leqslant 1}|u''(t)| \leqslant \frac{1}{B\Gamma(\alpha-1)}\times\max_{0\leqslant t\leqslant 1}|u(t)|.$$

Consequently, (4) holds.

(4) From (2.2.20), we have

$$u^{(3)}(t) = -\phi_q\left(\int_0^1 H(t,\tau)h(\tau)\mathrm{d}\tau\right).$$

The continuity of $h(t)$ and $H(t,\tau)$ implies that $u^{(3)}(t)$ is a negative, continuous function on $[0,1]$. Then

$$|u^{(3)}(t)| = \phi_q\left(\int_0^1 H(t,\tau)h(\tau)\mathrm{d}\tau\right)$$

By (H_2), we obtain

$$|u^{(3)}(t)| \leqslant \phi_q\left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau\right)$$

$$\leqslant \frac{1}{B} \times B_0 \left(\phi_q \left(\int_0^1 H(0,\tau)h(\tau)\mathrm{d}\tau \right) \right)$$
$$\leqslant \frac{1}{B} \times \left[\int_0^1 G(1,s)\phi_q \left(\int_0^1 H(s,\tau)h(\tau)d\tau \right) \mathrm{d}s \right.$$
$$+ \frac{\mu}{(1-\mu)} \int_0^1 G_1(\eta,s)\phi_q \left(\int_0^1 H(s,\tau)h(\tau)d\tau \right) \mathrm{d}s$$
$$\left. + \frac{\lambda}{(1-\mu)} + B_0 \left(\phi_q \left(\int_0^1 H(0,\tau)h(\tau)d\tau \right) \right) \right]$$
$$\leqslant \frac{1}{B} \times \max_{0\leqslant t\leqslant 1} |u(t)|.$$

Thus,

$$\max_{0\leqslant t\leqslant 1} |u^{(3)}(t)| \leqslant \frac{1}{B} \times \max_{0\leqslant t\leqslant 1} |u(t)|.$$

Using (1), we obtain

$$B\rho^{\alpha-1} \max_{0\leqslant t\leqslant 1} |u^{(3)}(t)| \leqslant \min_{\rho\leqslant t\leqslant 1} |u(t)|.$$

This completes the proof.

□

Remark 2.2.5. Let $h(t) \in C^+[0,1]$. if $u \in P$ is a solution of BVP (2.2.6)–(2.2.8), then u satisfies

$$\min_{\rho\leqslant t\leqslant 1} u(t) \geqslant \rho^{\alpha-1} M \parallel u \parallel,$$

where

$$M = \min \left\{ \frac{\alpha-2}{\alpha-1}, B, B\Gamma(\alpha-1) \right\}.$$

Define the cone K by

$$K = \left\{ u \in P : \min_{\rho\leqslant t\leqslant 1} u(t) \geqslant \rho^{\alpha-1} M \parallel u \parallel \right\}$$

and an integral operator $T_\lambda : E \to E$ by

$$T_\lambda u(t) = \int_0^1 G(t,s)\phi_q \left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau \right) \mathrm{d}s$$
$$+ \frac{\mu t}{(1-\mu)} \int_0^1 G_1(\eta,s)\phi_q \left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau \right) \mathrm{d}s$$
$$+ \frac{\lambda t}{(1-\mu)} + B_0 \left(\phi_q \left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau \right) \right) := F(t). \tag{2.2.21}$$

The fixed points of T_λ are solution of (2.1.1)-(2.1.3). Our aim is to show that $T_\lambda : K \to K$ is completely continuous, in order to use some fixed point Theorems.

Lemma 2.2.6. *[25] Let $c > 0, a > 0$. For any $x, y \in [0,c]$, we have*

1 *if* $a > 1$, *then* $|x^a - y^a| \leqslant ac^{\alpha-1}|x-y|$,

2 *if* $0 < a \leqslant 1$, *then* $|x^a - y^a| \leqslant |x-y|^{\alpha}$.

Lemma 2.2.7. $T_\lambda : K \to K$ *is completely continuous.*

Proof. Let $u \in K$; in view of the assumption of nonnegativeness and continuity of functions $G(t,s), H(t,s)$ and $a(t)f(u(t))$, we conclude that $T_\lambda : K \to K$ is continuous. Using the property of the fractional integrals and derivatives, we can get that

$$(T_\lambda' u)(t) = \int_0^1 G_t'(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu).} \tag{2.2.22}$$

Clearly, $T_\lambda'(u)$ is continuous and nonnegative.
and

$$\begin{aligned}(T_\lambda'' u)(t) &= D_{0^+}^2(T_\lambda u)(t) \\ &= D_{0^+}^{2-\alpha}D_{0^+}^{\alpha}(T_\lambda u)(t) \\ &= D_{0^+}^{2-\alpha}D_{0^+}^{\alpha}(F(t)) \\ &= -D_{0^+}^{2-\alpha}\left(\phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\ &= -I_{0^+}^{2-\alpha}\left(\phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right).\end{aligned} \tag{2.2.23}$$

It is clear that $T_\lambda''(u)$ is continuous and nonnegative on $[0,1]$.
Also, from (2.2.20) we get,

$$(T_\lambda^{(3)} u)(t) = -\phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right). \tag{2.2.24}$$

Observing $T_\lambda^{(3)}(u)$ is continuous.
So, $T_\lambda u$ is concave on $[0,1]$ and $T_\lambda u \in C^3[0,1]$. From Remark 2.2.5 we obtain $T_\lambda(K) \subset K$. Let $\Omega \subset K$ be bounded, then there exists $L > 0$ such that

$$\forall u \in \Omega : |f(u(t))| \leqslant L.$$

Then, $u \in \Omega$ and from Lemmas 2.2.2 and 2.2.3, we have

$$|T_\lambda u(t)| = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right)$$

$$
\begin{aligned}
&\leq \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\right) \\
&\leq \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right) \\
&\leq \left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s + A\right) \\
&\times L^{q-1}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right) + \frac{\lambda}{(1-\mu)} \\
&\leq \left(\frac{L^{q-1}}{\Gamma(\alpha)} - \frac{L^{q-1}}{\Gamma(\alpha+1)} + \frac{L^{q-1}(1-\eta)^{\alpha-1}}{\Gamma(\alpha)} + A\right) \\
&\quad \times \phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right) + \frac{\lambda}{(1-\mu)} \doteq \ell.
\end{aligned} \tag{2.2.25}
$$

$$
\begin{aligned}
|(T'_\lambda u)(t)| &= \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)} \\
&\leq \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)} \\
&\leq \int_0^1 \frac{\alpha-1}{\alpha-2}G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\right) \\
&\leq \frac{\alpha-1}{\alpha-2} \times \left[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s\right. \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s
\end{aligned}
$$

$$+\frac{\lambda}{(1-\mu)}+A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\Big] \doteq \frac{\alpha-1}{\alpha-2}\ell. \tag{2.2.26}$$

$$\begin{aligned}
|(T''_\lambda u)(t)| &= I_{0^+}^{2-\alpha}\left(\phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right)\\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\le \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s\\
&\le \left(\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{\alpha-3}\mathrm{d}s\right)\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\\
&\le \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\\
&\le \frac{1}{B\Gamma(\alpha-1)}\times B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\right)\\
&\le \frac{1}{B\Gamma(\alpha-1)}\times\Big[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s\\
&+\frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s
\end{aligned}$$

$$+\frac{\lambda}{(1-\mu)}+A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\Big] \doteq \frac{1}{B\Gamma(\alpha-1)}\ell. \tag{2.2.27}$$

$$\begin{aligned}
|(T)_\lambda^{(3)}u)(t)| &= \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\\
&\le \phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\\
&\le \frac{1}{B}B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\right)\\
&\le \frac{1}{B}\times\Big[\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s\\
&+\frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\mathrm{d}s
\end{aligned}$$

$$+\frac{\lambda}{(1-\mu)}+A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)Ld\tau\right)\Big] \doteq \frac{1}{B}\ell. \tag{2.2.28}$$

Thus, from (2.2.25), (2.2.26), (2.2.27) and (2.2.28), $\| T_\lambda u(t) \| \leqslant \mathbf{N}\ell$, for all $u \in \Omega$, where

$$\mathcal{N} = \max\left\{1, \frac{1}{B}, \frac{1}{B\Gamma(\alpha-1)}, \frac{\alpha-1}{\alpha-2}\right\}.$$

Hence, $T_\lambda(\Omega)$ is uniformly bounded.
On the other hand, let $u \in \Omega$, $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, from Lemmas 2.2.2, 2.2.6 and 2.2.3, we have

$$\begin{aligned}
|(T_\lambda u)(t_2) - (T_\lambda u)(t_1)| &= \int_0^1 G(t_2,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&+ \frac{\mu t_2}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s + \frac{\lambda t_2}{(1-\mu)} \\
&- \int_0^1 G(t_1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&- \frac{\mu t_1}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s - \frac{\lambda t_1}{(1-\mu)} \\
&\le \int_0^1 |G(t_2,s)-G(t_1,s)|\phi_q\left(\int_0^1 H(s,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&+ \frac{\mu|t_2-t_1|}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)Ld\tau\right)\mathrm{d}s + \frac{\lambda|t_2-t_1|}{(1-\mu)} \\
&\le \left(\int_0^1 |G(t_2,s)-G(t_1,s)|\mathrm{d}s + \frac{\mu|t_2-t_1|}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s\right) \\
&\quad \times L^{q-1}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right) + \frac{\lambda|t_2-t_1|}{(1-\mu)},
\end{aligned}$$

$$\begin{aligned}
|(T'_\lambda u)(t_2) - (T'_\lambda u)(t_1)| &= \int_0^1 G_1(t_2,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&- \int_0^1 G_1(t_1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\le \int_0^1 |G_1(t_2,s)-G_1(t_1,s)|\phi_q\left(\int_0^1 H(s,\tau)a(\tau)Ld\tau\right)\mathrm{d}s \\
&\le \left(\int_0^1 |G_1(t_2,s)-G_1(t_1,s)|\mathrm{d}s\right) \times L^{q-1}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right),
\end{aligned}$$

$$\begin{aligned}
|(T''_\lambda u)(t_2) - (T''_\lambda u)(t_1)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^{t_2}(t_2-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&- \frac{1}{\Gamma(\alpha-2)}\int_0^{t_1}(t_1-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha-2)}\int_0^{t_1}\left[(t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}\right] \\
&\quad \times \phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} |(t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}| \phi_q \left(\int_0^1 H(s,\tau) a(\tau) L d\tau \right) \mathrm{d}s$$
$$+ \frac{1}{\Gamma(\alpha-2)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-3} \phi_q \left(\int_0^1 H(s,\tau) a(\tau) L d\tau \right) \mathrm{d}s$$
$$\leq (I_1 + I_2) \times L^{q-1} \phi_q \left(\int_0^1 H(0,\tau) a(\tau) d\tau \right)$$
$$\leq (I_3 + I_4) \times L^{q-1} \phi_q \left(\int_0^1 H(0,\tau) a(\tau) d\tau \right),$$

where

$$I_1 = \frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} (t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3} \mathrm{d}s,$$
$$I_2 = \frac{1}{\Gamma(\alpha-2)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-3} \mathrm{d}s,$$
$$I_3 = \frac{1}{\Gamma(\alpha-1)} \left[t_2^{\alpha-2} - t_1^{\alpha-2} - (t_2-t_1)^{\alpha-2} \right],$$
$$I_4 = \frac{1}{\Gamma(\alpha-1)} t_1^{\alpha-2}.$$

If $q-1 > 1$, then

$$|(T_\lambda^{(3)} u)(t_2) - (T_\lambda^{(3)} u)(t_1)| = \left| \phi_q \left(\int_0^1 H(t_1,\tau) a(\tau) f(u(\tau))\, d\tau \right) - \phi_q \left(\int_0^1 H(t_2,\tau) a(\tau) f(u(\tau))\, d\tau \right) \right|$$
$$\leq (q-1) \phi_q (C) \left| \int_0^1 H(t_1,\tau) a(\tau) L d\tau - \int_0^1 H(t_2,\tau) a(\tau) L d\tau \right|$$
$$\leq (q-1) \phi_q (CL) \int_0^1 \Big| H(t_1,\tau) - H(t_2,\tau) \Big| a(\tau) d\tau.$$

If $q-1 \leq 1$, then

$$|(T_\lambda^{(3)} u)(t_2) - (T_\lambda^{(3)} u)(t_1)|$$

$$= \left| \phi_q \left(\int_0^1 H(t_1,\tau) a(\tau) f(u(\tau))\, d\tau \right) - \phi_q \left(\int_0^1 H(t_2,\tau) a(\tau) f(u(\tau))\, d\tau \right) \right|$$
$$\leq \phi_q \left(\left| \int_0^1 H(t_1,\tau) a(\tau) L d\tau - \int_0^1 H(t_2,\tau) a(\tau) L d\tau \right| \right)$$
$$\leq \phi_q (L)\, \phi_q \left(\int_0^1 \Big| H(t_1,\tau) - H(t_2,\tau) \Big| a(\tau) d\tau \right).$$

The continuity of $G(t,s)$, $G_1(t,s)$ and $H(t,s)$ implies that the right-side of the above inequality tends to zero if $t_2 \to t_1$. Therefore by Arzela-Ascoli Theorem, T_λ is completely continuous. □

Set,

$$\lambda_1 = \left[\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s + A\right) \times \phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]^{-1},$$

$$\lambda_2 = \left[\frac{(\alpha-1)}{(\alpha-2)}\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s\right) \times \phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]^{-1},$$

$$\lambda_3 = \left[\frac{A}{B\Gamma(\alpha-1)} \times \phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]^{-1},$$

$$\lambda_4 = \left[\frac{A}{B} \times \phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]^{-1},$$

$$\lambda_5 = \Bigg[\rho^{\alpha-1}M\Bigg(\int_\rho^1 G(1,s)(1-s)^{(q-1)(\beta-1)}\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)(1-s)^{(q-1)(\beta-1)}\mathrm{d}s\Bigg)\phi_q\left(\int_\rho^1 H(0,\tau)a(\tau)d\tau\right)\Bigg]^{-1}.$$

2.3 Existence of Solutions

Definition 2.3.1. A continuous function $f : \mathbb{R} \to \mathbb{R}_+$ is said superlinear if:

$$f_0 = 0, \quad f_\infty = \infty,$$

where

$$f_0 = \lim_{r\to 0^+}\frac{f(r)}{\phi_p(r)} \text{ and } f_\infty = \lim_{r\to\infty}\frac{f(r)}{\phi_p(r)}.$$

Theorem 2.3.2. *Suppose that f is superlinear. Then BVP (2.1.1)–(2.1.3) has at least one concave positive solution for λ small enough and has no positive solution for λ large enough.*

Proof. We divide the proof into two steps.

Step 1. First, we prove that the BVP (2.1.1)–(2.1.3) admits at least one concave and positive solution for sufficiently small $\lambda > 0$. Since $f_0 = 0$, for

$$\min\{\phi_p(\lambda_1), \phi_p(\lambda_2), \phi_p(\lambda_3), \phi_p(\lambda_4)\} > 0,$$

there exists $R_1 > 0$ such that

$$\frac{f(r)}{\phi_p(r)} \leqslant \min\{\phi_p(\lambda_1), \phi_p(\lambda_2), \phi_p(\lambda_3), \phi_p(\lambda_4)\}\,;\ r \in [0, R_1].$$

Therefore,

$$f(r) \leqslant \min\{\phi_p(r\lambda_1), \phi_p(r\lambda_2), \phi_p(r\lambda_3), \phi_p(r\lambda_4)\}\,;\ r \in [0, R_1]. \tag{2.3.1}$$

Let $\Omega_1 = \left\{u \in C^3[0,1] : \| u \| \leq R_1\right\}$ and λ satisfies

$$0 < \lambda \leqslant \frac{(1-\mu)R_1}{2}. \tag{2.3.2}$$

Then, for any $u \in K \cap \partial\Omega_1$, it follows from Lemma 2.2.4, Remark 2.2.5, equality (2.2.21) and inequalities (2.3.1)-(2.3.2) that,

$$\begin{aligned}
(T_\lambda u)(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\
&\leqslant \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\leqslant \frac{\lambda_1}{2}\left(\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\mathrm{d}s\right. \\
&\quad \left. + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\mathrm{d}s\right) \| u \| \\
&\quad + \frac{(1-\mu)R_1}{2(1-\mu)} + \frac{\lambda_1}{2}A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right) \| u \| \\
&\leqslant \frac{\lambda_1}{2}\left[\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s + A\right)\right. \\
&\quad \left. \times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \| u \| + \frac{R_1}{2} \\
&= \frac{\| u \|}{2} + \frac{\| u \|}{2} = \| u \|,
\end{aligned}$$

$$\begin{aligned}
(T'_\lambda u)(t) &= \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)} \\
&\leqslant \int_0^1 \frac{\alpha-1}{\alpha-2}G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)} \\
&\leqslant \frac{\lambda_2}{2}\left[\frac{(\alpha-1)}{(\alpha-2)}\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s\right)\right. \\
&\quad \left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \parallel u \parallel + \frac{R_1}{2} \\
&= \frac{\parallel u \parallel}{2} + \frac{\parallel u \parallel}{2} = \parallel u \parallel,
\end{aligned}$$

$$\begin{aligned}
|T''_\lambda u(t)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\leqslant \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\leqslant \lambda_3\left[\frac{A}{B\Gamma(\alpha-1)}\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \parallel u \parallel \\
&= \parallel u \parallel,
\end{aligned}$$

$$\begin{aligned}
|(T^{(3)}_\lambda u)(t)| &= \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\leqslant \phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\leqslant \lambda_4\left[\frac{A}{B}\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \parallel u \parallel \\
&= \parallel u \parallel .
\end{aligned}$$

Thus

$$\parallel T_\lambda u \parallel \leqslant \parallel u \parallel, \quad \text{for } u \in K\cap\partial\Omega_1. \tag{2.3.3}$$

On the other hand, since $f_\infty = \infty$, for $\phi_p(\lambda_5) > 0$, there exists $R_2 > R_1$ such that

$$\frac{f(r)}{\phi_p(r)} \geqslant \phi_p(\lambda_5), r \in [\rho^{\alpha-1}MR_2, \infty),$$

witch implies

$$f(r) \geqslant \phi_p(r\lambda_5), \quad \text{for } r \in [\rho^{\alpha-1}MR_2, \infty). \tag{2.3.4}$$

Set $\Omega_2 = \left\{u \in C^3[0,1] : \| u \| \leqslant R_2\right\}$. For any $u \in K \cap \partial\Omega_2$, by Remark 2.2.5 one has

$$\min_{\tau\in[\rho,1]} u(\tau) \geqslant \rho^{\alpha-1} M \| u \| = \rho^{\alpha-1} M R_2.$$

Thus, from (2.3.4) we can conclude that

$$\begin{aligned}
(T_\lambda u)(1) &= \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f(u(\tau))\,d\tau\right)\right) \\
&\geqslant \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\geqslant \int_\rho^1 G(1,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)\mathrm{d}s \\
&\geqslant \lambda_5\left[\int_\rho^1 G(1,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)\phi_p(u(\tau))\,d\tau\right)\mathrm{d}s\right. \\
&\quad \left. + \frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)\phi_p(u(\tau))\,d\tau\right)\mathrm{d}s\right] \\
&\geqslant \lambda_5\left[\int_\rho^1 \rho^{\alpha-1} M G(1,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)d\tau\right)\mathrm{d}s\right. \\
&\left. + \frac{\mu(\rho^{\alpha-1}M)}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)d\tau\right)\mathrm{d}s\right] \| u \|,
\end{aligned}$$

which implies that

$$\| T_\lambda u \| \geqslant \| u \|, \quad \text{for any} \quad u \in K \cap \partial\Omega_2 \tag{2.3.5}$$

Therefore, by (2.3.4), (2.3.5)and the first part of Theorem 1.5.11 we conclude that the operator T_λ has at least one fixed point $u \in K \cap \overline{\Omega_2} \backslash \Omega_2$, which is a concave and positive solution of the BVP (2.1.1)-(2.1.3).

Step 2. We verify that BVP (2.1.1)-(2.1.3) has no concave positive solution for λ large enough. Otherwise, there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, with $\lim_{n\to+\infty} \lambda_n = +\infty$, such that for any positive integer n, the BVP

$$H\begin{cases} D_{1^-}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha} u(t)\right)\right) + a(t)f(u(t)) = 0, \quad 0 < t < 1 \\ u(0) - B_0\left(D_{0^+}^{\alpha} u(0)\right) = 0, u''(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda_n \\ D_{0^+}^{\alpha} u(1) = 0, \left[\phi_p\left(D_{0^+}^{\alpha} u(0)\right)\right]' = 0, \left[\phi_p\left(D_{0^+}^{\alpha} u(1)\right)\right]'' = 0 \end{cases}$$

has a positive solution $u_n(t)$, the identity(2.2.21), permits us to write

$$\begin{aligned} u_n(1) = & \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\lambda_n}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\ \geqslant & \frac{\lambda_n}{(1-\mu)} \to +\infty, \quad (n\to\infty). \end{aligned}$$

Thus

$$\| u \| \to +\infty, \quad (n\to\infty)$$

Since $f_\infty = \infty$, for $\phi_p\left(4\rho^{\alpha-1}M\lambda_5\right) > 0$, there exists $\tilde{R} > 0$ such that $\frac{f(r)}{\phi_p(r)} \geqslant \phi_p\left(4\rho^{\alpha-1}M\lambda_5\right), r \in [\rho^{\alpha-1}M\tilde{R}, \infty)$, witch implies that

$$f(r) \geqslant \phi_p\left(2r\rho^{\alpha-1}M\lambda_5\right), \quad \text{for } r \in [\rho^{\alpha-1}M\tilde{R}, \infty),$$

Let n be large enough that $\| u \| \geqslant \tilde{R}$, then

$$\begin{aligned} \| u \| \geqslant & u_n(1) \\ = & \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\lambda_n}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\ \geqslant & \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ \geqslant & \int_\rho^1 G(1,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ & + \frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\ \geqslant & 2\lambda_5\left[\int_\rho^1 G(1,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)\phi_p\left(u(\tau)\right)d\tau\right)\mathrm{d}s\right. \\ & \left.+\frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_\rho^1 H(s,\tau)a(\tau)\phi_p\left(u(\tau)\right)d\tau\right)\mathrm{d}s\right] \end{aligned}$$

$$\begin{aligned}
&\geqslant 2\lambda_5 \left[\int_\rho^1 \rho^{2(\alpha-1)} M^2 G(1,s)(1-s)^{(q-1)(\beta-1)}\right.\\
&\qquad \times \phi_q\left(\int_\rho^1 H(0,\tau)a(\tau)d\tau\right) \mathrm{d}s\\
&+\frac{\mu(\rho^{2(\alpha-1)}M^2)}{(1-\mu)}\int_\rho^1 G_1(\eta,s)(1-s)^{(q-1)(\beta-1)}\\
&\qquad \left.\times \phi_q\left(\int_\rho^1 H(0,\tau)a(\tau)d\tau\right) \mathrm{d}s\right] \parallel u \parallel\\
&= 2 \parallel u \parallel,
\end{aligned}$$

which is contradiction. The proof is complete.

□

Moreover, if the functions f and B_0 are nondecreasing, the following theorem holds.

Theorem 2.3.3. *Suppose that f is superlinear. If f and B_0 are nondecreasing, then there exists a positive constant λ^* such that BVP (2.1.1)-(2.1.3) has at least one concave positive solution for $\lambda \in (0, \lambda^*)$ and has no concave positive solution for $\lambda \in (\lambda^*, \infty)$.*

Proof. Let $\Theta = \{\lambda : \text{BVP (2.1.1)–(2.1.3) has at least one positive solution}\}$ and $\lambda^* = \sup \Theta$; it follows from Theorem 2.3.2 that $0 < \lambda^* < \infty$. From the definition of λ^*, we know that for any $\lambda \in (0, \lambda^*)$, there exists a $\lambda_0 > \lambda$ such that

$$\begin{cases}
D_{1^-}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha}u(t)\right)\right) + a(t)f(u(t)) = 0, \quad 0 < t < 1\\
u(0) - B_0\left(D_{0^+}^{\alpha}u(0)\right) = 0, u''(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda_0\\
D_{0^+}^{\alpha}u(1) = 0, \left[\phi_p\left(D_{0^+}^{\alpha}u(0)\right)\right]' = 0, \left[\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right]'' = 0
\end{cases}$$

has a positive solution $u_0(t)$. Now we prove that for any $\lambda \in (0, \lambda_0)$, the BVP (2.1.1)–(2.1.3) has a positive solution. In fact, let

$$K(u_0) = \{u \in K : u(t) < u_0(t), t \in [0,1]\}$$

For any $\lambda \in (0, \lambda_0)$, $u \in K(u_0)$, it follows from (2.2.21) and the monotonicity of f that,

$$\begin{aligned}
(T_\lambda u)(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u_0(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u_0(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u_0(\tau)\right)d\tau\right)\right) \\
&= u_0(t).
\end{aligned}
$$

Thus, $T_\lambda(K(u_0)) \subseteq K(u_0)$. By Schauder's fixed point theorem we ensure that there exists a fixed point $u \in K(u_0)$, which is a positive solution of (2.1.1)–(2.1.3). The proof is complete. □

Now we consider the case where f is sublinear.

Definition 2.3.4. A continuous function $f : \mathbb{R} \to \mathbb{R}_+$ is said sublinear if:

$$f_0 = \infty, \quad f_\infty = 0,$$

where f_0 and f_∞ are mentioned in the Definition 2.3.1.

Theorem 2.3.5. *Suppose that f is sublinear. Then, the BVP (2.1.1)–(2.1.3) has at least one positive solution for any* $\lambda \in (0,\infty)$.

Proof. Since $f_0 = \infty$, there exists $R_1 > 0$ such that $f(r) \geq \phi_p(\lambda_5 r)$, for any $r \in [0, R_1]$. So for any $u \in K$ with $\| u \|= R_1$ and any $\lambda > 0$, we have

$$
\begin{aligned}
(T_\lambda u)(1) &= \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\
&\geq \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\geq \int_\rho^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\geq \lambda_5\left[\int_\rho^1 G(1,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)\phi_p\left(u(\tau)\right)d\tau\right)\mathrm{d}s\right. \\
&\quad \left. +\frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)\phi_p\left(u(\tau)\right)d\tau\right)\mathrm{d}s\right]
\end{aligned}
$$

$$\geqslant \lambda_5 \left[\int_\rho^1 \rho^{\alpha-1} M G(1,s) \phi_q \left(\int_\rho^1 H(s,\tau) a(\tau) d\tau \right) \mathrm{d}s \right.$$
$$\left. + \frac{\mu(\rho^{\alpha-1}M)}{(1-\mu)} \int_\rho^1 G_1(\eta,s) \phi_q \left(\int_\rho^1 H(s,\tau) a(\tau) d\tau \right) \mathrm{d}s \right] \| u \|$$
$$= \| u \| .$$

Hence, $\| T_\lambda u \| \geqslant \| u \|$. So, if we set $\Omega_1 = \{u \in K : \| u \| < R_1\}$, then

$$\| T_\lambda u \| \geqslant \| u \|, \quad \text{for} \quad u \in K \cap \partial\Omega_1. \tag{2.3.6}$$

Next we construct the set Ω_2. We consider two cases: f is bounded or f is unbounded.

Case (1) : Suppose that f is bounded, say $f(r) \leq S$ for all $r \in [0, \infty)$. In this case we choose

$$R_2 \geq \max \left\{ 2R_1, \phi_p(2S\lambda_1), \phi_p(2S\lambda_2, S\lambda_3), \phi_p(S\lambda_4), \frac{2\lambda}{1-\mu} \right\}$$

and then for $u \in K$ with $\| u \| = R_2$, we have

$$\begin{aligned} |(T_\lambda u)(t)| &= \int_0^1 G(t,s) \phi_q \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau))\, d\tau \right) \mathrm{d}s \\ &\quad + \frac{\mu t}{(1-\mu)} \int_0^1 G_1(\eta,s) \phi_q \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau))\, d\tau \right) \mathrm{d}s \\ &\quad + \frac{\lambda t}{(1-\mu)} + B_0 \left(\phi_q \left(\int_0^1 H(0,\tau) a(\tau) f(u(\tau))\, d\tau \right) \right) \\ &\leq S \left[\left(\int_0^1 G(1,s) \mathrm{d}s + \frac{\mu}{(1-\mu)} \int_0^1 G_1(\eta,s) \mathrm{d}s + A \right) \right. \\ &\quad \left. \times \phi_q \left(\int_0^1 H(0,\tau) a(\tau) d\tau \right) \right] + \frac{\lambda}{(1-\mu)} \\ &\leq S\lambda_1 + \frac{R_2}{2} \leq \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \| u \|, \end{aligned}$$

$$\begin{aligned} |(T'_\lambda u)(t)| &= \int_0^1 G'_t(t,s) \phi_q \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau))\, d\tau \right) \mathrm{d}s \\ &\quad + \frac{\mu}{(1-\mu)} \int_0^1 G_1(\eta,s) \phi_q \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau))\, d\tau \right) \mathrm{d}s + \frac{\lambda}{(1-\mu)} \\ &\leq S \left[\frac{\alpha-1}{\alpha-2} \left(\int_0^1 G(1,s) \mathrm{d}s + \frac{\mu}{(1-\mu)} \int_0^1 G_1(\eta,s) \mathrm{d}s \right) \right. \\ &\quad \left. \times \phi_q \left(\int_0^1 H(0,\tau) a(\tau) d\tau \right) \right] + \frac{\lambda}{(1-\mu)} \\ &\leq S\lambda_2 + \frac{R_2}{2} \leq \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \| u \|, \end{aligned}$$

$$\begin{aligned}
|(T''_\lambda u)(t)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\le S\left[\frac{A}{B\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \\
&= S\lambda_3 \le R_2 = \| u \|,
\end{aligned}$$

$$\begin{aligned}
|(T^{(3)}_\lambda u)(t)| &= \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\le S\left[\frac{A}{B}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] \\
&= S\lambda_4 \le R_2 = \| u \| .
\end{aligned}$$

So

$$\| T_\lambda u \| \le \| u \| .$$

Case (2) : Case where f is unbounded, since $f_\infty = 0$, there exists $R_0 > 0$ such that

$$f(r) \le \min\left\{\phi_p\left(\frac{r\lambda_1}{2}\right), \phi_p\left(\frac{r\lambda_2}{2}\right), \phi_p\left(r\lambda_3\right), \phi_p\left(r\lambda_4\right)\right\}, \quad \text{for } r \in [R_0, \infty), \tag{2.3.7}$$

Let

$$R_2 \ge \max\left\{2R_1, R_0, \frac{2\lambda}{1-\mu}\right\}$$

and be such that

$$f(r) \le f(R_2), \quad \text{for } r \in [0, R_2]$$

$$\begin{aligned}
|(T_\lambda u)(t)| &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\
&\le \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(R_2\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(R_2\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(R_2\right)d\tau\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda_1}{2}\left[\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s + A\right)\right.\\
&\qquad\left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]R_2 + \frac{R_2}{2}\\
&= \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \parallel u \parallel,
\end{aligned}$$

$$\begin{aligned}
|(T'_\lambda u)(t)| &= \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\lambda}{(1-\mu)}\\
&\leq \int_0^1 G'_t(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(R_2\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(R_2\right)d\tau\right)\mathrm{d}s\\
&\quad + \frac{\lambda}{(1-\mu)}\\
&\leq \frac{\lambda_2}{2}\left[\frac{\alpha-1}{\alpha-2}\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s\right)\right.\\
&\qquad\left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]R_2 + \frac{\lambda}{(1-\mu)}\\
&= \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \parallel u \parallel,
\end{aligned}$$

$$\begin{aligned}
|(T''_\lambda u)(t)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s\\
&\leq \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(R_2\right)d\tau\right)\mathrm{d}s\\
&\leq \lambda_3\left[\frac{A}{B\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]R_2\\
&= R_2 = \parallel u \parallel,
\end{aligned}$$

$$
\begin{aligned}
|(T_\lambda^{(3)}u)(t)| &= \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\le \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(R_2\right)d\tau\right) \\
&\le \lambda_4\left[\frac{A}{B}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right]R_2 \\
&= R_2 = \parallel u \parallel .
\end{aligned}
$$

Thus,

$$\parallel T_\lambda u \parallel \le \parallel u \parallel .$$

Therefore, in either case we may put $\Omega_2 = \{u \in K : \parallel u \parallel < R_2\}$, then

$$\parallel T_\lambda u \parallel \le \parallel u \parallel, \quad \text{for} \;\; u \in K \cap \partial\Omega_2. \tag{2.3.8}$$

So, it follows from (2.3.7), (2.3.8) and the second part of Theorem 1.5.11 that T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, then u^* is a positive solution of BVP (2.1.1)–(2.1.3).

□

2.4 Triple Solutions

To show the existence of multiple solutions we will use the Leggett-Williams fixed point theorem 1.5.15. To this end, we define the following subsets of a cone K

$$K_c = \{u \in K : \parallel u \parallel < c\} \quad , \; K(\varphi, b, d) = \{u \in K : b \le \varphi(u), \parallel u \parallel \le d\}$$

Theorem 2.4.1. *Suppose that there exist a, b, c with $0 < a < \rho^{\alpha-1}Mb < b \le c$ such that*

(C1) $f(u(t)) < \min\left\{\phi_p\left(\frac{a\lambda_1}{2}\right), \phi_p\left(\frac{a\lambda_2}{2}\right), \phi_p\left(a\lambda_3\right), \phi_p\left(a\lambda_4\right)\right\}, \qquad (t,u) \in [0,1] \times [0,a],$

(C2) $f(u(t)) \ge \phi_p\left(\rho^{\alpha-1}Mb\lambda_5\right), \qquad (t,u) \in [\rho,1] \times [\rho^{\alpha-1}Mb, b]$

(C3) $f(u(t)) \le \min\left\{\phi_p\left(\frac{c\lambda_1}{2}\right), \phi_p\left(\frac{c\lambda_2}{2}\right), \phi_p\left(c\lambda_3\right), \phi_p\left(c\lambda_4\right)\right\}, \qquad (t,u) \in [0,1] \times [0,c]$

(C4) $0 < \lambda \leqslant \frac{(1-\mu)c}{2}$.

Then (2.1.1)–(2.1.3) has at least three positive concave solutions u_1, u_2 and u_3 for λ small enough satisfying

$$\parallel u_1 \parallel < a \;,\; \rho^{\alpha-1}Mb < \varphi(u_2) \text{ and } \parallel u_3 \parallel > a \text{ with } \varphi(u_3) < \rho^{\alpha-1}bM,$$

and has no positive concave solution for λ large enough.

Proof. We prove that BVP (2.1.1)–(2.1.3) has at least three positive concave solution for sufficiently small $\lambda > 0$. By Lemma 2.2.6, $T_\lambda : K \to K$ is completely continuous.
Let

$$\varphi(u) = \min_{\rho \le t \le 1} u(t)$$

it is obvious that it is a nonnegative continuous concave functional on K with $\varphi(u) \le \| u \|$, for $u \in \overline{K_c}$.
Now we will show that the conditions of Theorem 1.5.15 are satisfied.
Suppose that $u \in \overline{K_c}$, witch implies, $\| u \| \le c$. For $t \in [0,1]$ by (2.2.21), Lemma 2.2.4, Lemma2.2.6, Remark 2.2.5, (C3) and (C4), we have

$$\begin{aligned}
(T_\lambda u)(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda t}{(1-\mu)} + B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right) \\
&\leqslant \int_0^1 G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\frac{c\lambda_1}{2}\right)^{p-1}d\tau\right)\mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\frac{c\lambda_1}{2}\right)^{p-1}d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda c(1-\mu)}{2(1-\mu)} + A\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\frac{c\lambda_1}{2}\right)^{p-1}d\tau)\right) \\
&\leqslant \frac{\lambda_1}{2}\left[\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s + A\right)\right. \\
&\quad \left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau)\right)\right]c + \frac{c}{2} \\
&\leqslant \frac{c}{2} + \frac{c}{2} = c,
\end{aligned}$$

$$\begin{aligned}
|(T_\lambda' u)(t)| &= \int_0^1 G_t'(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{(1-\mu)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^1 \frac{\alpha-1}{\alpha-2} G(1,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\frac{c\lambda_2}{2}\right)^{p-1} d\tau\right) \mathrm{d}s \\
&+ \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\frac{c\lambda_2}{2}\right)^{p-1} d\tau\right) \mathrm{d}s + \frac{\lambda c(1-\mu)}{2(1-\mu)} \\
&\leqslant \frac{\lambda_2}{2}\left[\frac{(\alpha-1)}{(\alpha-2)}\left(\int_0^1 G(1,s)\mathrm{d}s + \frac{\mu}{(1-\mu)}\int_0^1 G_1(\eta,s)\mathrm{d}s\right)\right. \\
&\qquad \left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] c + \frac{c}{2} \\
&\leqslant \frac{c}{2} + \frac{c}{2} = c,
\end{aligned}
$$

$$
\begin{aligned}
|(T_\lambda'' u)(t)| &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&\leqslant \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(c\lambda_3\right)^{p-1} d\tau\right)\mathrm{d}s \\
&\leqslant \lambda_3\left[\frac{A}{B\Gamma(\alpha-1)}\right. \\
&\qquad \left.\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] c, \\
&\leqslant c,
\end{aligned}
$$

$$
\begin{aligned}
|(T_\lambda^{(3)} u)(t)| &= \phi_q\left(\int_0^1 H(t,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right) \\
&\leqslant \phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(c\lambda_4\right)^{p-1} d\tau\right) \\
&\leqslant \lambda_4\left[\frac{A}{B}\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\right] c = c.
\end{aligned}
$$

Thus, we get

$$\| T_\lambda u(t) \| \leqslant c, \quad \text{for } \ u \in K_c.$$

This implies that $T : \overline{K_c} \to K_c$. By the same method, if $u \in \overline{K_a}$, then we can get $\| T_\lambda \| < a$ and therefore (**D2**) is satisfied. Next, we assert that $\{u \in K(\varphi, \rho^{\alpha-1}Mb, b) : \varphi(u) > \rho^{\alpha-1}Mb\} \neq \emptyset$ and $\varphi(Tu) > \rho^{\alpha-1}Mb$ for all $u \in K(\varphi, \rho^{\alpha-1}b, b)$. On the other hand, for $u \in K(\varphi, \rho^{\alpha-1}Mb, b)$, we have

$$\rho^{\alpha-1}Mb \le \varphi(u) = \min_{\rho \le t \le 1} u(t) \le \| u \| = b, \quad t \in [\rho, 1].$$

Thus, in view of (2.2.21), Lemmas 2.2.3– 2.2.4, Lemma 2.2.6, Remark 2.2.5, and (C2), we have

$$\begin{aligned}
\varphi(T_\lambda u) &= \min_{\rho\le t\le 1}\Bigg[\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&+\frac{\mu t}{(1-\mu)}\int_0^1 G_1(\eta,s)\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\mathrm{d}s \\
&+\frac{\lambda t}{(1-\mu)}+B_0\left(\phi_q\left(\int_0^1 H(0,\tau)a(\tau)f\left(u(\tau)\right)d\tau\right)\right)\Bigg] \\
&> \rho^{\alpha-1}M\int_\rho^1 G(1,s)(1-s)^{(q-1)(\beta-1)} \\
&\quad\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\rho^{\alpha-1}Mb\lambda_5\right)^{q-1}d\tau\right)\mathrm{d}s \\
&+\rho^{\alpha-1}M\frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)(1-s)^{(p-1)(\beta-1)} \\
&\quad\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)\left(\rho^{\alpha-1}Mb\lambda_5\right)^{q-1}d\tau\right)\mathrm{d}s \\
&> b\lambda_5\Bigg[\rho^{2(\alpha-1)}M^2\left(\int_\rho^1 G(1,s)(1-s)^{(q-1)(\beta-1)}\mathrm{d}s\right. \\
&\quad\left.+\frac{\mu}{(1-\mu)}\int_\rho^1 G_1(\eta,s)(1-s)^{(p-1)(\beta-1)}\mathrm{d}s\right) \\
&\times\phi_q\left(\int_0^1 H(0,\tau)a(\tau)d\tau\right)\Bigg] \\
&> b > \rho^{(\alpha-1)}Mb.
\end{aligned}$$

Thus, (**D**1) is satisfied. Finally, we assert that if $u\in K(\varphi,\rho^{\alpha-1}Mb,c)$ with $\| T_\lambda u \|> b$ then $\| T_\lambda u \|> \rho^{\alpha-1}Mb$. Indeed, suppose that $u\in K(\varphi,\rho^{\alpha-1}Mb,c)$ with $\| T_\lambda u \|> b$, then by Lemma 2.2.4 and Remark 2.2.5, we have

$$\varphi(T_\lambda u)=\min_{\rho\le t\le 1}(T_\lambda u)(t)\ge \rho^{\alpha-1}M\| T_\lambda u\|> \rho^{\alpha-1}Mb.$$

Thus, (**D**3) is satisfied. Hence, an application of Lemma 1.5.15 completes the proof. □

Corollary 2.4.2. *If there exist constants* $0<r_1<b_1<\rho^{1-\alpha}M^{-1}b_1\leqslant r_2<b_2<\rho^{1-\alpha}M^{-1}b_2\leqslant\cdots\leqslant r_n$, *for* $1\leqslant i\leqslant n-1$ *and the following conditions are satisfied:*

(I1) $f(u(t))<\min\left\{\phi_p\left(\frac{r_i\lambda_1}{2}\right),\phi_p\left(\frac{r_i\lambda_2}{2}\right),\phi_p\left(r_i\lambda_3\right),\phi_p\left(r_i\lambda_4\right)\right\};\ (t,u)\in[0,1]\times[0,r_i]$,

(I2) $f(u(t))>\phi_p\left(\rho^{\alpha-1}Mb_i\lambda_5\right);\ (t,u)\in[\rho,1]\times[\rho^{\alpha-1}Mb_i,b_i]$,

(I3) $0<\lambda\leqslant\frac{(1-\mu)r_1}{2}$.

Then (2.1.1)–(2.1.3) has at least $2n-1$ *positive concave solutions.*

Proof. If $n = 1$, by Condition (I3.4.2), (I3.4.2) and the proof of Theorem 3.4.1, we can confirm that $T_\lambda : \overline{K_c} \to K_c \subset \overline{K_c}$. From Schauder fixed-point theorem, the BVP (2.1.1)–(2.1.3) has at least one fixed point $u_1 \in \overline{K_c}$.

If $n = 2$, by Theorem 3.4.1, there exist at least three concave positive solutions u_2, u_3 and u_4. By induction method, we finish the proof. □

2.5 Some Examples

In this section we give some examples to illustrate the usefulness of our results.

Example 2.5.1. Let us consider the following fractional BVP

$$D_{1^-}^{\frac{5}{2}}\left(\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(t)\right)\right) + \frac{1}{\sqrt{t(1-t^2)}}u^{\frac{3}{2}}(t) = 0, \quad 0 < t < 1, \tag{2.5.1}$$

$$u(0) - D_{0^+}^{\alpha}u(0) = 0, u''(0) = 0, \quad u'(1) - \frac{1}{2\sqrt{2}}u'(\frac{1}{2}) = \lambda, \tag{2.5.2}$$

$$D_{0^+}^{\alpha}u(1) = 0, \left[\phi_{\frac{3}{2}}\left(D_{0^+}^{\alpha}u(0)\right)\right]' = 0, \left[\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right]'' = 0. \tag{2.5.3}$$

We can easily show that $f(u(t)) = u^{\frac{3}{2}}(t)$ satisfy:

$$f_0 = \lim_{u\to 0^+}\frac{f(u)}{\phi_{\frac{3}{2}}(u)} = \lim_{u\to 0^+} u(t) = 0, \quad f_\infty = \lim_{u\to\infty}\frac{f(u)}{\phi_{\frac{3}{2}}(u)} = \lim_{u\to 0^+} u(t) = \infty$$

obviously, for a.e $t \in [0, 1]$, we have

$$\int_0^1 H(0,\tau)a(\tau)\mathrm{d}\tau = \frac{1}{\Gamma(5/2)}\int_0^1 \frac{\frac{3}{2}\tau^{\frac{1}{2}} - \tau^{\frac{3}{2}}}{\sqrt{\tau(1-t^2)}}\mathrm{d}\tau = \frac{1}{\Gamma(5/2)}\int_0^1 \frac{\frac{3}{2} - \tau}{\sqrt{(1-\tau^2)}}\mathrm{d}\tau \simeq 1.020201672$$

So conditions hold, then we can choose $R_2 > R_1$, and for λ satisfies $0 < \lambda \leqslant \frac{e-1}{2e}R_1 < R_2$, then we can choose

$$\Omega_1 = \{u \in K :\| u \|< R_1\}, \Omega_2 = \{u \in K :\| u \|< R_2\}$$

and by Theorem 2.3.2, we can show that the BVP (2.5.1)–(2.5.3) admits at least one concave positive solution $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ for λ small enough and has no concave positive solution for λ large enough.

Example 2.5.2. Let us consider the following fractional BVP

$$D_{1^-}^{\frac{5}{2}}\left(\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(t)\right)\right) + \frac{1}{\sqrt{t(1-t^2)}}\exp(-u^2(t) = 0), \quad 0 < t < 1, \tag{2.5.4}$$

$$u(0) - D_{0^+}^{\alpha}u(0) = 0, u''(0) = 0, \quad u'(1) - \frac{1}{2\sqrt{2}}u'(\frac{1}{2}) = \lambda, \tag{2.5.5}$$

$$D_{0^+}^{\frac{5}{2}}u(1)=0,\left[\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(0)\right)\right]'=0,\left[\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(1)\right)\right]''=0. \tag{2.5.6}$$

We can easily show that $f(u(t))=u^{\frac{3}{2}}(t)$ satisfy:

$$f_0=\lim_{u\to0^+}\frac{f(u)}{\phi_{\frac{3}{2}}(u)}=\lim_{u\to0^+}\frac{1}{\exp(u^2(t))u^{\frac{1}{2}}(t)}=\infty,$$

$$f_\infty=\lim_{u\to\infty}\frac{f(u)}{\phi_{\frac{3}{2}}(u)}=\lim_{u\to0^+}\frac{1}{\exp(u^2(t))u^{\frac{1}{2}}(t)}=0$$

obviously ,for a.e $t\in[0,1]$,we have

$$\int_0^1 H(0,\tau)a(\tau)\mathrm{d}\tau=\frac{1}{\Gamma(5/2)}\int_0^1\frac{\frac{3}{2}\tau^{\frac{1}{2}}-\tau^{\frac{3}{2}}}{\sqrt{\tau(1-t^2)}}\mathrm{d}\tau=\frac{1}{\Gamma(5/2)}\int_0^1\frac{\frac{3}{2}-\tau}{\sqrt{(1-\tau^2)}}\mathrm{d}\tau\simeq1.020201672.$$

From Theorem 2.3.5, we can show that the BVP (2.5.4)–(2.5.6) has at least one concave positive solution for any $\lambda\in(0,\infty)$.

Example 2.5.3. Consider the following fractional BVP

$$D_{1^-}^{\frac{7}{3}}\left(\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(t)\right)\right)+\frac{1}{\sqrt{t(1-t^2)}}f(u(t))=0,\quad 0<t<1, \tag{2.5.7}$$

$$u(0)-\frac{2}{\sqrt{\pi}}D_{0^+}^{\frac{5}{2}}u(0)=0,u''(0)=0,\quad u'(1)-\frac{7}{10}u'(\frac{1}{2})=\lambda, \tag{2.5.8}$$

$$D_{0^+}^{\frac{5}{2}}u(1)=0,\left[\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(0)\right)\right]'=0,\left[\phi_{\frac{3}{2}}\left(D_{0^+}^{\frac{5}{2}}u(1)\right)\right]''=0, \tag{2.5.9}$$

where $q=3$, $\rho=\frac{1}{3}$,$B=\frac{2}{\pi}$, $\quad A=\frac{3}{\sqrt{\pi}}$ and

$$f(t,u)=\begin{cases}14u^2, & u\le1,\\ 13+u^{1/4}, & u>1.\end{cases}$$

Through a simple calculation, we obtain
$M=\frac{1}{3}$, $\lambda_1\simeq0.4936$, $\lambda_2\simeq0.4773$, $\lambda_3\simeq0.5410$, $\lambda_4\simeq0.3915$ and $\lambda_5\simeq1.15526$.
Choosing $a=141$, $b=18$ and $c=1296$, we get

$$f(u)<f(\tfrac{1}{14})\simeq0.07\ldots<\min\left\{\phi_p\left(\tfrac{a\lambda_1}{2}\right),\phi_p\left(\tfrac{a\lambda_2}{2}\right),\phi_p\left(a\lambda_3\right),\phi_p\left(a\lambda_4\right)\right\}$$

$$\simeq0.1136024\ldots;\ u\in[0,\frac{1}{14}],$$

$$f(u(t))>13+(2\sqrt{3})^{1/4}\simeq14.36426160\ldots>\phi_p\left(\rho^{\alpha-1}Mb\lambda_5\right)$$

$$\simeq1.155113195453685\ldots;\ u\in[\frac{18}{3^{2.5}},18],$$

$$f(u(t)) < f(1296) \simeq 19.000\ldots < \min\left\{\phi_p\left(\frac{c\lambda_1}{2}\right), \phi_p\left(\frac{c\lambda_2}{2}\right), \phi_p\left(c\lambda_3\right), \phi_p\left(c\lambda_4\right)\right\}$$

$$\simeq 21.64058872\ldots;\ u \in [0, 1296];\ 0 < \lambda \leqslant \frac{(1-\mu)a}{2} = \frac{1}{60}.$$

Then the conditions (C1– C3) are satisfied. Therefore, it follows from Theorem 2.4.1 that (2.5.7)–(2.5.9) has at least three positive solutions u_1, u_2 and u_3 such that

$$\| u_1 \| < \frac{1}{14}, \ \frac{18}{3^{5/2}} < \varphi(u_2) \text{ and } \| u_3 \| > \frac{1}{14} \text{ with } \varphi(u_3) < \frac{18}{3^{5/2}}.$$

Chapter 3

Existence of positive solutions for p-Laplacian boundary value problems of fractional differential equations

3.1 Introduction

Last decades witnessed an increased number of theoretical studies and practical applications of fractional differential equations (𝔽𝔻𝔼s) in science, engineering, biology, etc. [44, 59, 62, 64, 73, 75]. In particular, fractional p-Laplacian (𝔽pL) has been used in modeling different problems [35].

In this chapter, we studied the following fractional boundary value problem (𝔽𝔹𝕍ℙ):

$$\begin{cases} {}^{\rho_2;CK}\mathcal{D}^{\sigma_2}_{\grave{\imath}^-}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathrm{q}\right)\right)(\tau)+\hbar(\tau)\wp(\mathrm{q}(\tau))=0, \qquad \grave{a}<\tau<\grave{\imath}, \\ \mathrm{q}(\grave{a})-F_\circ\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathrm{q}(\grave{a})\right)=0, \\ \delta^2_{\rho_1}\mathrm{q}(\grave{a})=0, \\ \delta^1_{\rho_1}\mathrm{q}(\grave{\imath})=\mu\delta^1_{\rho_1}\mathrm{q}(\eta)+\lambda, \\ {}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathrm{q}(\grave{\imath})=-\delta^1_{\rho_2}\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathrm{q}\right)\right](\grave{a})=\delta^2_{\rho_2}\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathrm{q}\right)\right](\grave{\imath})=0, \end{cases} \tag{3.1.1}$$

where ${}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}$ and ${}^{\rho_2;CK}\mathcal{D}^{\sigma_2}_{\grave{\imath}^-}$, $(\rho_1, \rho_2 \in \mathbb{R}\setminus\{1\})$ are the right and left sided Caputo-Katugampola fractional derivatives (CK𝔽𝔻), $2<\sigma_1,\sigma_2\leqslant 3$, ϕ_p is the pL operator, i.e., $\phi_p(\xi)=|\xi|^{p-2}\xi$, $p>1$,

$$\delta^k_\rho=\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k,$$

$F_\circ$ is a continuous even function, $\wp$, $\hbar$ are continuous and positive. $\eta\in(\grave{a},\grave{\imath})$, $0\le\mu<1$, and $\lambda\ge 0$.

Using some fixed point theorems and under some additional assumptions, we prove some important results and obtain the existence of at least three solutions, this problem was studied recently in [26].

Su *et al.* studied the existence of positive solution for a nonlinear four-point singular $\mathbb{FBVP}$

$$\begin{cases} \left(\phi_p\left(q'\right)\right)'(\tau)+\hbar(\tau)\wp(q(\tau))=0, \qquad 0<\tau<1, \\ \eta_1\phi_p(q(0))-\eta_2\phi_p\left(q'(\xi)\right)=0, \\ \eta_3\phi_p(q(1))+\eta_4\phi_p\left(q'(\lambda)\right)=0, \end{cases} \tag{3.1.2}$$

by using the fixed point index theory, where $\eta_1,\eta_3>0$, $\eta_2,\eta_4\geq 0$, $0<\xi<\lambda<1$ and $\hbar:(0,1)\to[0,\infty)$ [72]. Also, he applied the theory to study the existence of positive solutions for the nonlinear third-order two-point singular $\mathbb{BVP}$

$$\begin{cases} \left(\phi_p\left(q^{(n-1)}\right)\right)'(\tau)+\hbar(\tau)\wp(q(\tau))=0, \quad 0<\tau<1, \\ q(0)=q'(0)=\cdots=q^{(n-3)}(0)=q^{(n-1)}(0)=0, \\ q(1)=\sum_{i=1}^{m-2}\eta_i q\left(\lambda_i\right), \end{cases} \tag{3.1.3}$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_{m-2}<1, \eta_i>0$ with $\sum_{i=1}^{m-2}\eta_i\lambda_i^{n-2}<1$ [71]. Chai in [25], considered the nonlinear $\mathbb{FBVP}$

$$\begin{cases} {}^{RL}D_{0^+}^{\sigma_1}\left(\phi_p\left({}^{RL}D_{0^+}^{\sigma_2}q\right)\right)(\tau)+\wp(\tau,q(\tau))=0, \quad 0<\tau<1, \\ q(0)=0, \\ q(1)+\eta\left({}^{RL}D_{0^+}^{\sigma_3}q(1)\right)=0, \\ {}^{RL}D_{0^+}^{\sigma_2}q(0)=0, \end{cases} \tag{3.1.4}$$

on a cone and obtained some results and positive solutions, where $1<\sigma_2\leq 2$, $0<\sigma_1,\sigma_3\leq 1$, $0\leq\sigma_2-\sigma_3-1$, $\eta>0$ and pL operator is defined as $\phi_p(\xi)=|\xi|^{p-2}\xi$, $p>1$. In 2018, Bai used the Guo-Krasnoselskii fixed point theorem and the Banach contraction mapping principle to prove the existence and uniqueness of positive solutions for the following $\mathbb{FBVP}$

$$\begin{cases} \left(\phi_p\left({}^{RL}D_{0^+}^{\sigma_1}q\right)\right)'(\tau)+\wp(\tau,q(\tau))=0, \qquad 0<\tau<1, \\ q(0)={}^{RL}D_{0^+}^{\sigma_1}q(0)=0, \\ D_{0^+}^{\sigma_2}q(0)=D_{0^+}^{\sigma_2}q(1)=0, \end{cases} \tag{3.1.5}$$

where $0<\sigma_2\leq 1$, $2<\sigma_1<2+\sigma_2$, ${}^{RL}D_{0^+}^{\sigma_1}$ and $D_{0^+}^{\sigma_2}$ are the Riemann–Liouville and Caputo fractional derivatives of orders σ_1,σ_2, respectively, $p>1$, and $\wp:[\tau_1,\tau_2]\times\mathbb{R}\to\mathbb{R}$ is a continuous function [13].

Using the coincidence degree theory, Tang *et al.* gave a new result on the existence of positive solutions to the $\mathbb{FBVP}$

$$\begin{cases} D_{0^+}^{\sigma_1}\left(\phi_p D_{0^+}^{\sigma_2}\mathrm{q}\right)(\tau) = \wp\left(\tau, \mathrm{q}(\tau), D_{0^+}^{\sigma_2}\mathrm{q}(\tau)\right), \\ \quad \mathrm{q}(0) = 0, \\ \quad D_{0^+}^{\sigma_2}\mathrm{q}(0) = D_{0^+}^{\sigma_2}\mathrm{q}(1), \end{cases} \tag{3.1.6}$$

where $0 < \sigma_1, \sigma_2 \leq 1$, $1 < \sigma_1 + \sigma_2 \leq 2$ and $D_{0^+}^{\sigma_1}$, $D_{0^+}^{\sigma_2}$ denote the Caputo fractional derivatives [81]. Torres studied the existence and multiplicity for a mixed-order three-point $\mathbb{BVP}$ of $\mathbb{FDE}$ evolving Caputo's differential operator and the boundary conditions with integer order derivatives

$$\begin{cases} \left(\phi_p\left(D_{0^+}^{\sigma}\mathrm{q}\right)\right)'(\tau) + \hbar(\tau)\wp(\tau, \mathrm{q}(\tau)) = 0, \qquad 0 < \tau < 1, \\ \quad D_{0^+}^{\sigma}\mathrm{q}(0) = \mathrm{q}(0) = \mathrm{q}''(0) = 0, \\ \quad \mathrm{q}'(1) = \eta\mathrm{q}'(\lambda), \end{cases} \tag{3.1.7}$$

where $\eta, \lambda \in (0,1)$, $\sigma \in (2,3]$ [82]. Base on the coincidence degree theory, Chen *et al.* gave new results about the problem

$$\begin{cases} D_{0^+}^{\sigma_1}\phi_p\left(D_{0^+}^{\sigma_2}\mathrm{q}(\tau)\right) = \wp\left(\tau, \mathrm{q}(\tau), D_{0^+}^{\sigma_2}\mathrm{q}(\tau)\right), \qquad \tau \in [0,1], \\ \quad D_{0^+}^{\sigma_2}\mathrm{q}(0) = D_{0^+}^{\sigma_2}\mathrm{q}(1) = 0, \end{cases} \tag{3.1.8}$$

where $0 < \sigma_1, \sigma_2 \leq 1$, $(1 < \sigma_1 + \sigma_2 \leq 2)$ [27].

The rest of the chapter is organized as follows. Section 3.2 presents some basic definitions, lemmas, and preliminary results. In Section 3.3, we derive some conditions on the parameter λ to obtain the existence of at least one positive solution, we derive an interval for λ which ensures the existence of ρ_1-concave positive solutions of the $\mathbb{FBVP}$. In Section 3.4, we discuss the existence of multiple positive solutions. Finally, we give some illustrative examples in Section 3.5.

The following technical hypotheses will be used latter.

(H1) $\hbar$ does not vanish identically on any closed sub-interval of $(\grave{a}, \grave{\imath})$.

(H2) $F_\circ$ is even and continuous on $\mathbb{R}$, and $\exists\, A, B > 0$:

$$Bv^{p-1} \leqslant F_\circ(v) \leqslant Av^{p-1}, \qquad (v \in \mathbb{R}^+).$$

3.2 Main results

We present some important lemmas which assist to prove our main results. Consider the linear generalized $\mathbb{BVP}$ associated to (3.1.1)

$$\begin{cases} {}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\tau)+\mathrm{w}(\tau)=0, \qquad \grave{a}<\tau<\grave{\imath}, \\ \mathrm{q}(\grave{a})-F_\circ\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\grave{a})\right)=0, \\ \delta_{\rho_1}^2\mathrm{q}(\grave{a})=0, \\ \delta_{\rho_1}^1\mathrm{q}(\grave{\imath})-\mu\delta_{\rho_1}^1\mathrm{q}(\eta)=\lambda. \end{cases} \tag{3.2.1}$$

Lemma 3.2.1. *For* $\mathrm{w}\in C(J)$*, the integral solution of* (3.2.1) *is given by*

$$\mathrm{q}(\tau)=\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\mathrm{w}(\xi)\,d\xi+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\mathrm{w}(\xi)\,d\xi$$
$$+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right)+F_\circ\left(\mathrm{w}(\grave{a})\right), \tag{3.2.2}$$

for $\tau,\xi\in J$ *where*

$$\mathcal{G}_1(\tau,\xi)=\begin{cases}\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1} \\ \qquad -\frac{1}{\Gamma(\sigma_1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\xi^{\rho_1-1}, \qquad \xi\le\tau, \\ \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}, \quad \tau\le\xi,\end{cases} \tag{3.2.3}$$

and

$$\mathcal{G}_2(\tau,\xi)=\begin{cases}\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1} \\ \qquad -\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}, \qquad \xi\le\tau, \\ \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}, \quad \tau\le\xi.\end{cases} \tag{3.2.4}$$

Proof. By applying (1.3.19), the equation (3.2.1) becomes

$$\mathrm{q}(\tau)=-l_0-l_1\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)-l_2\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^2$$
$$-\frac{\rho_1^{1-\sigma_1}}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}(\grave{a}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}\xi^{\rho_1-1}\mathrm{w}(\xi)\,d\xi,$$

for some arbitrary constants $l_0, l_1, l_2 \in \mathbb{R}$. From the boundary conditions of (3.2.1) we get

$$\begin{aligned}
q(\tau) &= F_\circ\left(w(\grave{a})\right) + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1} w(\xi)\,d\xi \\
&\quad + \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\frac{1}{(1-\mu)\Gamma(\sigma_1 - 1)}\left[\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1} w(\xi)\,d\xi\right. \\
&\quad \left. - \mu\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1} w(\xi)\,d\xi\right], \\
&= F_\circ\left(w(\grave{a})\right) + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1} w(\xi)\,d\xi \\
&\quad + \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\left[\frac{1}{\Gamma(\sigma_1 - 1)}\right. \\
&\quad \left. + \frac{\mu}{\Gamma(\sigma_1 - 1)(1-\mu)}\right]\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1} w(\xi)\,d\xi \\
&\quad - \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\frac{\mu}{(1-\mu)\Gamma(\sigma_1 - 1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1} w(\xi)\,d\xi.
\end{aligned}$$

Splitting the second integral in two parts permits us to write

$$\begin{aligned}
q(\tau) = \frac{1}{\Gamma(\sigma_1)}&\left[(\sigma_1 - 1)\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\int_{\grave{a}}^{\tau}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\sigma_1 - 1} w(\xi)\,d\xi\right. \\
&- \int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1} w(\xi)\,d\xi \\
&\left. +(\sigma_1 - 1)\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\int_{\tau}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1} w(\xi)\,d\xi\right] \\
&+ \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right)\frac{1}{\Gamma(\sigma_1 - 1)}\left[\mu\left(\int_{\grave{a}}^{\tau}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1} w(\xi)\,d\xi\right.\right.
\end{aligned}$$

$$+\int_{\tau}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}\mathrm{w}(\xi)\,d\xi\Bigg)$$

$$-\mu\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\rho-2}\xi^{\rho_1-1}\mathrm{w}(\xi)\,d\xi\Bigg]+F_{\circ}\left(\mathrm{w}(\grave{a})\right)+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right),$$

$$=\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\mathrm{w}(\xi)+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{(1-\mu)\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\mathrm{w}(\xi)\,d\xi$$

$$+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{(1-\mu)\rho_1}\right)+F_{\circ}\left(\mathrm{w}(\grave{a})\right).$$

The converse follows by direct computation. The proof is completed. □

Now, consider the generalized $\mathbb{BVP}$ associated to (3.1.1)

$$\begin{cases} {}^{\rho_2;CK}\mathcal{D}_{\grave{\imath}^+}^{\sigma_2}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\tau)\right)\right)=\mathrm{w}(\tau), \qquad \grave{a}<\tau<\grave{\imath},\\ \mathrm{q}(\grave{a})-F_{\circ}\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\grave{a})\right)=0,\\ \delta_{\rho_1}^2\mathrm{q}(\grave{a})=0,\\ \delta_{\rho_1}^1\mathrm{q}(\grave{\imath})-\mu\delta_{\rho_1}^1\mathrm{q}(\eta)=\lambda,\\ {}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}(\grave{\imath})=\delta_{\rho_2}^1\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}\right)\right](\grave{a})=\delta_{\rho_2}^2\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1}\mathrm{q}\right)\right](\grave{\imath})=0. \end{cases} \tag{3.2.5}$$

Lemma 3.2.2. *For* $\mathrm{w}(\tau)\in C^+(J)$, *the* $\mathbb{BVP}$ (3.2.5) *has a unique solution*

$$\mathrm{q}(\tau)=\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi$$

$$+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi$$

$$+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_{\circ}\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,d\xi\right)\right), \tag{3.2.6}$$

where

$$\mathcal{H}(\tau,\xi)=\begin{cases}\frac{1}{\Gamma(\sigma_2-1)}\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}\\ \quad-\frac{1}{\Gamma(\sigma_2)}\left(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^{\sigma_2-1}\xi^{\rho_2-1}, & \tau\le\xi,\\ \frac{1}{\Gamma(\sigma_2-1)}\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}, & \xi\le\tau,\end{cases} \tag{3.2.7}$$

$\mathcal{G}_1(\tau,\xi)$, $\mathcal{G}_2(\tau,\xi)$ *are defined in the previous Lemma 3.2.1 and* $\bar{p}=\frac{p}{p-1}$.

Proof. From Lemma 1.3.22, the equation (3.2.5) is equivalent to the equation

$$\phi_{\bar{p}}\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau)\right) = -l_0 - l_1\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right) - l_2\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^2 + {}^{\rho_2;\grave{\imath}^-}\mathbb{I}^{\sigma_2}_{K}w(\tau),$$

for some constants $l_0, l_1, l_2 \in \mathbb{R}$. Using the second boundary condition, we get

$$\begin{aligned}\phi_{\bar{p}}\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau)\right) &= {}^{\rho_2;\grave{\imath}^-}\mathbb{I}^{\sigma_2}_{K}w(\tau)\\ &\quad - \left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)\frac{1}{\Gamma(\sigma_2-1)}\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}w(\xi)\,d\xi\\ &= -\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\tau,\xi)w(\xi)\,d\xi.\end{aligned}$$

Consequently,

$${}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau) = -\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\tau,\xi)w(\xi)\,d\xi\right).$$

Thus, the problem (3.2.5) can be written as

$$\begin{cases} {}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau) + \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\tau,\xi)w(\xi)\,d\xi\right) = 0, \qquad \tau\in(\grave{a},\grave{\imath}),\\ q(\grave{a}) - F_\circ\left({}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\grave{a})\right) = 0,\\ \delta^2_{\rho_1}q(\grave{a}) = 0,\\ \delta^1_{\rho_1}q(\grave{\imath}) = \mu\delta^1_{\rho_1}q(\eta) + \lambda, \end{cases} \tag{3.2.8}$$

which, according to Lemma 3.2.1, has a unique solution of the form (3.2.6). □

Lemma 3.2.3. *The functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$, Eqs. (3.2.3), (3.2.4), and (3.2.7), satisfy the following.*

(i) $\mathcal{G}_1(\tau,\xi)$*,* $\mathcal{G}_2(\tau,\xi)$ *and* $\mathcal{H}(\tau,\xi)$ *are continuous on* $[\grave{a},\grave{\imath}]\times[\grave{a},\grave{\imath}]$*.*

(ii) $\forall(\tau,\xi)\in[\grave{a},\grave{\imath}]\times[\grave{a},\grave{\imath}]$,

$$\begin{aligned}\mathcal{G}_1(\tau,\xi) &\leqslant \left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\frac{\grave{\imath}^{\rho_1-1}}{\Gamma(\sigma_1-1)}\quad\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\,d\xi\\ &= \left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\Gamma(\sigma_1)\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1} - \frac{1}{\Gamma(\sigma_1+1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1},\\ \mathcal{G}_2(\tau,\xi) &\leqslant \left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\frac{\grave{\imath}^{\rho_1-1}}{\Gamma(\sigma_1-1)}\quad\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\,d\xi\\ &= \frac{1}{\Gamma(\sigma_1)}\left(\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1} - \left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\right),\end{aligned}$$

$$\mathcal{H}(\tau,\xi) \leqslant \left(\frac{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-1} \frac{\grave{\imath}^{\rho_2-1}}{\Gamma(\sigma_2-1)} \int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\,\mathrm{d}\xi$$

$$= \frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\rho_2\Gamma(\sigma_2)} \left(\left(\frac{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-1} - \frac{1}{\sigma_2}\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\rho_2}\right)^{\sigma_2-1}\right).$$

(iii) $\forall(\tau,\xi) \in [\grave{a},\grave{\imath}]^2: \quad \mathcal{G}_1(\tau,\xi) \geqslant 0$, $\mathcal{G}_2(\tau,\xi) \geqslant 0$, $\mathcal{H}(\tau,\xi) \geqslant 0$;

(iv) $\forall\xi \in J$, *the function* $\tau \to \mathcal{G}_1(\tau,\xi)$ *is increasing and* $\tau \to \mathcal{H}(\tau,\xi)$ *is decreasing. In addition,* $\forall(\tau,\xi) \in (\grave{a},\grave{\imath})^2$ *we have*

$$\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} \mathcal{G}_1(\grave{\imath},\xi) \leqslant \mathcal{G}_1(\tau,\xi)$$

and

$$\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1} \mathcal{H}(\grave{a},\xi) \leqslant \mathcal{H}(\tau,\xi);$$

(v) $\forall(\tau,\xi) \in (\grave{a},\grave{\imath})^2$, *we have*

$$\frac{\tau^{\rho_1-1}\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\left[1-\left(\frac{\tau}{\grave{\imath}}\right)^{\rho_1(\sigma_1-2)}\right]\mathcal{G}_1(\grave{\imath},\xi) \leqslant \mathcal{G}'_{1\tau}(\tau,\xi) \leqslant \frac{\sigma_1-1}{\sigma_1-2}\frac{\tau^{\rho_1-1}\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\mathcal{G}_1(\grave{\imath},\xi).$$

Proof. Using the definitions of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$, (i) and (ii) are obtained straightforwardly. For property (iii), we only consider the case $\xi \leq \tau$ as the other case is straightforward. When $\xi \leq \tau$, we have

$$\begin{aligned}
\mathcal{G}_1(\tau,\xi) &\geqslant \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\grave{a}^{\rho_1-1} \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\grave{a}^{\rho_1-1} \\
&\geqslant \left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\grave{a}^{\rho_1-1}\left[\frac{1}{\Gamma(\sigma_1-1)}-\frac{1}{\Gamma(\sigma_1)}\right] \\
&\geqslant 0,
\end{aligned}$$

because $\Gamma(\sigma_1-1) \leqslant \Gamma(\sigma_1)$, for $2 < \sigma_1 \leq 3$. Similarly, we can easily prove that $\mathcal{G}_2(\tau,\xi) \geqslant 0$ and $\mathcal{H}(\tau,\xi) \geqslant 0$, $\forall(\tau,\xi) \in J^2$. Now, for property (iv), we first check that $\mathcal{G}_1(\tau,\xi)$ is nondecreasing w.r.t. $\tau \in J$.

$$\frac{\partial\mathcal{G}_1}{\partial\tau}(\tau,\xi) = \begin{cases} \frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1} \\ \qquad -\frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}, & \xi \leqslant \tau, \\ \frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\xi^{\rho_1-1}, & \tau \leqslant \xi. \end{cases} \tag{3.2.9}$$

Thus, $\mathcal{G}_1(\tau,\xi)$ is increasing with respect to $\tau \in J$ and therefore, $\mathcal{G}_1(\tau,\xi) \leqslant \mathcal{G}_1(\grave{\imath},\xi)$, for $\grave{a} \leqslant \tau,\xi \leqslant \grave{\imath}$. Furthermore, for $\tau \leqslant \xi$, we have

$$\begin{aligned}\frac{\partial \mathcal{H}(\tau,\xi)}{\partial \tau} &= -\frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}\\ &\quad+\frac{(\sigma_2-1)\tau^{\rho_2-1}}{\Gamma(\sigma_2)}\left(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1},\\ &=\frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\xi^{\rho_2-1}\left[\left(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}-\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\right],\\ &\leqslant \frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\xi^{\rho_2-1}\left[\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}-\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\right]=0,\end{aligned}$$

and for $\xi \leqslant \tau$, we have

$$\frac{\mathcal{H}(\tau,\xi)}{\partial \tau}=\frac{-\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}\leqslant 0.$$

Thus, $\mathcal{H}(\tau,\xi)$ is nonincreasing with respect to τ. Consequently $\mathcal{H}(\tau,\xi) \leqslant \mathcal{H}(\grave{a},\xi)$, $\forall \tau,\xi \in J$. On the other hand, when $\tau \geqslant \xi$,

$$\begin{aligned}\frac{\mathcal{G}_1(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)} &= \frac{(\sigma_1-1)(\tau^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}\\ &= \frac{(\sigma_1-1)(\tau^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\alpha-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\tau^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}.\end{aligned}$$

As

$$\left(\frac{\tau^{\rho}-\xi^{\rho}}{\tau^{\rho}-\grave{a}^{\rho}}\right)^{\sigma} \leq \left(\frac{\grave{\imath}^{\rho}-\xi^{\rho}}{\grave{\imath}^{\rho}-\grave{a}^{\rho}}\right)^{\sigma},$$

for $\sigma > 0$, we obtain

$$\begin{aligned}\frac{\mathcal{G}_1(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)} &\geqslant \frac{(\sigma_1-1)(\tau^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}\left(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}\\ &\geqslant \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}\\ &\quad\times\frac{(\sigma_1-1)\left(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}\right)^{\sigma_1-1}(\tau^{\rho_1}-\grave{a}^{\rho_1})^{2-\sigma_1}(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}\end{aligned}$$

that is

$$\geqslant \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}$$

$$\times \frac{\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\tau^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}(\sigma_1-1)\,(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})\,(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}$$

$$\geqslant \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}.$$

For $\tau \leqslant \xi$, we have

$$\frac{\mathcal{G}_1(\tau,\xi)}{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}} = \frac{\rho_1^{\sigma_1-1}\,\xi^{\rho_1-1}}{\Gamma(\sigma_1-1)}\,(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}\,\frac{1}{(t^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-2}},$$

which is a nonincreasing function as $\sigma_1 \geq 0$. Consequently,

$$\frac{\mathcal{G}_1(\tau,\xi)}{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}} \geq \frac{\mathcal{G}(\grave{\imath},\xi)}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}},$$

which implies

$$\mathcal{G}_1(\tau,\xi) \geqslant \left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\mathcal{G}_1(\grave{\imath},\xi).$$

Using similar techniques, one can prove that

$$\mathcal{H}(\tau,\xi) \geqslant \left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma-1}\mathcal{H}(\grave{a},\xi),$$

for $\grave{a} \leqslant \xi, \tau < \grave{\imath}$. Therefore (iv) of Lemma 3.2.3 holds. Finally, for property (v), we can consider two cases. Nevertheless, we prove the results for the case $\xi \leq \tau$ only. The simpler case $\grave{a} \leq \tau \leq \xi < \grave{\imath}$ can be treated similar arguments. When $\xi \leq \tau$ we have

$$\frac{\mathcal{G}_1{}'_\tau(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)}\,\frac{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1(\sigma_1-1)} = \frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\sigma_1-1)\,(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}}.$$

Consequently,

$$\begin{aligned}\frac{\mathcal{G}_1{}'_\tau(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)}\,\frac{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1(\sigma_1-1)} &\leq \frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\sigma_1-1)\,(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}}\\ &\leq \frac{1}{(\sigma_1-1)-\frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}}\\ &\leq \frac{1}{(\sigma_1-2)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\mathcal{G}_1{}'_\tau(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)}\frac{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1} &= \frac{(\sigma_1-1)\left[(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}\right]}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})}}\\ &\geq 1-\frac{(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}\\ &\geq 1-\left(\frac{\tau}{\grave{\imath}}\right)^{\rho_1(\sigma_1-2)}\left(\frac{1-\left(\frac{\xi}{\tau}\right)^{\rho_1}}{1-\left(\frac{\xi}{\grave{\imath}}\right)^{\rho_1}}\right)^{\sigma_1-2}\\ &\geq 1-\left(\frac{\tau}{\grave{\imath}}\right)^{\rho_1(\sigma_1-2)}.\end{aligned}$$

Thus, the proof is completed. □

Now, consider the Banach space $\mathbb{E}=C^3_{\rho_1}(J)$. Suppose ${}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau)$ is continuous on J for all $q\in\mathbb{E}$, then from the Definition 1.3.22 and Lemma 1.3.27 we can define the norm on $\mathbb{E}$ as follows:

$$\|q\|=\begin{cases}\max\left\{\check{M}_1,\max_{\tau\in J}\left|{}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}q(\tau)\right|\right\}, & 2<\sigma_1<3,\\ \max\left\{\check{M}_1,\max_{\tau\in J}\left|\delta^3_{\rho_1}q(\tau)\right|\right\}, & \sigma_1=3,\end{cases}$$

in which

$$\check{M}_1=\max\left\{\max_{\tau\in J}|q(\tau)|,\max_{\tau\in J}|\delta^1_{\rho_1}q(\tau)|,\max_{\tau\in J}|\delta^2_{\rho_1}q(\tau)|\right\},$$

and the cone

$$K=\left\{q\in\mathbb{E}\,:\,q\text{ is nonnegative, increasing and }\rho_1-\text{concave}\right\}.$$

Lemma 3.2.4. *Assume (H2) and let q be the unique solution of the $\mathbb{BVP}$ (3.2.5) associated to a given $w(\tau)\in C^+(J)$. Then $q\in K$, and the following inequalities hold for $\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]\subset(\grave{a},\grave{\imath})$.*

$$\max_{\tau\in J}|q(\tau)|\leqslant\left(\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\left(\frac{\grave{\imath}^{\rho_2}-\grave{\imath}_\circ^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_1-1}\right)^{-1}q(\tau),\tag{3.2.10}$$

$$\max_{\tau\in J}\left|\delta^1_{\rho_1}q(\tau)\right|\leqslant\frac{\sigma_1-1}{\sigma_1-2}\frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\max_{\tau\in J}|q(\tau)|,\tag{3.2.11}$$

$$\begin{aligned}\max_{\tau\in J}|\delta^2_{\rho_1}q(\tau)|\leqslant&\left(\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,d\xi\right)^{-1}\\ &\times\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\max_{\tau\in J}|q(\tau)|,\end{aligned}\tag{3.2.12}$$

$$\max_{\tau\in J}\left|{}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{CK}q(\tau)\right| \leqslant \left(\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,d\xi\right)^{-1} \times \max_{\tau\in J}|q(\tau)|, \qquad \forall\sigma_1\in(2,3], \tag{3.2.13}$$

$$\min_{\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]} q(\tau) \geqslant \left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} \breve{M}_2\|q\|, \tag{3.2.14}$$

where

$$Z(\tau)=\phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right),$$

and

$$\breve{M}_2=\min\left\{1,\frac{\sigma_1-2}{\sigma_1-1}\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right), \min\left\{\Gamma(\sigma_1-1)\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{2-\sigma_1},1\right\}\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,d\xi\right\}. \tag{3.2.15}$$

Proof. From Lemma (3.2.2), we have

$$\begin{aligned} q(\tau) &= \int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi \\ &\quad+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi \\ &\quad+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\right). \end{aligned}$$

(1) The functions $\mathcal{G}_1,\mathcal{G}_2$ and $\mathcal{H}$ are non-negative (Lemma (3.2.3)-(*iii*)). In addition, $F_\circ(v)$ is non-negative for $v\geq 0$ (thanks to (H2)). Thus, q is also non-negative. Furthermore, as $\mathcal{G}_1$ is increasing w.r.t. τ (Lemma (3.2.3)-(*iv*), so it is the function q. To Prove that q is ρ_1-concave, we need to show that $\delta^1_{\rho_1}q(\tau)$ is decreasing on J (Remark 1.5.13), which can be obtained form the negativity of the derivative

$$\left(\delta^1_{\rho_1}q(\tau)\right)'=-\frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-2)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-3}\xi^{\rho_1-1}\,\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi \\ \leq 0.$$

(2) As q is non-negative and increasing, we have

$$\max_{\tau\in J}|q(\tau)| = q(\grave{\imath}) = \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$+\lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\right).$$

For $\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]$, using (iv) of Lemma 3.2.3 and the fact that

$$\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)<1,$$

we get

$$q(\tau)\geqslant \int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$+\mu\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$+\lambda\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)$$

$$+\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\right).$$

Consequently,

$$q(\tau)\geqslant\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\max_{t\in J}|q(\tau)|,$$

and thus, (3.2.10) hold.

(3) We have

$$\delta^1_{\rho_1}q(\tau)=\tau^{1-\rho_1}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1{}'_\tau(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$+\frac{\mu}{(1-\mu)}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)w(s)\,ds\right)d\xi+\frac{\lambda}{(1-\mu)}.$$

From Lemma 3.2.3 ((*iii*) and (*v*)), we can deduce that $\delta^1_{\rho_1} q(\tau) \geq 0$ and

$$\delta^1_{\rho_1} q(\tau) \leqslant \int_{\grave{a}}^{\grave{\imath}} \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi$$

$$+ \frac{\mu}{(1-\mu)} \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi + \frac{\lambda}{(1-\mu)}$$

$$\leqslant \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \Bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi$$

$$+ \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ss \right) d\xi$$

$$+ \lambda \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \Bigg]$$

$$\leqslant \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \Bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi$$

$$+ \lambda \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right)$$

$$+ \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi$$

$$+ F_\circ \left(\phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) w(\xi)\, d\xi \right) \right) \Bigg]$$

$$\leqslant \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} q(\grave{\imath}).$$

Thus, we obtain (3.2.11).

(4) A straightforward calculus gives

$$\delta^2_{\rho_1} q(\tau) = -\frac{1}{\Gamma(\sigma_1 - 2)} \int_{\grave{a}}^{\tau} \left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 3} \xi^{\rho_1 - 1} \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) w(s)\, ds \right) d\xi.$$

Then, we get

$$\left| \delta^2_{\rho_1} q(\tau) \right| \leqslant \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) w(\xi)\, d\xi \right) \frac{1}{\Gamma(\sigma_1 - 2)} \int_{\grave{a}}^{\tau} \left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 3} \xi^{\rho_1 - 1}\, d\xi,$$

$$\leqslant \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) w(\xi)\, d\xi \right) \frac{1}{\Gamma(\sigma_1 - 1)} \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2}.$$

Thus,

$$\max_{\tau\in J}\left|\delta^2_{\rho_1}q(\tau)\right| \leqslant \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}.$$

By multiplying both sides of the previous inequality by

$$\phi_{\bar{p}}\left(\left(\frac{\grave{l}^{\rho_2}-\xi^{\rho_2}}{\grave{l}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right),$$

we get

$$\phi_{\bar{p}}\left(\left(\frac{\grave{l}^{\rho_2}-\xi^{\rho_2}}{\grave{l}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\max_{\tau\in J}\left|\delta^2_{\rho_1}q(\tau)\right|$$

$$\leqslant \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\left(\frac{\grave{l}^{\rho_2}-\xi^{\rho_2}}{\grave{l}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2},$$

using Lemma (3.2.3)-(iv), we get

$$\phi_{\bar{p}}\left(\left(\frac{\grave{l}^{\rho_2}-\xi^{\rho_2}}{\grave{l}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\max_{\tau\in J}\left|\delta^2_{\rho_1}q(\tau)\right|$$

$$\leqslant \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\tau,\xi)w(\xi)\,d\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}. \quad (3.2.16)$$

We multiplying both sides by $\mathcal{G}_1(\tau,\xi)$ and integrate over J w.r.t. ξ, we get

$$\max_{\tau\in J}\left|\delta^2_{\rho_1}q(\tau)\right|\int_{\grave{a}}^{\grave{l}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\left(\frac{\grave{l}^{\rho_2}-\xi^{\rho_2}}{\grave{l}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)d\xi$$

$$\leqslant \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\int_{\grave{a}}^{\grave{l}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi,$$

$$\leqslant \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\left[\int_{\grave{a}}^{\grave{l}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi\right.$$

$$+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)$$

$$+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{l}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\xi,s)w(s)\,ds\right)d\xi$$

$$\left.+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{l}}\mathcal{H}(\grave{a},\xi)w(\xi)\,d\xi\right)\right)\right] = \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{l}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}q(\tau)$$

$$\leqslant \frac{1}{\Gamma(\sigma_1 - 1)} \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2} \max_{\tau \in J} |q(\tau)|.$$

Furthermore, for $\tau \in [\grave{a}_\circ, \grave{\imath}_\circ]$

$$\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau, \xi) \phi_{\bar{p}} \left(\left(\frac{\grave{\imath}^{\rho_2} - \xi^{\rho_2}}{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}} \right)^{\sigma_2 - 1} \right) d\xi \geqslant \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \right)^{\alpha - 1} Z(\grave{\imath}_\circ) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \, d\xi,$$

and

$$\max_{\tau \in J} |\delta_{\rho_1}^2 q(\tau)| \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau, \xi) \phi_{\bar{p}} \left(\left(\frac{\grave{\imath}^{\rho_2} - \xi^{\rho_2}}{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}} \right)^{\sigma_2 - 1} \right) d\xi$$

$$\geqslant \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \right)^{\sigma_1 - 1} Z(\grave{\imath}_\circ) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \, d\xi \max_{\tau \in J} \left| \delta_{\rho_1}^2 q(\tau) \right|,$$

Thus, we obtain (3.2.12).

(5) From the first equation in (3.2.8), one can see that

$$ {}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1} q(\tau) = -\phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau, \xi) w(\xi) \, d\xi \right), \qquad (2 < \sigma_1 \leqslant 3). \tag{3.2.17}$$

Thus,

$$\max_{\tau \in J} \left| {}^{\rho_1;CK}\mathcal{D}_{\grave{a}^+}^{\sigma_1} q(\tau) \right| \leqslant \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) w(\xi) \, d\xi \right), \qquad (2 < \sigma_1 \leqslant 3).$$

As in (2), we can deduce (3.2.13).

(6) Equation (3.2.14) is a direct consequence of the previous results.

□

Then, for a given $[\grave{a}_\circ, \grave{\imath}_\circ] \subset (\grave{a}, \grave{\imath})$, we define the cone

$$Y = \left\{ q \in K : \min_{\tau \in [\grave{a}_\circ, \grave{\imath}_\circ]} q(\tau) \geqslant \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \right)^{\sigma_1 - 1} \breve{M}_2 \|q\| \right\},$$

and the integral operator $\mathcal{N}_\lambda : Y \to \mathbb{E}$ defined for $\tau \in [\grave{a}_\circ, \grave{\imath}_\circ]$ by

$$\begin{aligned} \mathcal{N}_\lambda(q)(\tau) = & \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp(q(s)) \, ds \right) d\xi \\ & + \mu \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp(q(s)) \, ds \right) d\xi \\ & + \lambda \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) + F_\circ \left(\phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) \wp(q(\xi)) \, d\xi \right) \right). \end{aligned} \tag{3.2.18}$$

When (H2) holds, we have $\mathcal{N}_\lambda(Y) \subset Y$ and fixed points of $\mathcal{N}_\lambda$ are solutions of (3.1.1). To use some fixed point Theorems, we need to show that $\mathcal{N}_\lambda$ is completely continuous.

Lemma 3.2.5. *Assume (H2) is true. Then, $\mathcal{N}_\lambda : \mathrm{Y} \to \mathrm{Y}$ is continuous and compact.*

Proof. The continuity of $\mathcal{N}_\lambda$ is a consequence of the continuity and positiveness of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}$, $\hbar$ and $\wp$. To prove that $\mathcal{N}_\lambda$ is compact, let us consider a bounded subset $\Omega \subset \mathrm{Y}$. Then, there exists $L > 0$ such that for any $\mathrm{q} \in \Omega$ we have $|\wp(\mathrm{q}(\tau))| \leqslant L$. For any $\mathrm{q} \in \Omega$, as $\mathcal{N}_\mathrm{q}$ is positive and $\mathcal{G}_1$ is increasing w.r.t. τ, we have

$$\max_{\tau \in J} \mid \mathcal{N}_\lambda(\mathrm{q}(\tau)) \mid = \mathcal{N}_\lambda(\mathrm{q}(\grave{\imath})).$$

Consequently, using the previous inequality and the hypothesis (H2), we get

$$\begin{aligned} \max_{\tau \in J} \mid \mathcal{N}_\lambda(\mathrm{q}(\tau)) \mid \leq & \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s) \hbar(s) L \, \mathrm{d}s \right) \mathrm{d}\xi \\ & + \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s) \hbar(s) L \, \mathrm{d}s \right) \mathrm{d}\xi \\ & + \lambda \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) + A \int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) L \, \mathrm{d}\xi =: \bar{L}. \end{aligned} \tag{3.2.19}$$

Then, as in Lemma 3.2.4, we obtain $\|\mathcal{N}_\lambda \mathrm{q}\| \leqslant \breve{M}_3 \, \bar{L}$, where

$$\begin{aligned} \breve{M}_3 = \max \Bigg\{ & 1, \frac{\sigma_1 - 1}{\sigma_1 - 2} \left(\frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \right), \\ & \max \left\{ \frac{1}{\Gamma(\sigma_1 - 1)} \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2}, 1 \right\} \left[\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}} \right)^{\sigma_1 - 1} Z(\grave{\imath}_\circ) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \, \mathrm{d}\xi \right]^{-1} \Bigg\}. \end{aligned}$$

Hence, $\mathcal{N}_\lambda(\Omega)$ is uniformly bounded. Furthermore, by using Lemmas (3.2.2)–(3.2.3) and Lebesgue dominated convergence theorem, we deduce the equicontinuouty of $\mathcal{N}_\lambda(\Omega)$. Therefore, $\mathcal{N}_\lambda$ is completely continuous by Arzelà-Ascoli Theorem. □

3.3 Existence of solutions in a cone

In this section, we derive an interval for λ which ensures the existence of ρ_1-concave positive solutions of the $\mathbb{FBVP}$.

Theorem 3.3.1. *Assume that all conditions (H1) and (H2) hold, and that there exist $0 < \ell_1 < \ell_2$, and*

$$m_1 \in \left(0, \breve{M}_4\right), \qquad m_2 \in \left(\Lambda_6, \infty\right), \tag{3.3.1}$$

here $\breve{M}_4 = \min\left\{\frac{\Lambda_1}{4}, \frac{\Lambda_2}{4}, \frac{\Lambda_3}{2}, \Lambda_4, \Lambda_5\right\}$ such that

(H3) $\forall \mathrm{q} \in [0, \ell_1]$ we have $\wp(\mathrm{q}) \leqslant \min\left\{\phi_p\left(m_1 \ell_1\right), m_1 \ell_1\right\}$;

(H4) $\forall \mathrm{q} \in [\gamma\ell_2, \ell_2]$ *we have* $\wp(\mathrm{q}) \geqslant \phi_p\,(m_2\ell_2)$.

Then, the $\mathbb{FBVP}$ (3.1.1) *has at least one* ρ_1*-concave positive solution for* $\lambda > 0$ *small enough, where*

$$\gamma := \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \breve{M}_2, \tag{3.3.2}$$

and

$$\Lambda_1 := \left[A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right]^{-1},$$

$$\Lambda_2 := \left[\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \rho_1\mu}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi\right) \times \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1},$$

$$\Lambda_3 := \left[\frac{\sigma_1 - 1}{\sigma_1 - 2}\frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi + \frac{\mu}{1-\mu}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1},$$

$$\Lambda_4 := \left[\frac{1}{\Gamma(\sigma_1 - 1)}\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1},$$

$$\Lambda_5 := \left[\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1},$$

$$\Lambda_6 := \left[\gamma\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} Z(\grave{\imath}_\circ)\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}. \tag{3.3.3}$$

Proof. Let $\Omega_{\ell_1} = \{\mathrm{q} \in K : \|\mathrm{q}\| \leq \ell_1\}$ and λ satisfying

$$0 < \lambda \leqslant \frac{1}{2}(1-\mu)\ell_1\ \min\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right\}, \tag{3.3.4}$$

so that

$$2\lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \leq \ell_1,$$

and $2\lambda \leq \ell_1(1-\mu)$. Let $\mathrm{q} \in K \cap \partial\Omega_{\ell_1}$, i.e., $\|\mathrm{q}\| = \ell_1$. From (H2) and (H3), we get

$$F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi\right)\right) \leq A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi$$
$$\leq m_1\ell_1 A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi,$$

$$\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi\right) \leq m_1\ell_1\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\hbar(\xi)\,\mathrm{d}\xi\tau\right).$$

But,

$$\max_{\tau\in J} \mid \mathcal{N}_\lambda(\mathrm{q}(\tau)) \mid = \mathcal{N}_\lambda(\mathrm{q}(\grave{\imath}))$$

$$= \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\right)\mathrm{d}\xi$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\ \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\right)\mathrm{d}\xi$$

$$+\lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi\right)\right).$$

Then,

$$\max_{\tau\in J} \mid \mathcal{N}_\lambda(\mathrm{q}(\tau)) \mid \leqslant \frac{\Lambda_2\ell_1}{4}\left[\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\right)\right.$$
$$\left.\times\,\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]+\frac{\ell_1}{2}+\frac{\Lambda_1\ell_1 A}{4}\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi.$$

Consequently,

$$\max_{\tau\in J} \mid \mathcal{N}_\lambda(\mathrm{q}(\tau)) \mid \leqslant \frac{\ell_1}{4}+\frac{\ell_1}{2}+\frac{\ell_1}{4} = \|\mathrm{q}\|.$$

Similarly, we obtain

$$\max\left\{\max_{k\in\{1,2\}}\ \max_{\tau\in J}\left|\delta^k_{\rho_1}\mathcal{N}_\lambda(\mathrm{q}(\tau))\right|,\ \max_{\tau\in J}\left|{}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathcal{N}_\lambda(\mathrm{q}(\tau))\right|\right\} \leqslant \|\mathrm{q}\|.$$

Therefore, we conclude that $\|\mathcal{N}_\lambda\mathrm{q}\| \leqslant \|\mathrm{q}\|$, $\forall \mathrm{q} \in K \cap \partial\Omega_{\ell_1}$. Then, Theorem 1.5.10 implies that

$$\mathbf{i}(\mathcal{N}_\lambda, \Omega_{\ell_1}, K) = 1. \tag{3.3.5}$$

On the other hand, let us consider $\Omega_{\ell_2} = \{q \in K : \|q\| \leqslant \ell_2\}$. Then, for any $q \in K \cap \partial\Omega_{\ell_2}$ by Lemma (3.2.4) one has $\ell_2 \geqslant \min_{\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]} q(\tau) \geqslant \gamma\ell_2$. Using hypothesis (H4), we get

$$\begin{aligned}
\mathcal{N}_\lambda(q(\grave{\imath})) \geqslant & \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \left[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\right) d\xi \right. \\
& + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\right) d\xi \\
& \left. + \lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(q(\xi)\right)\, d\xi\right)\right)\right] \\
\geqslant & \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} m_2\ell_2\gamma\, Z(\grave{\imath}_\circ)\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \right. \\
& \left. + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\, d\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\, d\xi\right) \\
\geqslant & \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \Lambda_6\ell_2\gamma\, Z(\grave{\imath}_\circ)\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \right. \\
& \left. + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\, d\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\, d\xi\right) \\
:= & \ \ell_2 = \|q\|.
\end{aligned}$$

Which implies that $\|\mathcal{N}_\lambda q\| \geqslant \|q\|$ for any $q \in K \cap \partial\Omega_{\ell_2}$. Hence Theorem 1.5.10 implies that

$$\mathbf{i}(\mathcal{N}_\lambda, \Omega_{\ell_2}, K) = 0. \tag{3.3.6}$$

Therefore, by Eqs. (3.3.5), (3.3.6) and $\ell_1 < \ell_2$, we have $\mathbf{i}(\mathcal{N}_\lambda, \overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1}, K) = 1$. By employing Thm 1.5.11, one can see the operator $\mathcal{N}_\lambda$ has at least one fixed point $q \in K \cap \overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1}$, which is ρ_1-concave positive solution of $\mathbb{FBVP}$ (3.1.1). □

Theorem 3.3.2. *Assume that all conditions (H1), (H2) and (H4) hold. Then, the* $\mathbb{FBVP}$ *(3.1.1) has no* ρ_1*-concave positive solution for* λ *large enough.*

Proof. Suppose that $\exists \check{N} \in \mathbb{N}$ and $(\lambda_j)_j$ such that $\lim_{j\to\infty} \lambda_j = +\infty$ and the $\mathbb{FBVP}$ (3.1.1) has ρ_1-

concave positive solution q_j, $(j \geq \check{N})$, i.e.,

$$q_j(\tau) = \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\mu\bigg(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\lambda_j\bigg(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)+F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(u(\xi)\right)\, d\xi\bigg)\bigg).$$

Thus,

$$q_j(\grave{\imath}) \geqslant \bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{\sigma_1-1}\bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\lambda_j\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)+F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(q(\xi)\right)\, d\xi\bigg)\bigg)\bigg].$$

Consequently,

$$q_j(\grave{\imath}) \geqslant \bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{\alpha-1}\lambda_j\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg).$$

Without loss of generality, we can suppose that $\check{N}$ is large enough so to get for $j \geq \check{N}$:

$$\lambda_j > j\bigg(\frac{\rho_1-\mu\rho_1}{\grave{\imath}^{\rho}-\grave{a}^{\rho}}\bigg)\bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{1-\sigma_1} \tag{3.3.7}$$

Then, we have $q_j(\grave{\imath}) > j$. Consequently, $\lim_{j\to+\infty}\|q_j\| = +\infty$. Using (H4), we deduce that there exist $m_2 > \Lambda_6$ and $\ell_2 > 0$ such that $\wp(q) \geqslant \phi_p\left(m_2\ell_2\right)$, $\forall q \in [\gamma\ell_2, \ell_2]$. Again, we can choose $\check{N}$ large enough to get $\|q_j\| \geqslant \ell_2$, $\forall j \geq \check{N}$. By writing $m_2 = \Lambda_6 + \varpi$ where $\varpi > 0$, we get

$$\|q_j\| \geqslant q_j(\grave{\imath}) \geqslant \bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{\sigma_1-1}\bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(q(s)\right)\, ds\bigg)\, d\xi$$

$$+\lambda\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)+F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(q(\xi)\right)\, d\xi\bigg)\bigg)\bigg]$$

$$\geqslant (\Lambda_6+\varpi)\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\,d\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\phi_p\left(q(\xi)\right)\,d\xi\bigg)$$

$$\geqslant \|q_j\|(\Lambda_6+\varpi)\gamma\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\,d\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi\bigg)$$

$$=\|q_j\|(1+\varpi\Lambda_6^{-1}),$$

which leads to a contradiction $\|q_j\|\varpi\Lambda_6^{-1}\leqslant 0$. The proof is completed. □

Remark 3.3.3. Let

$$\wp_0 := \lim_{q\to 0^+}\frac{\wp(q)}{\min\left\{\phi_p\left(q\right),q\right\}},\qquad \wp_\infty=\lim_{q\to\infty}\frac{\wp(q)}{\phi_p\left(q\right)}. \tag{3.3.8}$$

If $\wp_0=0$ and $\wp_\infty=\infty$ hold, then the conditions (H3) and (H4) hold respectively. Moreover, if the functions $\wp$ and $F_\circ$ are nondecreasing, the following theorem holds.

Theorem 3.3.4. *Assume that the hypotheses of Theorem 3.3.1 hold and that $\wp$ and $F_\circ$ are nondecreasing. Then, there exists $\lambda^*>0$ such that the $\mathbb{FBVP}$ (3.1.1) has at least one ρ-concave positive solution for $\lambda\in(0,\lambda^*)$ and has no ρ_1-concave positive solution for $\lambda\in(\lambda^*,\infty)$.*

Proof. Let

$$\acute{Y}=\Big\{\lambda\,:\,\text{the } \mathbb{FBVP}(3.1.1) \text{ has at least one } \rho_1-\text{concave positive solution}\Big\}\subset\mathbb{R}_+^*,$$

and $\lambda^*=\sup\acute{Y}$. It follows from Theorem 3.3.1 that $\acute{Y}\neq\emptyset$ and thus λ^* exists. We denote by q_0 the solution of $\mathbb{FBVP}$ (3.1.1) associated to λ_0 and

$$\mathcal{K}(q_0)=\Big\{q\in K\,:\,q(\tau)<q_0(\tau),\,\forall\tau\in J\Big\}.$$

Let $\lambda\in(0,\lambda_0)$ and $q\in\mathcal{K}(q_0)$. It follows from the definition of $\mathcal{N}_\lambda$ (3.2.18) and the monotonicity of f that for any $\tau\in J$,

$$\mathcal{N}_\lambda(q(\tau))\leq\mathcal{N}_\lambda(q_0(\tau))=q_0(\tau).$$

Thus, $\mathcal{N}_\lambda(\mathcal{K}(q_0))\subseteq\mathcal{K}(q_0)$. Now, Shaulder's fixed point theorem implies that there exists a fixed point $q\in\mathcal{K}(q_0)$, such that it is a positive solution of (3.1.1). The proof is completed. □

Theorem 3.3.5. *Suppose that conditions (H1) and (H2) hold. Assume that $\wp$ also satisfies:*

(H5) $\wp_0 = \varpi_1 \in [0, \min\{k^{p-1}, k\})$, $k = \frac{1}{4}\check{M}_4$;

(H6) $\wp_\infty = \varpi_2 \in \left(\left(\frac{2\Lambda_6}{\gamma}\right)^{p-1}, \infty\right)$.

Then, the $\mathbb{FBVP}$ (3.1.1) *has at least one* ρ_1-*concave positive solution for* λ *small enough.*

Proof. Firstly, from the definition of $\wp_0$, for all $\epsilon > 0$ there exists an adequate small positive number $\bar{\delta}(\epsilon)$ such that

$$\wp(\mathrm{q}) \leqslant (\epsilon + \varpi_1) \min\left\{\mathrm{q}^{p-1}, \mathrm{q}\right\} \leqslant (\epsilon + \varpi_1) \min\left\{\bar{\delta}^{p-1}, \bar{\delta}\right\},$$

$\forall \mathrm{q} \in [0, \bar{\delta}(\epsilon)]$. Then, for $\epsilon = \min\{k^{p-1}, k\} - \varpi_1$, we have

$$\begin{aligned}
\wp(\mathrm{q}) &\leqslant \min\left\{k^{p-1}, k\right\} \min\left\{\bar{\delta}(\epsilon)^{p-1}, \bar{\delta}(\epsilon)\right\} \\
&\leqslant \min\left\{k^{p-1}\bar{\delta}(\epsilon)^{p-1}, k\bar{\delta}(\epsilon)\right\} \\
&\leqslant \min\left\{\left(2k\bar{\delta}(\epsilon)\right)^{p-1}, 2k\bar{\delta}(\epsilon)\right\}.
\end{aligned}$$

It's enough to take $\ell_1 = \bar{\delta}(\epsilon)$ and $m_1 = 2k \in \left(0, \check{M}_4\right)$, i.e., the condition (H3) holds. Next, since (H6) hold, then for every $\epsilon > 0$, there exists an adequate big positive number $\ell_2 \neq \ell_1$ such that

$$\wp(\mathrm{q}) \geqslant (\varpi_2 - \epsilon)\mathrm{q}^{p-1} \geqslant (\varpi_2 - \epsilon)\left(\gamma\ell_2\right)^{p-1}, \qquad (\mathrm{q} \geqslant \gamma\ell_2).$$

Hence, for $\epsilon = \varpi_2 - \left(\frac{2\Lambda_6}{\gamma}\right)^{p-1}$, we get

$$\wp(\mathrm{q}) \geqslant \left(\frac{2\Lambda_6}{\gamma}\right)^{p-1} (\gamma\ell_2)^{p-1} = (2\Lambda_6\ell_2)^{p-1}. \tag{3.3.9}$$

By considering $m_2 = 2\Lambda_6 > \Lambda_6$, the condition (H4) holds by Theorem 3.3.1, we complete the proof. □

3.4 Several Solutions in a cone

In order to show existence of multiple solutions we will use the Leggett-Williams fixed point theorem 1.5.15. For this we define the following subsets of a cone K,

$$\Omega_c = \left\{\mathrm{q} \in K : \|\mathrm{q}\| < c\right\},$$
$$\Omega_\varphi(b, d) = \left\{\mathrm{q} \in K : b \le \varphi(\mathrm{q}), \|\mathrm{q}\| \le d\right\}.$$

Theorem 3.4.1. *Suppose that conditions (H1) and (H2) hold, if there exist* $å, b, c$ *with* $0 < å < \gamma b < b \le c$, *such that*

(H7) $\wp(\mathrm{q}(\tau)) < \min\left\{\phi_p\left(m_1 å\right), m_1 å\right\}$ *for* $(\tau, \mathrm{q}) \in J \times [0, å]$;

(H8) $\wp(\mathrm{q}(\tau)) \geqslant \phi_p\left(m_2 \gamma b\right)$, *for* $(\tau, \mathrm{q}) \in [\grave{a}_\circ, \grave{\imath}_\circ] \times [\gamma b, b]$;

(H9) $\wp(\mathrm{q}(\tau)) \leqslant \min\left\{\phi_p\left(m_1 c\right), m_1 c\right\}$ *for* $(\tau, \mathrm{q}) \in J \times [0, c]$;

(H10) $0 < \lambda < \frac{(1-\mu)å}{2} \min\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right\}$;

where the constants m_2 *and* m_1 *are defined in* (3.3.1)*. Then, the* $\mathbb{FBVP}$ (3.1.1) *has at least three positive* ρ_1*-concave solutions* $\mathrm{q}_1, \mathrm{q}_2$ *and* q_3 *satisfying* $\|\mathrm{q}_1\| < å$, $\gamma b < \varphi(\mathrm{q}_2)$ *and* $\|\mathrm{q}_3\| > å$ *with* $\varphi(\mathrm{q}_3) < b\gamma$ *for* λ *small enough.*

Proof. We prove that, the $\mathbb{FBVP}$ (3.1.1) has at least three positive ρ_1-concave solutions for $\lambda > 0$ small enough. By Lemma 3.2.5, $\mathcal{N}_\lambda : \mathrm{Y} \to \mathrm{Y}$ is completely continuous.
Let $\varphi(\mathrm{q}) = \min_{\tau \in [\grave{a}_\circ, \grave{\imath}_\circ]} \mathrm{q}(\tau)$. Obviously, $\varphi(\mathrm{q})$ is nonnegative, continuous and concave functional on K with $\varphi(\mathrm{q}) \leq \|\mathrm{q}\|$, for $\mathrm{q} \in \overline{\Omega_c}$. Now we will show that, all conditions of Theorem 1.5.15 are satisfied. Suppose that $\mathrm{q} \in \overline{\Omega_c}$, this is, $\|\mathrm{q}\| \leq c$. For $\tau \in J$ by Eq. (3.2.18), Lemma 3.2.4, 2.2.6, we acquire

$$\begin{aligned}
\max_{\tau \in J} |\, \mathcal{N}_\lambda(\mathrm{q}(\tau))\,| = & \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \mathrm{q}(\tau) \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\left(\mathrm{q}(s)\right)\, \mathrm{d}s \right) \mathrm{d}\xi \\
& + \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\left(\mathrm{q}(s)\right)\, \mathrm{d}s \right) \mathrm{d}\xi \\
& + \lambda \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1} \right) + F_\circ \left(\phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) \wp\left(\mathrm{q}(\xi)\right)\, \mathrm{d}\xi \right) \right).
\end{aligned}$$

From (H2), (H9) and (H10), we get

$$\begin{aligned}
\max_{\tau \in J} |\, \mathcal{N}_\lambda(\mathrm{q}(\tau))\,| \leqslant & \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s) \hbar(s) \wp\left(\mathrm{q}(s)\right)\, \mathrm{d}s \right) \mathrm{d}\xi \\
& + \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1} \right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s) \hbar(s) \wp\left(\mathrm{q}(s)\right)\, \mathrm{d}s \right) \mathrm{d}\xi \\
& + \frac{c}{2} + A \int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) \wp\left(\mathrm{q}(\xi)\right)\, \mathrm{d}\xi \\
\leqslant & \, m_1 c \Bigg[\Bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\, \mathrm{d}\xi
\end{aligned}$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1\mu}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\Big)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)$$

$$+A\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\Big]+\frac{c}{2}$$

$$\leqslant\frac{\Lambda_2 c}{4}\Big[\Big(\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi$$

$$+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1\mu}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\Big)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\Big]$$

$$+\frac{A\Lambda_1 c}{4}\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi+\frac{c}{2}$$

$$=\frac{c}{4}+\frac{c}{4}+\frac{c}{2}=c,$$

and

$$\max\left\{\max_{k\in\{1,2\}}\max_{\tau\in J}\left|\delta^k_{\rho_1}\mathcal{N}_\lambda(\mathrm{q}(\tau))\right|,\max_{\tau\in J}\left|{}^{\rho_1;CK}\mathcal{D}^{\sigma_1}_{\grave{a}^+}\mathcal{N}_\lambda(\mathrm{q}(\tau))\right|\right\}\leqslant\|\mathrm{q}\|.$$

Therefore, we have

$$\|\mathcal{N}_\lambda\mathrm{q}(\tau)\|\leqslant c,\qquad(\forall\mathrm{q}\in\Omega_c).$$

This implies that $\mathcal{N}_\lambda:\overline{\Omega_c}\to\overline{\Omega_c}$. By the same method, if $\mathrm{q}\in\overline{\Omega_{\mathring{a}}}$, then we can get $\|\mathcal{N}_\lambda\mathrm{q}(\tau)\|<\mathring{a}$, therefore (**D**2) has been checked. Next, we assert that

$$\left\{\mathrm{q}\in\Omega_\varphi(\gamma b,b)\;:\;\varphi(\mathrm{q})>\gamma b\right\}\neq\emptyset$$

and $\varphi(\mathcal{N}_\lambda(\mathrm{q}))>\gamma b\,\forall\mathrm{q}\in\Omega_\varphi(\gamma b,b)$. In fact, the constant function $\frac{\gamma b+b}{2}\in\Omega_\varphi(\gamma b,b)$ and $\varphi\left(\frac{\gamma b+b}{2}\right)>\gamma b$. On the other hand, for $\mathrm{q}\in\Omega_\varphi(\gamma b,b)$, we have

$$\gamma b\leq\varphi(\mathrm{q})=\min\mathrm{q}(\tau)\leq\|\mathrm{q}\|=b,\qquad(\forall t\in[\grave{a}_\circ,\grave{\imath}_\circ]).$$

Thus, in view of (3.2.18), Lemma 3.2.3, 3.2.4, 2.2.6 and (H8), we have

$$\varphi(\mathcal{N}_\lambda\mathrm{q})=\min_{t\in[\grave{a}_\circ,\grave{\imath}_\circ]}\Big[\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\right)\mathrm{d}\xi$$

$$+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\right)\mathrm{d}\xi$$

$$+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi\right)\right)\Big]$$

$$
\begin{aligned}
&\geq \gamma\Bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \\
&\quad + \mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\left(\mathrm{q}(s)\right)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \\
&\quad + \lambda\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg) + F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\left(\mathrm{q}(\xi)\right)\,\mathrm{d}\xi\bigg)\bigg)\Bigg] \\
&> \gamma m_2 b\Bigg[\gamma\bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{\sigma_1-1} Z(\grave{\imath}_\circ)\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \\
&\quad + \mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\bigg)\Bigg] \\
&> \gamma\Lambda_6 b\Bigg[\gamma\bigg(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\bigg)^{\sigma_1-1} Z(\grave{\imath}_\circ)\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \\
&\quad + \mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\bigg)\Bigg] \\
&= \gamma b.
\end{aligned}
$$

Thus, (**D**1) has been verified. Finally, we need to show that if $\mathrm{q} \in \Omega_\varphi(\gamma b, b)$ with $\|\mathcal{N}\lambda\mathrm{q}\| > b$, then $\|\mathcal{N}_\lambda\mathrm{q}\| > \gamma b$. In fact, to see this, suppose that $\mathrm{q} \in \Omega_\varphi(\gamma b, b)$ with $\|\mathcal{N}_\lambda\mathrm{q}\| > b$, then through Lemma 3.2.4, we have

$$\varphi(\mathcal{N}_\lambda\mathrm{q}) = \min_{\grave{a}_\circ \leq t \leq \grave{\imath}_\circ} (\mathcal{N}_\lambda\mathrm{q})(\tau) \geq \gamma\|\mathcal{N}_\lambda\mathrm{q}\| > \gamma b.$$

Thus, (**D**3) is satisfied. Hence, an application of Theorem 1.5.15 completes the proof. □

Corollary 3.4.2. *Suppose conditions (H1) and (H2) hold. If there exist constants*

$$0 < r_1 < b_1 < \gamma b_1 \leqslant r_2 < b_2 < \gamma b_2 \leqslant \cdots \leqslant r_n,$$

for $1 \leqslant j \leqslant n-1$ *and the following conditions are satisfied:*

(H11) $\wp(\mathrm{q}(\tau)) < \min\left\{\phi_p\left(m_1 r_j\right), m_1 r_j\right\}$, *for* $(\tau, \mathrm{q}) \in J \times [0, r_j]$,*;*

(H12) $\wp(\mathrm{q}(\tau)) > \phi_p\left(m_2 b_j\right)$ *for* $(\tau, \mathrm{q}) \in [\grave{a}_\circ, \grave{\imath}_\circ] \times [\gamma b_j, b_j]$*;*

(H13) $0 < \lambda < \frac{(1-\mu) r_1}{2} \max\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right\}$,

then, the $\mathbb{FBVP}$ (3.1.1) *has at least* $2n-1$ *positive* ρ_1*-concave solutions.*

Proof. By the induction method, we get the proof. □

3.5 Applications

In this section, we give some examples to illustrate the usefulness of our main results.

Example 3.5.1. Let us consider the following $\mathbb{FBVP}$

$$\begin{cases} {}^{\rho_2;CK}\mathcal{D}^{5/2}_{e^{2-}}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}\mathrm{q}\right)\right)(\tau) \\ \qquad + \dfrac{\mathrm{q}^{3/2}(\tau)}{\sqrt{(2.2-\ln(\tau))\,(\ln(\tau)-0.9)}} = 0, \qquad e<\tau<e^2, \\ \qquad \mathrm{q}(e) - \sqrt{\left|{}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}\mathrm{q}(e)\right|} = 0, \\ \qquad \delta^2_{0^+}\mathrm{q}(e) = 0, \\ \qquad \delta^1_{0^+}\mathrm{q}(e^2) = \frac{1}{2}\delta^1_{0^+}\mathrm{q}(e^{\frac{8}{5}}) + \lambda, \\ {}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}\mathrm{q}(e^2) = -\delta^1_{0^+}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}\mathrm{q}\right)\right)(e) \\ \qquad = \delta^2_{0^+}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}\mathrm{q}\right)\right)(e^2) = 0. \end{cases} \tag{3.5.1}$$

Here $J=[e,e^2]$, $\sigma_1=\sigma_2=\frac{5}{2}\in(2,3]$, $\mu=\frac{1}{2}\in(0,1)$, $\eta=e^{\frac{8}{5}}\in J$, $[\grave{a}_\circ,\grave{\imath}_\circ]=[e^{3/2},e^{7/4}]\subset J$. We put $\rho_1=0.5\in\mathbb{R}\setminus\{1\}$, $\rho_2=1.3\in\mathbb{R}\setminus\{1\}$, $p=\frac{3}{2}$ and so $\bar{p}=3$, $A=\frac{3}{2}$, $B=\frac{1}{2}$. ${}^{\rho_1;CK}\mathcal{D}^{5/2}_{e^+}$ and ${}^{\rho_2;CK}\mathcal{D}^{5/2}_{e^{2-}}$ are the left and right-sided CK$\mathbb{FD}$, $F_\circ(v)=\sqrt{|\ v\ |}$ and

$$\hbar(\tau)=\frac{1}{\sqrt{(2.2-\ln(\tau))\,(\ln(t)-0.9)}}.$$

We can easily show that (H1), (H2) hold and from (3.3.8), we get $\wp(\mathrm{q}(\tau))=(\mathrm{q}(\tau))^{3/2}$ satisfy

$$\wp_0=\lim_{\mathrm{q}\to 0^+}\frac{\wp(\mathrm{q})}{\min\left\{\phi_{\frac{3}{2}}(\mathrm{q}),\mathrm{q}\right\}}=\lim_{\mathrm{q}\to 0^+}\frac{\mathrm{q}^{\frac{3}{2}}}{\min\left\{\frac{\mathrm{q}}{\sqrt{|\mathrm{q}|}},\mathrm{q}\right\}}=0,$$

$$\wp_\infty=\lim_{\mathrm{q}\to\infty}\frac{\wp(\mathrm{q})}{\phi_{\frac{3}{2}}(\mathrm{q})}=\lim_{\mathrm{q}\to\infty}\frac{\mathrm{q}^{\frac{3}{2}}}{\mathrm{q}|\mathrm{q}|^{\frac{3}{2}-2}}=\lim_{\mathrm{q}\to\infty}\frac{\mathrm{q}^{\frac{1}{2}}}{|\mathrm{q}|^{\frac{-1}{2}}}=\infty.$$

Tableau 3.1 – *Numerical values of* $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi$, $\breve{M}_2$, γ, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi$, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi$ *and* $\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)$ *in Example 3.5.1 for* $\tau \in J$.

τ	$\int_{\grave{a}}^{\tau} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi$	$\breve{M}_2$	γ	$\int_{\grave{a}}^{\tau} \mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi$	$\int_{\grave{a}}^{\tau} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi$	$\phi_{\bar{p}}\left(\int_{\grave{a}}^{\tau} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)$
2.7183	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.0377	0.2357	0.0039	0.0011	0.3462	3.7320	13.9275
3.3947	0.5121	0.0085	0.0025	0.6944	10.0873	101.7541
3.7937	0.8293	0.0138	0.0040	1.0416	18.4145	339.0940
4.2395	1.1850	0.0197	0.0057	1.3835	28.8166	830.3948
4.7377	1.5738	0.0261	0.0076	1.7148	41.5311	1724.8351
5.2945	1.9852	0.0329	0.0095	2.0276	56.8307	3229.7341
5.9167	2.3994	0.0398	0.0115	2.3105	74.9520	5617.7976
6.6120	2.7785	0.0461	0.0133	2.5445	95.9883	9213.7513
7.3891	3.0207	0.0501	0.0145	2.6809	119.6935	14326.5251

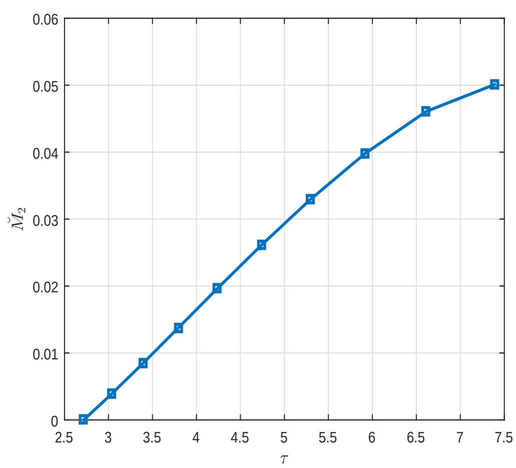
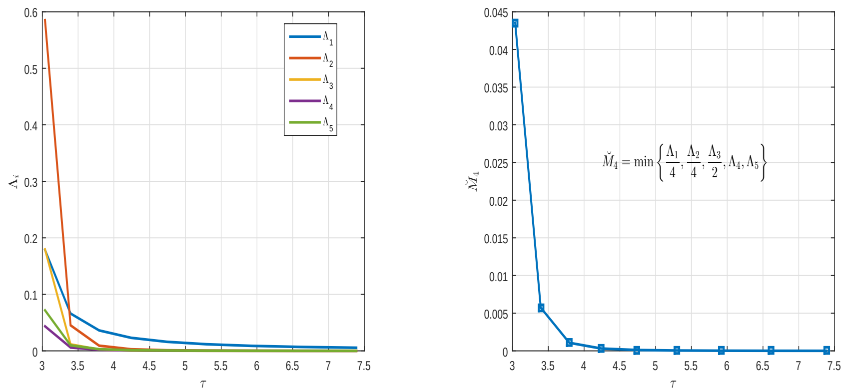


Figure 3.1 – *2D-graph of* $\breve{M}_2$ *for* $\tau \in [e, e^2]$ *in Example 3.5.1.*

Tableau 3.2 – *Numerical values of* Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6 *and* $\breve{M}_4$ *in Example 3.5.1 for* $\tau \in J$.

τ	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	$\breve{M}_4$
2.7183	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*
3.0377	0.178637	0.585071	0.179564	0.043506	53.867322	0.071800	0.043506
3.3947	0.066090	0.045238	0.011463	0.005955	1.916555	0.009828	0.005731
3.7937	0.036203	0.009195	0.002150	0.001787	0.240558	0.002949	0.001075
4.2395	0.023135	0.002839	0.000621	0.000730	0.051977	0.001204	0.000311
4.7377	0.016052	0.001104	0.000227	0.000351	0.015226	0.000580	0.000114
5.2945	0.011731	0.000499	0.000097	0.000188	0.005456	0.000310	0.000049
5.9167	0.008895	0.000252	0.000047	0.000108	0.002278	0.000178	0.000023
6.6120	0.006945	0.000140	0.000025	0.000066	0.001090	0.000109	0.000012
7.3891	0.005570	0.000085	0.000015	0.000042	0.000612	0.000070	0.000007



(a) $\Lambda_i \; (i = 1,2,3,4,5)$ (b) $\breve{M}_4$

Figure 3.2 – *Graphical representation of $\Lambda_i \; (i = 1,2,3,4,5)$ and $\breve{M}_4$ for $\tau \in J$ in Example 3.5.1.*

Then obviously, $Z(\grave{\imath}_\circ) \approx 0.05549$,

$$\breve{M}_4 = \min\left\{\frac{\Lambda_1}{4}, \; \frac{\Lambda_2}{4}, \; \frac{\Lambda_3}{2}, \; \Lambda_4, \; \Lambda_5\right\} \simeq 0.00007, \qquad \Lambda_6 \approx 0.000007.$$

Tables 3.1 and 3.2 show the numerical results in Example 3.5.1 for $\tau \in J$. So by assume that $\lambda = 1.5$ and $\ell_1 = 12$, all conditions of Theorem 3.3.1 hold, then we can choose $\ell_2 > \ell_1$ and for λ satisfies

$$0 < \lambda \leqslant \frac{1}{2}(1-\mu)\ell_1 \min\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right\} = 2.4542789 < \ell_2,$$

$$2\lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \approx 3.42259 \leq 12 = \ell_1,$$

and $2\lambda \leq \ell_1(1-\mu) = 10.5$ such that

$$\Omega_{\ell_1} = \{q \in K : \|q\| < \ell_1\}, \qquad \Omega_2 = \{q \in K : \|q\| < \ell_2\}.$$

Figures 3.1, 3.2 and 3.3 show graphical representation of the variables in Example 3.5.1 for $\tau \in J$. Then, we can show that, the $\mathbb{FBVP}$ (3.5.1) has at least positive solution $q \in K \cap (\overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1})$ for λ small enough.

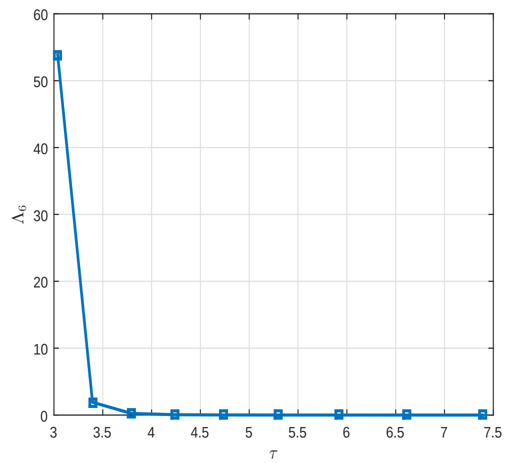


FIGURE 3.3 – *2D-graph of Λ_6 for $\tau \in J$ in Example 3.5.1.*

Example 3.5.2. Let us consider the following $\mathbb{FBVP}$

$$\begin{cases} {}^{\rho_2;CK}\mathcal{D}_{e^-}^{5/2}\left(\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}\mathrm{q}\right)\right)(\tau) \\ \qquad +\dfrac{5\sqrt{\pi}}{4}\ln(\tau)\wp(\mathrm{q}(\tau)) = 0, \qquad 1<\tau<e, \\ \mathrm{q}(1) - F_\circ\left({}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}\mathrm{q}(1)\right) = 0, \\ \delta_{\frac{1}{2}}^{2}\mathrm{q}(1) = 0, \\ \delta_{\frac{1}{2}}^{1}\mathrm{q}(e) = \frac{1}{2}\delta_{\frac{1}{2}}^{1}\mathrm{q}(\sqrt{e}) + \lambda \\ {}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}\mathrm{q}(e) = 0, \\ -\delta_{0+}^{1}\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}\mathrm{q}\right)\right](1) = 0, \\ \delta_{0+}^{2}\left[\phi_p\left({}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}\mathrm{q}\right)\right](e) = 0, \end{cases} \tag{3.5.2}$$

Here $J = [1, e]$, $\sigma_1 = \sigma_2 = \frac{5}{2} \in (2, 3]$, $\mu = \frac{1}{2} \in (0, 1)$, $\eta = \sqrt{e} \in J$, $[\grave{a}_\circ, \grave{\imath}_\circ] = [\sqrt{e}, \sqrt[4]{e}] \subset J$. We put $\rho_1 = 0.5 \in \mathbb{R} \setminus \{1\}$, $\rho_2 = 2 \in \mathbb{R} \setminus \{1\}$, $p = \frac{3}{2}$ and so $\bar{p} = 3$, $A = \frac{3}{2}$, $B = \frac{1}{2}$. ${}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}$ and ${}^{\rho_1;CK}\mathcal{D}_{1^+}^{5/2}$ are the left and right-sided CK$\mathbb{FD}$, $F_\circ(v) = \sqrt{|\, v \,|}$ and $\hbar(\tau) = \frac{5\sqrt{\pi}}{4}\ln(\tau)$, and

$$\wp(\mathrm{q}) = \begin{cases} 6\mathrm{q}^2, & \mathrm{q} \leq 1, \\ 5 + \mathrm{q}^{1/4}, & \mathrm{q} > 1. \end{cases}$$

TABLEAU 3.3 – *Numerical values of* $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,d\xi$, $\breve{M}_2$, γ, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath},\xi)\,d\xi$, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi$ *and* $\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi\right)$ *in Example 3.5.2 for* $\tau \in J$.

τ	$\int_{\grave{a}}^{\tau} \mathcal{G}_1(\grave{\imath},\xi)\,d\xi$	$\breve{M}_2$	γ	$\int_{\grave{a}}^{\tau} \mathcal{G}_2(\grave{\imath},\xi)\,d\xi$	$\int_{\grave{a}}^{\tau} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi$	$\phi_{\bar{p}}\left(\int_{\grave{a}}^{\tau} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi\right)$
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.1052	0.0602	0.1059	0.0307	0.0384	0.0119	0.0001
1.2214	0.1299	0.2284	0.0662	0.0771	0.0793	0.0063
1.3499	0.2091	0.3677	0.1065	0.1157	0.2572	0.0661
1.4918	0.2975	0.4325	0.1253	0.1539	0.6196	0.3839
1.6487	0.3942	0.4325	0.1253	0.1913	1.2649	1.5999
1.8221	0.4975	0.4325	0.1253	0.2273	2.3154	5.3612
2.0138	0.6046	0.4325	0.1253	0.2610	3.9084	15.2758
2.2255	0.7105	0.4325	0.1253	0.2913	6.1656	38.0151
2.4596	0.8056	0.4325	0.1253	0.3162	9.1216	83.2040
2.7183	0.8654	0.4325	0.1253	0.3307	12.5716	158.0444

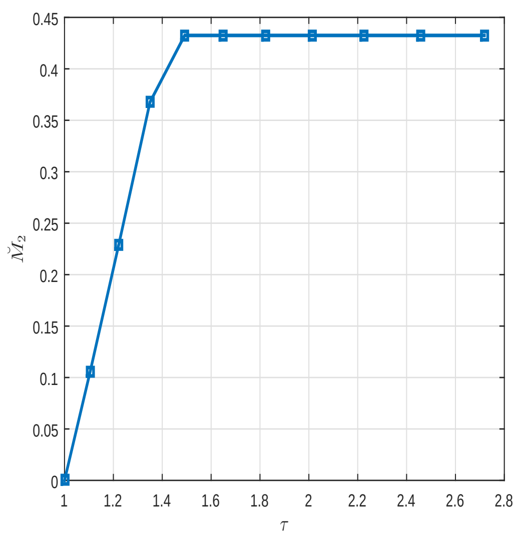


FIGURE 3.4 – *2D-graph of* $\breve{M}_2$ *for* $\tau \in [1, e]$ *in Example 3.5.2.*

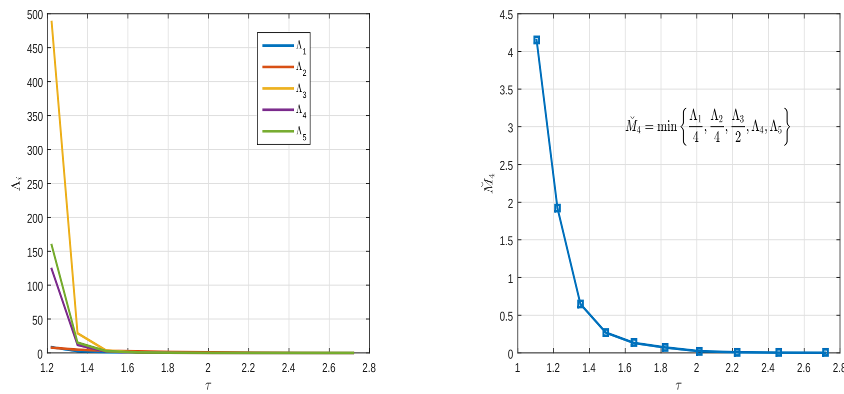
Through a simple calculation, we have $\int_1^e \mathcal{H}(e,\xi)\hbar(\xi)\,d\xi \approx 12.5716$,

$$\gamma = \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \breve{M}_2 = \left(\frac{e^{0.25} - 1}{e^{0.5} - 1}\right)^{\frac{3}{2}} \times 0.4325 = 0.1253.$$

Tables 3.3 and 3.4 show the numerical results.

$$\breve{M}_4 = \min\left\{\frac{\Lambda_1}{4},\ \frac{\Lambda_2}{4},\ \frac{\Lambda_3}{2}, \Lambda_4,\ \Lambda_5\right\} \simeq 0.001499,$$

and $\Lambda_6 \simeq 1.583636$.



(a) $\Lambda_i\ (i = 1, 2, 3, 4, 5)$ (b) $\check{M}_4$

FIGURE 3.5 – *Graphical representation of $\Lambda_i\ (i = 1, 2, 3, 4, 5)$ and $\check{M}_4$ for $\tau \in J$ in Example 3.5.1.*

Figures 3.4, 3.5 and 3.6 show graphical representation of the variables in Example 3.5.2 for $\tau \in J$. Choosing $\mathring{a} = 10^{-2}$, $b = \frac{11}{10}$,$c = 10^5$, $m_1 = 0.001 \in (0, \check{M}_4)$, $m_2 = 13 \in (\Lambda_6, \infty) = (1.583636, \infty)$, we get

$$\wp(\mathrm{q}) < \wp\left(10^{-2}\right) = 6 \times 10^{-4} < \min\left\{\phi_p\left(\mathring{a}m_1\right), \mathring{a}m_1\right\} = \mathring{a}m_1 = 8 \times 10^{-4} \in [0, 10^{-2}],$$

$$\wp(\mathrm{q}(\tau)) > 5 + (m_2\gamma b)^{1/4} = 5 + \left(0.1253 \times 13 \times \frac{11}{10}\right)^{1/4} \simeq 1.1570 > \phi_p\left(\gamma b m_2\right)$$
$$\simeq 0.739467251 \in \left[\frac{11}{10}\gamma, \frac{11}{10}\right],$$

$$\wp(\mathrm{q}(\tau)) < \wp(10^4) = 15 < \min\left\{\phi_p\left(cm_1\right),\ cm_1\right\} = \phi_p\left(cm_1\right) = \sqrt{8000}.001\mathrm{q} \in [0, 10^4],$$

$$0 < \lambda \leqslant \frac{(1-\mu)\mathring{a}}{2} = 2.5 \times 10^{-3}.$$

TABLEAU 3.4 – *Numerical values of Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6 and $\check{M}_4$ in Example 3.5.2 for $\tau \in J$.*

τ	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	$\check{M}_4$
1.0000	Inf	Inf	Inf	Inf	Inf	Inf	Inf
1.1052	56.016486	16.599423	46453.302726	5493.069851	1132.699598	7060.155197	4.149856
1.2214	8.405999	7.690366	487.818288	123.697821	243.285192	158.986839	1.922592
1.3499	2.592400	4.749445	28.980434	11.764874	93.355843	15.121205	0.648100
1.4918	1.075949	3.242105	3.526105	2.026594	54.177100	2.604749	0.268987
1.6487	0.527061	2.217810	0.641322	0.486300	37.060645	0.625034	0.131765
1.8221	0.287924	1.382433	0.152208	0.145124	23.101100	0.186526	0.071981
2.0138	0.170572	0.744218	0.044104	0.050933	12.436232	0.065463	0.022052
2.2255	0.108126	0.361888	0.015127	0.020467	6.047314	0.026305	0.007563
2.4596	0.073086	0.175974	0.006109	0.009351	2.940615	0.012019	0.003055
2.7183	0.053030	0.094769	0.002998	0.004923	1.583636	0.006327	0.001499

Then, the conditions (H7),(H8) and (H9) are satisfied. Therefore, it follows from Theorem 3.4.1 that, the $\mathbb{FBVP}$ (3.5.2) has at least three $\frac{1}{2}$-concave positive solutions q_1, q_2 and q_3 such that $\|q_1\| < 10^{-2}$, $\frac{11}{10}\gamma < \varphi(q_2)$ and $\|q_3\| > 10^{-2}$ with $\varphi(q_3) < \frac{11}{10}\gamma$.

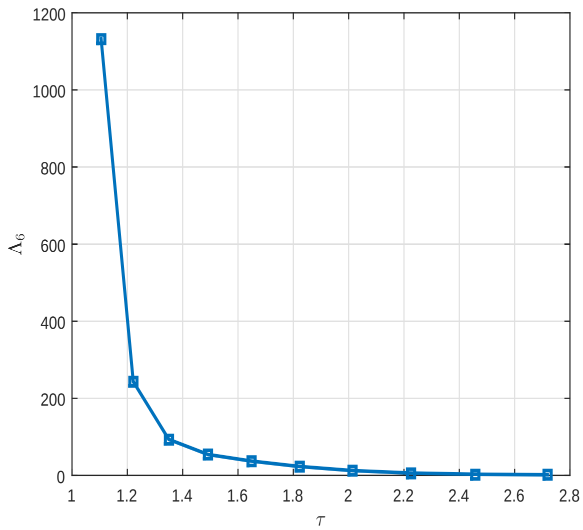


FIGURE 3.6 – *2D-graph of* Λ_6 *for* $\tau \in J$ *in Example 3.5.2.*

We want to inform the reader that, the data represented in the tables and curves mentioned and shown in both Example 3.5.1 and Example 3.5.2 have been set by using MATLAB.

3.6 Conclusion

The chapter presents a new boundary value of two-sided fractional differential equations involving generalized-Caputo fractional derivatives which we investigate the existence and the multiplicity of ρ-concave positive solutions of it. We made some additional assumptions to we prove some important results and obtain the existence of at least three solutions by using some fixed point theorems. this results can extended in some works such as [68, 74, 75].

Chapter 4

Existence and uniqueness of solutions for p-Laplacian boundary value problems of fractional impulsive differential equations

4.1 Introduction

In this chapter we focus to deal with the existence and uniqueness results for boundary-value problem of following nonlinear $\psi-$ Caputo fractional impulsive differential equations:

$$\begin{cases} {}^{\psi_2;C}_{\ \ t}\mathcal{D}^{\beta}_{T^-}\left(\rho(t)\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)\right)(t)+s(t)\phi_p\left(u(t)\right)=f(t,u(t)) \quad a<t<T, \\ \Delta\left(u\left(t_k\right)\right)=I^1_k\left(u\left(t_k\right)\right), \Delta\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)\left(t_k\right)=I^2_k\left(u\left(t_k\right)\right), k=1,2,\ldots,m \\ \qquad u(a)=u_0+\lambda\left|{}^{\psi_3}_{\ t}\mathcal{I}^{\gamma}_{a^+}\eta(t)\mid u(t)|^{p-1}\right|^{p^\star-1}_{t=T}, {}^{\psi_1;C}\mathcal{D}^{\alpha}_{a^+}u(T)=u_1 \end{cases} \tag{4.1.1}$$

where $p,p^\star>1$, $0<\alpha,\beta,\gamma\leqq 1,\phi_p$ is a p -Laplacian operator, $s(t),\rho(t),\eta(t)\in C([a,T],\mathbb{R}^\star_+)$, $f\in C([a,T]\times\mathbb{R},\mathbb{R})$, $u_0,u_1,\lambda\in\mathbb{R}$, for $k=1,2,\ldots,m,i=1,2,$ $I^i_k\in C(\mathbb{R},\mathbb{R})$ $,0<a=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=T$. $\Delta u\left(t_k\right)=u\left(t^+_k\right)-u\left(t^-_k\right), u\left(t^+_k\right)$ and $u\left(t^-_k\right)$ denote the right and the left limits of $u(t)$ at $t=t_k$ $(k=1,2,\ldots,m)$ respectively and $\Delta\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)\left(t_k\right)$ has a similar meaning for $\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)\left(t_k\right)$.
Through the last decade of the past century, the theory of differential equations that involve fractional derivatives of non-integer order has witnessed a wide intensive improvement and many applications such as physics, mechanics, electricity, control theory, rheology, signal and image processing, aerodynamics, and other fields. This field has attracted the attention of many authors constantly in the study of fractional differential equations is based upon that, the fact of fractional calculus service as a great tool in common usage for the applications of such constructions in

various sciences, the description of properties of diverse materials, processes and important part of the physical mathematics and also a large part of the literature is related to fractional differential equations. Moreover, the fractional-order models are more represents powerful tools, factual, and usefulness than the integer-order models. For more details and applications about fractional calculus, see[17, 19, 42, 49, 59, 62, 64, 70, 79, 88] and references therein. However, the need to describe more accurately these phenomena paves the way to suggest several types of fractional operators.

Very recently, Almeida in [11] given a version generalized of Caputo FOD with some interesting properties. For some particular cases of ψ, one can realize that ψ–Caputo FD can be reduced to some well-known classical kinds of Caputo fractional operator [44, 48, 50, 54, 70]. Models based on generalized fractional derivatives may be more accurate than the models based on classical fractional derivatives.

At the same time, the fractional calculus and p-Laplacian operator appear naturally in the applied fields of sciences and is extensively used in the mathematical, modeling of physical and natural phenomena such as turbulent filtration in porous media, blood flow problems, rheology, modeling of viscoelasticity, and material science, it is worth studying the fractional p-Laplacian differential equations. As is well known, the formulation of the ordinary p-Laplacian operator was put forward by Leibenson in 1945 [51], there are numerous research of fractional boundary value problems with p-Laplacian operators have been established ([13, 27, 53, 71, 72, 81, 82, 86]).

There are many processes and phenomena in the real world, which are subjected during their development to short-term external influences. But, this duration is negligible compared with the total duration of the studied phenomena and processes. Therefore, it can be seen that these external effects are "instantaneous", i.e. they are in the form of impulses. Furthermore, the differential equations with impulsive effects arise from many phenomena in the real world and describe the dynamics of processes in which sudden, discontinuous jumps occur.

For the best background concerning the basic theory and some applications of impulsive differential equations, we refer the interested readers to [20, 29, 30, 32, 39, 40, 56, 76, 77, 78, 80]. In the literature, there have been many different tools and approaches to the study of the existence of solutions to impulsive fractional differential equations, such as topological degree theory, fixed point theory, upper and lower solution's method, monotone iterative technique and so on (see, for example, [7, 10, 15, 28] and the references therein).

Recently, L. Menasria *et al* [61]. By virtue of variational method and critical point theory, they investigated the existence of weak solutions for a p-Laplacian impulsive differential equation with boundary conditions

$$\begin{cases} -\left(\rho(t)\phi_p\left(u'\right)\right)'(t)+s(t)\phi_p\left(u(t)\right)=f(t,u(t)) \quad 0<t<T, \\ \Delta\phi_p\left(u'\right)\left(t_k\right)=I_k\left(u\left(t_k\right)\right), k=1,2,\ldots,m, \\ u(0)=u(T)=0, \end{cases} \tag{4.1.2}$$

where $p>1$, ϕ_p is a p -Laplacian operator, $s(t),\rho(t)\in L^{\infty}([0,T])$, $f\in C([a,T]\times\mathbb{R},\mathbb{R})$, for $k=1,2,\ldots,m,\quad I_k\in C(\mathbb{R},\mathbb{R})$,$0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=T$. $\Delta\left(\phi_p\left(u'(t_k)\right)\right)=$

$\phi_p\left(u'\left(t_k^+\right)\right)-\phi_p\left(u'\left(t_k^-\right)\right), u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ denote the right and the left limits of $u(t)$ at $t=t_k$ $(k=1,2,\ldots,m)$ respectively.

In [55] Z. Liu, L. Lu and I. Szántó considered the solvability of the fractional differential equation model

$$\begin{cases} {}^C_t\mathcal{D}^{\beta}_{0^+}\left(\phi_p\left({}^C_t\mathcal{D}^{\alpha}_{0^+}u\right)\right)(t)=f(t,u(t)) \quad 0<t<1, \\ \Delta\left(u\left(t_k\right)\right)=I_k\left(u\left(t_k\right)\right), \Delta\phi_p\left({}^C_t\mathcal{D}^{\alpha}_{+}u\right)\left(t_k\right)=b_k, k=1,2,\ldots,m, \\ u(0)=u_0, {}^C\mathcal{D}^{\alpha}_{0^+}u(0)=u_1, \end{cases} \tag{4.1.3}$$

where $p>1$, $0<\alpha,\beta,\gamma\leqq 1, 1<\alpha+\beta,\gamma\leqq 2, \phi_p$ is a p -Laplacian operator, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $u_0,u_1,\lambda\in\mathbb{R}$, for $k=1,2,\ldots,m, i=1,2,$ $I_k\in C(\mathbb{R},\mathbb{R})$ $,0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1$. $\Delta u\left(t_k\right)=u\left(t_k^+\right)-u\left(t_k^-\right), u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ denote the right and the left limits of $u(t)$ at $t=t_k$ $(k=1,2,\ldots,m)$ respectively and $\Delta\phi_p\left({}^C_t\mathcal{D}^{\alpha}_{a^+}u\right)\left(t_k\right)$ has a similar meaning for $\phi_p\left({}^C_t\mathcal{D}^{\alpha}_{a^+}u\right)\left(t_k\right)$.

By applying the Banach contraction principle, they obtained some results on the existence and uniqueness of solutions for the model.

The survey chapter is organized as follows. In Section 4.2, we present some background material for our problems. In Section 4.3, by applying Schauder's, Schaefer's fixed point theorems, and Banach contraction mapping principle, we will prove the existence and uniqueness of solutions for the problem (4.1.1). Finally, two examples are given in Section 4.4 to illustrate the usefulness and the impact of our main results.

4.2 Preliminaries Notes on Fractional Derivatives and background materials

In this section, we introduce some definitions and preliminary needed in our proofs later.

Lemma 4.2.1. *Let $\phi_p:\mathbb{R}\to\mathbb{R}$ be a p-Laplacian operator, $\phi_p(x)=|x|^{p-2}x$, $x\in\mathbb{R}$. Then $\frac{d}{dx}\phi_p(x)=(p-1)|x|^{p-2}(x\neq 0$ if $1<p<2)$.*

The basic properties of the p -Laplacian operator are the following:

1. *The p-Laplacian operator is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and its inverse is $\phi_{p^\star}(x)=|x|^{p^\star-2}x$, with $p^\star=\frac{p}{p-1}$*
2. *If $1<p<2, xy>0, |x|,|y|\geqq m>0$, then*
$$\left|\phi_p(x)-\phi_p(y)\right|\leqq(p-1)m^{p-2}|x-y|.$$
3. *If $p\geqq 2, |x|,|y|\leqq M$, then*
$$\left|\phi_p(x)-\phi_p(y)\right|\leqq(p-1)M^{p-2}|x-y|.$$

In this section, we stady the existence of solutions and uniqueness results of (4.1.1) and prove it, wich has integral an representation include the inverse of a given Nemytskii operator.

Before starting and proving the major results, we shall give and introduce the following lemmas

4.2.1 Fractional functional differential equations

Lemma 4.2.2. *Let $\alpha \in (0,1)$, $\psi_1 \in C^n(J)$ and $w \in C(J,\mathbb{R})$. Then the linear initial value problem*

$$\begin{cases} {}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u(t) = w(t) \quad a < t < T, \\ u(\hat{a}) = u_a, \quad \hat{a} > a, \end{cases} \tag{4.2.1}$$

has a unique solution $u \in C(J,\mathbb{R})$ defined by the integral structure

$$u(t) = u_a - \frac{1}{\Gamma(\alpha)}\int_a^{\hat{a}} \psi_1'(s)(\psi(\hat{a}) - \psi_1(s))^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^{t} \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}h(s)ds.$$

Proof. By applying Lemma 1.3.28 and 1.3.25 in Lemma 1.3.29, the equation (4.2.1) is equivalent to the following integral equation,

$$u(t) = -u(a) + \frac{1}{\Gamma(\alpha)}\int_a^{t} \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}h(s)ds, \tag{4.2.2}$$

boundary condition of (4.2.1) permit us to deduce the value,

$$-u(a) = u_a - \frac{1}{\Gamma(\alpha)}\int_a^{\hat{a}} \psi_1'(s)(\psi(\hat{a}) - \psi_1(s))^{\alpha-1}h(s)ds,$$

then, the unique solution of (4.2.1) is given by the formula,

$$u(t) = u_a - \frac{1}{\Gamma(\alpha)}\int_a^{\hat{a}} \psi_1'(s)(\psi(\hat{a}) - \psi_1(s))^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^{t} \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}h(s)ds.$$

□

Lemma 4.2.3. *Let $\beta \in (0,1)$, $\psi_2 \in C^n(J)$ and $\varpi \in C(J,R)$. Then the linear initial value problem*

$$\begin{cases} {}^{\psi_2;C}_{\ \ t}\mathcal{D}^{\beta}_{T^-}u(t) = \varpi(t) \quad a < t < T, \\ u(\hat{T}) = u_T, \quad \hat{T} < T, \end{cases} \tag{4.2.3}$$

has a unique solution $u \in C(J,R)$ defined by the integral structure

$$u(t) = u_T - \frac{1}{\Gamma(\beta)}\int_{\hat{T}}^{T} \psi_2'(s)(\psi_2(s) - \psi(\hat{T}))^{\beta-1}\varpi(s)ds + \frac{1}{\Gamma(\beta)}\int_t^{T} \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1}\varpi(s)ds.$$

Proof. By applying Lemma 1.3.28 and 1.3.25 in Lemma 1.3.29, the equation (4.2.3) is equivalent to the following integral equation,

$$u(t) = -u(T) + \frac{1}{\Gamma(\beta)}\int_t^{T} \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1}\varpi(s)ds, \tag{4.2.4}$$

boundary condition of (4.2.3) permit us to deduce the value,

$$-u(T) = u_T - \frac{1}{\Gamma(\beta)} \int_{\hat{T}}^{T} \psi_2'(s)(\psi_2(s) - \psi(\hat{T}))^{\beta-1}\varpi(s)ds,$$

then, the unique solution of (4.2.3) is given by the the formula,

$$u(t) = u_T - \frac{1}{\Gamma(\beta)} \int_{\hat{T}}^{T} \psi_2'(s)(\psi_2(s) - \psi(\hat{T}))^{\beta-1}\varpi(s)ds + \frac{1}{\Gamma(\beta)} \int_{t}^{T} \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1}\varpi(s)ds.$$

□

4.3 Major results

Lemma 4.3.1. *Let $\alpha \in (0,1]$ and $h : J \to \mathbb{R}$ be continuous. A function $u \in PC(J,\mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = \begin{cases} u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1}\eta(s) \left|u(s)\right|^{p-1} \mathrm{d}s \right|^{p^\star-1} \\ \quad + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}h(s)\mathrm{d}s \\ \quad\quad + \sum_{j=0}^{k} I_j^1\left(u\left(t_j\right)\right) H(t-t_j), \quad \text{for } \forall t \in J, k = 0,1,2,\ldots,m, \end{cases} \tag{4.3.1}$$

if and only if u is a solution of the following impulsive problem

$$\begin{cases} {}^{\psi_1;C}{}_t\mathcal{D}^\alpha_{a^+}u(t) = h(s) \quad a < t < T, \\ \Delta\left(u\left(t_k\right)\right) = I_k^1\left(u\left(t_k\right)\right), k = 1,2,\ldots,m \\ u(a) = u_0 + \lambda \left| {}^{\psi_3}{}_t\mathcal{I}^\gamma_{a^+}\eta(t) \mid u(t)|^{p-1} \right|^{p^\star-1}_{t=T}, \end{cases} \tag{4.3.2}$$

where H the Heaviside function and $I_0^1 = 0$.

Proof. Let $h \in C(J,\mathbb{R})$. Assume that $u(t)$ is a solution of impulsive problem 4.3.2 . If $t \in [a,t_1]$ then

$${}^{\psi_1;C}{}_t\mathcal{D}^\alpha_{a^+}u(t) = h(t) \tag{4.3.3}$$

by applying Lemma 4.2.2 we get

$$u(t) = u(a) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}h(s)ds,$$

If $t \in (t_1,t_2]$ then

$${}^{\psi_1;C}{}_t\mathcal{D}^\alpha_{a^+}u(t) = h(t) \text{ with } \Delta u\left(t_1\right) = I_1^1\left(u\left(t_1\right)\right)$$

then Lemma 4.2.2 implies

$$\begin{aligned}
u(t) &= u(t_1^+) - \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds,\\
&= u\left(t_1^-\right) + I_1^1\left(u\left(t_1\right)\right) - \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds,\\
&= u(a) + I_1^1\left(u\left(t_1\right)\right) + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds.
\end{aligned}$$

If $t \in (t_2, t_3]$ then by using Lemma 4.2.2 one more, we get

$$\begin{aligned}
u(t) &= u(t_2^+) - \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds,\\
&= u\left(t_2^-\right) + I_2^1\left(u\left(t_1\right)\right) - \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)ds,\\
&= u(a) + I_1^1\left(u\left(t_1\right)\right) + I_2^1\left(u\left(t_2\right)\right) + \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)\mathrm{d}s.
\end{aligned}$$

If $t \in J_k$, for $k = 1, 2, \cdots, m$, then again from Lemma 4.2.2 and note that the boundary value condition

$$u(a) = u_0 + \lambda \left|{}^{\psi_3}{}_t\mathcal{I}_{a^+}^{\gamma}\eta(t) \mid u(t)|^{p-1}\right|_{t=T}^{p^\star-1},$$

one can see that

$$u(t) = u(a) + \sum_{j=1}^{k} I_j^1\left(u\left(t_j\right)\right) + \int_a^t \psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}h(s)\mathrm{d}s.$$

$\therefore$ by simple calculations we get:

$$u(a) = u_0 + \lambda \left|\frac{1}{\Gamma(\gamma)}\int_a^T \psi_3'(s)(\psi_3(T)-\psi_3(s))^{\gamma-1}\eta(s)\,|u(s)|^{p-1}\,\mathrm{d}s\right|^{p^\star-1}$$

hence, from the proposition 1.1.3 we get (4.3.1).

Conversely, assume that u satisfies (4.3.1). If $t \in [a, t_1]$ then $u(a) = u_0 + \lambda \left|{}^{\psi_3}{}_t\mathcal{I}_{a^+}^{\gamma}\eta(t) \mid u(t)|^{p-1}\right|_{t=T}^{p^\star-1}$ and using the fact that ${}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha}$ is the left inverse of ${}^{\psi_1}{}_t\mathcal{I}_{a^+}^{\alpha}$ we get (4.3.3). If $t \in J_k, k = 1, 2, \cdots, m$ and using the fact of the (ψ-C) Caputo derivative of a constant is equal to zero, we obtain ${}^{\psi_1;C}{}_t\mathcal{D}_{a^+}^{\alpha}u(t) = h(t), t \in (t_k, t_{k+1}]$ and $\Delta(u(t_k)) = I_k^1(u(t_k))$. This completes the proof. □

Lemma 4.3.2. *Let $\varphi(t) \in C(J,\mathbb{R}), \alpha, \beta \in (0,1]$. Then the fractional impulsive differential equation*

$$\begin{cases} {}^{\psi_2;C}{}_t\mathcal{D}^{\beta}_{T^-}\left(\rho(t)\phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}^{\alpha}_{a^+}u\right)\right)(t) = \varphi(t), t \in J, t \neq t_k, \\ \Delta\left(u\left(t_k\right)\right) = I^1_k\left(u\left(t_k\right)\right), \Delta\phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}^{\alpha}_{a^+}u\left(t_k\right)\right) = I^2_k(u(t_k)), k = 1,2,\ldots,m, \\ u(a) = u_0 + \lambda\left|{}^{\psi_3}{}_t\mathcal{I}^{\gamma}_{a^+}\eta(t)\mid u(t)|^{p-1}\right|^{p^\star-1}_{t=T}, {}^{\psi_1;C}{}_t\mathcal{D}^{\alpha}_{a^+}u(T) = u_1, \end{cases} \tag{4.3.4}$$

is equivalent to the following integral equation:

$$u(t) = \begin{cases} u_0 + \lambda\left|\frac{1}{\Gamma(\gamma)}\int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1}\eta(s)\,|u(s)|^{p-1}\,\mathrm{d}s\right|^{p^\star-1}, \\ \quad + \frac{1}{\Gamma(\alpha)}\int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}\phi_{p^\star}(\mathfrak{F}\varphi(s))\mathrm{d}s, \\ \quad\quad + \sum_{j=0}^{k} I^1_j\left(u\left(t_j\right)\right)H(t - t_j) \quad t \in J, k = 0,1,\ldots,m, \end{cases} \tag{4.3.5}$$

where

$$\mathfrak{F}\varphi(t) = \frac{1}{\rho(t)}\begin{cases} \rho(T)\phi_p\left(u_1\right) + \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}\varphi(\tau)\mathrm{d}\tau \\ + \sum_{j=0}^{k} I^2_j(u(t_j)H(t - t_j),\ t \in\ J, k = 0,1,\ldots,m \end{cases}, \tag{4.3.6}$$

H is the Heaviside function and $I^i_0 = 0$, $i = 1,2$.

Proof. Let $\alpha, \beta \in (0,1]$ and $\varphi \in C(J,\mathbb{R})$. Let $v(t) = \phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}^{\alpha}_{a^+}u(t)\right)$, then from 4.3.4 we get the following impulsive problem

$${}^{\psi_2;C}{}_t\mathcal{D}^{\beta}_{T^-}\rho(t)v(t) = \varphi(t) \quad a < t < T, t \neq t_k, \tag{4.3.7}$$

$$\Delta\left(v\left(t_k\right)\right) = b_k, k = 1,2,\ldots,m, \tag{4.3.8}$$

$$v(T) = \phi_p\left(u_1\right), \tag{4.3.9}$$

where $b_k = I^2_k(u(t_k), k = 1,2,\ldots,m$, $b_0 = 0$
Then by applying Lemma 4.2.3, (4.3.7)-(4.3.9) are equivalent to

$$\rho(t)v(t) = \begin{cases} \rho(T)\phi_p\left(u_1\right) + \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}\varphi(\tau)\mathrm{d}\tau \\ + \sum_{j=0}^{k} b_j(u(t_j)H(t - t_j),\ t \in\ J, k = 1,\ldots,m. \end{cases} \tag{4.3.10}$$

Hence, by identification $v(t) = \phi_p\left({}^{\psi_1;C}{}_t\mathcal{D}^{\alpha}_{a^+}u(t)\right) = \mathfrak{F}(\varphi(t)), \quad t \in J, k = 0,1,2,\ldots,m$, the problem

(4.3.4) is equivalent to the following:

$$\begin{cases} {}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u(t) = \phi_{p^\star}\left(\mathfrak{F}(\varphi(t))\right), \quad a < t < T, t \neq t_k, \\ \Delta\left(u\left(t_j\right)\right) = I_j^1\left(u\left(t_j\right)\right), k = 1, 2, \dots, m, \\ u(a) = u_0 + \lambda \left| {}^{\psi_3}{}_t\mathcal{I}^{\gamma}_{a^+}\eta(t) \mid u(t)|^{p-1} \right|^{p^\star - 1}_{t=T}. \end{cases} \tag{4.3.11}$$

Then, by applying Lemma 4.2.2 again, we get the desired result. □

4.3.1 Existence and uniqueness results

Lemma 4.3.3. *Assume that $f \in C(J \times \mathbb{R}, \mathbb{R})$. Then $u(t) \in PC(J, \mathbb{R})$ is a solution of the boundary value problem (4.1.1) if and only if $u(t)$ is a solution of the integral equation*

$$u(t) = \begin{cases} u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1}\eta(s) \left|u(s)\right|^{p-1} \mathrm{d}s \right|^{p^\star - 1} \\ \quad + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}\phi_{p^\star}(\mathfrak{F}\mathcal{N}u(s))\mathrm{d}s \\ \quad + \sum_{j=0}^{k} I_j^1\left(u\left(t_j\right)\right) H(t - t_j) \quad t \in J, k = 0, 1, \dots, m, \end{cases} \tag{4.3.12}$$

where

$$\mathfrak{F}\mathcal{N}u(t) = \frac{1}{\rho(t)} \begin{cases} \rho(T)\phi_p\left(u_1\right) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}\mathcal{N}u(\tau)\mathrm{d}\tau \\ \quad + \sum_{j=0}^{k} I_j^2(u(t_j)H(t - t_j), \ t \in J, k = 1, \dots, m, \end{cases} \tag{4.3.13}$$

and $\mathcal{N}$ is the Nemytskii operator associated to (4.1.1) defined by

$$\mathcal{N}(u(t)) = f(t, u(t)) - s(t)\phi_p\left(u(t)\right), \quad t \in J, t \neq t_k, \text{ for } k = 1, 2, \cdots, m, u \in PC(J, \mathbb{R}). \tag{4.3.14}$$

Now, we consider the integral operator $\mathcal{L} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as

$$\mathcal{L}u(t) = \begin{cases} u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1}\eta(s) \left|u(s)\right|^{p-1} \mathrm{d}s \right|^{p^\star - 1} \\ + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}\phi_{p^\star}(\mathfrak{F}\mathcal{N}u(s))\mathrm{d}s \\ \quad + \sum_{j=0}^{k} I_j^1\left(u\left(t_j\right)\right) H(t - t_j) \quad t \in J, k = 0, 1, \dots, m. \end{cases} \tag{4.3.15}$$

Clearly, a fixed point of the operator $\mathcal{L}$ is a solution of the problem (4.1.1)

Lemma 4.3.4. *The operator $\mathcal{L} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.*

Proof. Firstly, we will prove that, the operator $\mathcal{L}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous. Let $\{u_n\} \subseteqq PC(J,\mathbb{R})$ be a sequence with $u_n \to u$ in $PC(J,\mathbb{R})$, one can see that in view of the hypothesis of continuity of $f(t,u)$, I_k^i, for $i = 1,2$ and the first item of Lemma 4.2.1

$$\lim_{n\to\infty} \mathcal{L}u_n(t) = \lim_{n\to\infty} \left(\begin{array}{c} u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1} \eta(s) \left| u_n(s) \right|^{p-1} \mathrm{d}s \right|^{p^\star - 1} \\ + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1} \phi_{p^\star}(\mathfrak{F}\mathcal{N}u_n(s))\mathrm{d}s \\ + \sum_{j=0}^{k} I_j^1 \left(u_n\left(t_j\right)\right) H(t-t_j), \end{array} \right)$$

$$= u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1} \eta(s) \left| u_n(s) \right|^{p-1} \mathrm{d}s \right|^{p^\star - 1}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1} \phi_{p^\star}(\mathfrak{F}\mathcal{N}u_n(s))\mathrm{d}s$$

$$+ \sum_{j=0}^{k} I_j^1 \left(u_n\left(t_j\right)\right) H(t-t_j) = \mathcal{L}u(t),$$

uniformly for $t \in J$, $k = 0,1,\ldots,m$. This shows that $\mathcal{L}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous.

Next, we show that $\mathcal{L}$ is compact, let $\Omega = \{u \in PC(J,\mathbb{R}), \| u \| < \mathcal{R}\}$, then from the continuity of f and I_k^i, there exist $M_0, M_1, M_2 > 0$ such that $| f(t,u(t) | \leqq M_0$ and $| I_k^i(u(t_k) | \leqq M_i$ $(k = 1,2,\cdots,m$, $i = 1,2)$ for $t \in J$ and any $u \in \Omega$. Then we have

$$| \mathfrak{F}\mathcal{N}u(t) | = \frac{1}{|\rho(t)|} \left| \rho(T)\phi_p(u_1) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1} f(\tau, u(\tau) - s(\tau)\phi_p(u(\tau))\ \mathrm{d}\tau \right.$$
$$\left. + \sum_{j=0}^{k} I_j^2(u(t_j)H(t-t_j),\ t \in J, k = 1,\ldots,m \right|, \tag{4.3.16}$$

$$\leqq M_3 \left(\rho(T)\phi_p(| u_1 |) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(\tau)(\psi_2(s) - \psi_2(t))^{\beta-1} \left(| f(\tau,u(\tau) | + s(\tau)\phi_p(| u(\tau) |) \right) \mathrm{d}\tau \right.$$
$$\left. + \sum_{j=1}^{m} | I_j^2(u(t_j) | \right),$$
$$\leqq M_3 \left(\rho(T) | u_1 |^{p-1} + \frac{M_0 + M_5 \mathfrak{R}^{p-1}}{\Gamma(\beta+1)} (\psi_2(T) - \psi_2(a))^\beta + mM_2 \right) := L,$$

where $M_3 = \frac{1}{\min_{t\in J}}$, $M_5 = \max_{t\in J}(s(t)$, M_0 and M_2 are given above, which implies that,

$$\begin{aligned}
\mid \mathcal{L}u(t) \mid \leqq \mid u_0 \mid + \mid \lambda \mid & \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1}\eta(s)\, |u(s)|^{p-1}\, \mathrm{d}s \right|^{p^\star-1} + \sum_{j=1}^{m} \mid I_j^1(u(t_j) \mid \\
& + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1}\phi_{p^\star}\left(|\mathfrak{F}\mathcal{N}u(s)|\right) \mathrm{d}s \\
\leqq \mid u_0 \mid + \mid \lambda \mid & \frac{\mathfrak{R}M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T) - \psi_3(a))^{\gamma} + mM_1 + \frac{1}{\Gamma(\alpha+1)}(\psi_1(T) - \psi_1(a))^{\alpha} L^{p^\star-1} := L^\star,
\end{aligned}$$

here $M_4 = \max_{t\in J} \eta(t)$.

Then, we get that $\mid \mathcal{L}u \mid \leqq L^\star$ for any $u \in \Omega$. This means that $\mathcal{L}(\Omega)$ is uniformly bounded in $PC(J, \mathbb{R})$.

Next, we need to prove that $\mathcal{L}(\Omega) \subset PC(J, \mathbb{R})$ is equicontinuous in J_k by PC-type Arzelá-Ascoli theorem, for that let $u \in \Omega$ and $\tau_1, \tau_2 \in [a, t_1]$ such that $a \leqq \tau_1 < \tau_2 \leqq t_1$, then we have

$$\begin{aligned}
\mid \mathcal{L}u(\tau_2) - \mathcal{L}u(\tau_1) \mid = & \left| \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} \psi_1'(s) \left((\psi_1(\tau_2) - \psi_1(s))^{\alpha-1} - (\psi_1(\tau_1) - \psi_1(s))^{\alpha-1} \right) \phi_{p^\star}\left(\mathfrak{F}\mathcal{N}u(s)\right) \mathrm{d}s \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi_1'(s)(\psi_1(\tau_2) - \psi_1(s))^{\alpha-1}\phi_{p^\star}\left(\, \mathfrak{F}\mathcal{N}u(s)\right) \mathrm{d}s \right| \\
\leqq & \frac{L^{p^\star-1}}{\Gamma(\alpha)} \left(\int_a^{\tau_1} \left| \psi_1'(s) \left((\psi_1(\tau_2) - \psi_1(s))^{\alpha-1} - (\psi_1(\tau_1) - \psi_1(s))^{\alpha-1} \right) \right| \mathrm{d}s \right. \\
& \left. + \int_{\tau_1}^{\tau_2} \psi_1'(s)(\psi_1(\tau_2) - \psi_1(s))^{\alpha-1} \mathrm{d}s \right) \\
= & \frac{L^{p^\star-1}}{\Gamma(\alpha)} \left(2\left(\psi_1(\tau_2) - \psi_1(\tau_1)\right)^{\alpha} - (\psi_1(\tau_2) - \psi_1(a))^{\alpha} + (\psi_1(\tau_1) - \psi_1(a))^{\alpha} \right).
\end{aligned}$$

Then, simalarly for the interval J_k, $t_k \leqq \tau_1 < \tau_2 \leqq t_{k+1}$, $k = 1, 2, \cdots, m$ we get

$$\mid \mathcal{L}u(\tau_2) - \mathcal{L}u(\tau_1) \mid \leqq \frac{L^{p^\star-1}}{\Gamma(\alpha)} \left(2\left(\psi_1(\tau_2) - \psi_1(\tau_1)\right)^{\alpha} - (\psi_1(\tau_2) - \psi_1(a))^{\alpha} + (\psi_1(\tau_1) - \psi_1(a))^{\alpha} \right).$$

As $\psi_1(t)^\alpha$ is uniformly continuous on J_k and $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. Therefore $T(\Omega)$ is equicontinuous. Therefore, the PC-type Arzelá -Ascoli theorem permit us to say that $\mathcal{L}(\Omega)$ is relatively compact in $PC(\,J, \mathbb{R})$. □

In order to stady the existence and uniqueness results of solutions to problem (4.1.1), we list the following assumptions that:

(H$_1$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative constants $r, e \in \mathbb{R}$, $0 \leq \ell < p - 1$ such that $|f(t, u)| \leq r + e|u|^{\ell}$, $t \in J, u \in \mathbb{R}$;

(H$_2$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, there exist constants $r^i, e^i \geqq 0$, $0 \leqq \ell^1 < 1$ and $0 \leqq \ell_k^2 < p - 1$ such that $|I_k^i(u)| \leqq r_k^i + e_k^i|u|^{\ell_k^i}, u \in \mathbb{R}$.

($\mathbf{H}_3$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative constants $r, e, \mathcal{R}_1 \in \mathbb{R}$ and $0 \leq \ell < p-1$ such that $|f(t,u)| \leq r + e|u|^{\ell}$, $t \in J, u \in [0, \mathcal{R}_1]$;

($\mathbf{H}_4$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, there exist constants $r_k^i, e_k^i, \mathcal{R}_1 \geqq 0$, $0 \leqq \ell_k^1 < 1$ and $0 \leqq \ell_k^2 < p-1$ such that $|I_k^i(u) \leqq r_k^i + e_k^i|u|^{\ell_k^i}, u \in [0, \mathcal{R}_1]$;

($\mathbf{H}_3^{\star}$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative constants $r, e, \mathcal{R}_1 \in \mathbb{R}$ such that $|f(t,u)| \leq r + e|u|^{p-1}$, $t \in J, u \in [0, \mathcal{R}_1]$;

($\mathbf{H}_4^{\star}$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, there exist constants $r_k^i, e_k^i, \mathcal{R}_1 \geqq 0$ and $0 \leqq \ell_k^1 < 1$ such that $|I_k^i(u) \leqq r_k^i + e_k^i|u|^{p-1}, u \in [0, \mathcal{R}_1]$;

($\mathbf{H}_5$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative constant $L \in \mathbb{R}$ such that

$$|f(t,u) - f(t,v)| \leqq L|u - v|, \qquad t \in J, u, v \in \mathbb{R};$$

($\mathbf{H}_6$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, there exist constants $L_k^i > 0$ such that

$$|I_k^i(u) - I_k^i(u)| \leqq L_k^i|u - v|, \qquad u, v \in \mathbb{R}$$

($\mathbf{H}_5^{\star}$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative function $\Phi(t) \in C(\,J)$ and there exist nonnegative constant $L \in \mathbb{R}$ such that

$$\begin{aligned}
&0 < f(t,u) - s(t)\phi_p\left(u(t)\right) \leqq \Phi(t), \qquad t \in J, u \in \mathbb{R},\\
&|f(t,u) - f(t,v)| \leqq L|u - v|, \qquad t \in J, u, v \in \mathbb{R}\\
&u_0, u_1 > 0, \lambda \geqq 0;
\end{aligned}$$

($\mathbf{H}_6^{\star}$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$, the exist a positive functions $\Psi_k^i \in C(J, \mathbb{R})$ and there exist constants $L_k^i, e_k^1 > 0$ such that

$$\begin{aligned}
&0 \leqq I_k^1(u) \leqq \Psi_k^1(t) + e_k^1 \parallel u \parallel, \qquad (t,u) \in \mathbb{R} \times \mathbb{R},\\
&0 \leqq I_k^2(u) \leqq \Psi_k^2(t), \qquad (t,u) \in \ J \times \mathbb{R},\\
&|I_k^i(u) - I_k^i(u)| \leqq L_k^i|u - v|, \qquad u, v \in \mathbb{R};
\end{aligned}$$

($\mathbf{H}_5^{\star\star}$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative constants $L \in \mathbb{R}$ and e such that

$$\begin{aligned}
&0 < f(t,u) - s(t)\phi_p\left(u(t)\right), \qquad t \in J, u \in \mathbb{R},\\
&0 < |f(t,u)| \leqq e|u|^{p-1}, \qquad t \in J, u \in \mathbb{R},\\
&|f(t,u) - f(t,v)| \leqq L|u - v|, \qquad t \in J, u, v \in \mathbb{R}\\
&u_0 > u_1 > 0, \lambda \geqq 0;
\end{aligned}$$

($\mathbf{H}_6^{\star\star}$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$,the exist a positive functions $\Psi_k^1 \in C(J, \mathbb{R})$ and there exist constants $L_k^i, e_k^i > 0$ such that

$$\begin{aligned}
&0 \leqq I_k^1(u) \leqq \Psi_k^1(t) + e_k^1 \parallel u \parallel, \qquad (t,u) \in \mathbb{R} \times \mathbb{R},\\
&0 \leqq I_k^2(u) \leqq e_k^2|u|^{p-1}, \qquad (t,u) \in \ J \times \mathbb{R},\\
&|I_k^i(u) - I_k^i(u)| \leqq L_k^i|u - v|, \qquad u, v \in \mathbb{R};
\end{aligned}$$

($\mathbf{H}_5^{\star\star\star}$) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative function $\Pi(t) \in C(J)$ and there exist nonnegative constant $L \in \mathbb{R}$ such that such that

$$\begin{aligned}
&-\Pi(t) \leqq f(t,u) - s(t)\phi_p\left(u(t)\right) < 0, \qquad t \in J, u \in \mathbb{R},\\
&|f(t,u) - f(t,v)| \leqq L|u-v|, \qquad t \in J, u, v \in \mathbb{R}\\
&u_0, u_1 < 0, \lambda \leqq 0;
\end{aligned}$$

($\mathbf{H}_6^{\star\star\star}$) for $k = 1, 2, \cdots, m$, $i = 1, 2$, $I_k^i \in C(\mathbb{R}, \mathbb{R})$,the exist a positive functions $\chi_k^i \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $L_k^i, e_k^1 > 0$ such that

$$\begin{aligned}
&-\chi_k^1(t) - e_k^1 \parallel u \parallel \leqq I_k^1(u) \leqq 0, \qquad (t,u) \in \ J \times \mathbb{R},\\
&-\chi_k^2(t) \leqq I_k^2(u) \leqq 0, \qquad (t,u) \in \ J \times \mathbb{R},\\
&|I_k^i(u) - I_k^i(u)| \leqq L_k^i |u-v|, \qquad u, v \in \mathbb{R};
\end{aligned}$$

Theorem 4.3.5 (The first existence result)**.** *Suppose that* $(\boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2)$ *hold. If*

$$\frac{|\ \lambda\ |\ M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T) - \psi_3(a))^\gamma < 1, \tag{4.3.17}$$

then, the problem (4.1.1) *has at least one solution.*

Proof. Firstly, the Lemma 4.3.4 implies that the integral operator $T\mathcal{L}(\ J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. Next, suppose that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold, then we show the set $E(\mathcal{L}) = \{u \in PC(J, \mathbb{R}) : u = \sigma\mathcal{L}u \text{ for some } \sigma \in [0,1]\}$ is bounded. Let $u \in E(\mathcal{L})$, then we have $u = \sigma\mathcal{L}u$ for each $t \in J$, $k = 0, 1, 2, \cdots, m$ we have

$$\begin{aligned}
|\ \mathfrak{F}\mathcal{N}u(t)\ | \leqq M_3 \Bigg(&\rho(T)\phi_p\left(|\ u_1\ |\right) + \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1}\left(|\ f(\tau, u(\tau)\ |\right.\\
&\left. + s(\tau)\phi_p\left(|\ u(\tau)\ |\right)\right) \mathrm{d}\tau + \sum_{j=1}^m |\ I_j^2(u(t_j)\ |\Bigg),\\
\leqq M_3 \Bigg(&\rho(T)\ |\ u_1\ |^{p-1} + \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1}\Big(r + e|u(\tau)|^\ell\\
&+ s(\tau)\phi_p\left(|\ u(\tau)\ |\right)\Big) \mathrm{d}\tau + \sum_{j=1}^m r_k^2 + e_k^2|u(t_k)|^{\ell_k^2}\Bigg),\\
\leqq M_3 \Bigg(&\rho(T)\ |\ u_1\ |^{p-1} + \frac{r + e \parallel u \parallel^\ell + M_5 \parallel u \parallel^{p-1}}{\Gamma(\beta+1)}(\psi_2(T) - \psi_2(a))^\beta\\
&+ \sum_{j=1}^m r_j^2 + e_j^2 \parallel u \parallel^{\ell_j^2}\Bigg),
\end{aligned}$$

we can conclude that there exists a positive constant ϖ such that $\| \mathfrak{F}\mathcal{N}u \| < \varpi$. Therefore

$$
\begin{aligned}
\mid u(t) \mid = \sigma \mid \mathcal{L}u(t) \mid = \sigma \Big| u_0 + \Big| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T)-\psi_3(s))^{\gamma-1}\eta(s)\, |u(s)|^{p-1}\, \mathrm{d}s \Big|^{p^\star-1} \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}\phi_{p^\star}(\mathfrak{F}\mathcal{N}u(s))\, \mathrm{d}s \Big| \\
\leqq \sigma \mid u_0 \mid + \sigma \Big| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T)-\psi_3(s))^{\gamma-1}\eta(s)\, |u(s)|^{p-1}\, \mathrm{d}s \Big|^{p^\star-1} \\
+ \frac{\sigma}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}\phi_{p^\star}(|\mathfrak{F}\mathcal{N}u(s)|)\, \mathrm{d}s + \sigma \sum_{j=1}^m \mid I_j^1(u(t_j) \mid \\
\leqq \mid u_0 \mid + \mid \lambda \mid \frac{\| u \| M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^\gamma + \frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^\alpha \varpi^{p^\star-1} \\
+ \sum_{j=1}^m r_j^1 + e_j^1 \| u \|^{\ell_j^1},
\end{aligned}
$$

in consequence,

$$
\begin{aligned}
\| u(t) \| \leqq \mid u_0 \mid + \mid \lambda \mid \frac{\| u \| M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^\gamma + \frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^\alpha \varpi^{p^\star-1} \\
+ \sum_{j=1}^m r_j^1 + e_j^1 \| u \|^{\ell_j^1}, \text{ for } t \in J,\ k = 1, 2, \cdots, m.
\end{aligned}
$$

By taking into account that $0 \leqq \ell_j^1 < 1$ and $\frac{|\lambda| M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^\gamma < 1$, we can deduce that there exists a positive constant $\varpi^\star$ such that $\| u \| \leqq \varpi^\star$ for any solution of the functional equation $u = \sigma \mathcal{L}u$, $0 < \sigma < 1$. Hence, by Theorem 1.5.9, as result we get the existence of a fixed point for $\mathcal{L}$, which implies the existence of at least of one solution of the problem (4.1.1) □

Before start the second existence result, we put the following, for the sake of convenience :

$$
\begin{aligned}
&C^1 = M_3 \left[\rho(T) \mid u_1 \mid^{p-1} + \sum_{j=1}^m r_j^2 + \frac{r}{\Gamma(\beta+1)}(\psi_2(T)-\psi_2(a))^\beta \right]; \\
&C^2 = \frac{4eM_3}{\Lambda^{p-1}\Gamma(\beta+1)}(\psi_2(T)-\psi_2(a))^\beta;\ C^3 = \frac{M_5 M_3}{\Lambda^{p-1}\Gamma(\beta+1)}(\psi_2(T)-\psi_2(a))^\beta; \\
&C_j^4 = \frac{4M_3 e_j^2 m}{\Lambda^{p-1}};\ C^5 = \mid u_0 \mid + \sum_{j=1}^m r_j^1,\ C^6 = 4 \mid \lambda \mid \frac{M_4^{p^\star-1}}{\Gamma(\gamma+1)}; \\
&C_j^7 = 4m \sum_{j=1}^m e_j^1,\ \Lambda = 4\frac{\Gamma(\alpha+1)}{(\psi_1(T)-\psi_1(a))^\alpha}, \text{ for } j = 1,\ 2,\ \cdots,\ m.
\end{aligned}
$$

Theorem 4.3.6 (The second existence result)**.** *Suppose that* $(\boldsymbol{H}_3)$ *and* $(\boldsymbol{H}_4)$ *hold. If*

$$C^2, C^6 \leqq 1, \tag{4.3.18}$$

then, the problem (4.1.1) *has at least one solution.*

Proof. We shall prove that BVP(4.1.1) has at least one solution. Suppose that $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ hold and C^2, C^6 satisfies (4.3.18), let $\Omega_1 = \{u \in PC(J, \mathbb{R}), \| u \| < \mathcal{R}_1\}$, such that

$$\mathcal{R}_1 \geqq \max \left\{ \frac{4\left(C^1\right)^{p^\star}}{\Lambda}, \; \left(C^3\right)^{\frac{1}{p-1-\ell}}, \; \left(C_j^4\right)^{\frac{1}{p-1-\ell_j^2}}, \; 4C^5, \; \left(C_j^7\right)^{\frac{1}{1-\ell_j^1}} \right\}, \text{ for } j = 1,\, 2,\, \cdots,\, m,$$

for $\forall u \in \Omega$, $t \in J$, we have

$$\begin{aligned}
|\, \mathfrak{F}\mathcal{N}u(t) \,| \leqq{}& M_3 \Bigg(\rho(T)\phi_p\left(|\, u_1 \,|\right) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1} \times \\
&\qquad \left(|\, f(\tau, u(\tau) \,| + s(\tau)\phi_p\left(|\, u(\tau) \,|\right)\right) \mathrm{d}\tau + \sum_{j=1}^m |\, I_j^2(u(t_j) \,| \Bigg), \\
\leqq{}& M_3 \Bigg(\rho(T) \,|\, u_1 \,|^{p-1} + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(s)(\psi_2(s) - \psi_2(t))^{\beta-1} \times \\
&\qquad \left(r + e|u(\tau)|^{\ell} + s(\tau)\phi_p\left(|\, u(\tau) \,|\right) \right) \mathrm{d}\tau + \sum_{j=1}^m r_k^2 + e_k^2 |u(t_k)|^{\ell_k^2} \Bigg), \\
\leqq{}& M_3 \Bigg(\rho(T) \,|\, u_1 \,|^{p-1} + \sum_{j=1}^m r_j^2 + \sum_{j=1}^m e_j^2 \,\|\, u \,\|^{\ell_j^2} \\
&\qquad + \frac{r + e \,\|\, u \,\|^{\ell} + M_5 \,\|\, u \,\|^{p-1}}{\Gamma(\beta+1)} (\psi_2(T) - \psi_2(a))^{\beta} \Bigg) \\
\leqq{}& C^1 + \frac{\Lambda^{p-1}}{4} C^2 \mathcal{R}_1^{\ell} + C^3 \frac{\Lambda^{p-1}}{4} \mathcal{R}_1^{p-1} + \sum_{j=1}^m \frac{\Lambda^{p-1}}{4m} C_j^4 \Lambda^{p-1} \mathcal{R}_1^{\ell_j^2}, \\
\leqq{}& \frac{(\Lambda \mathcal{R}_1)^{p-1}}{4} + \frac{\Lambda^{p-1}}{4} \mathcal{R}_1^{p-1-\ell} \mathcal{R}_1^{\ell} + C^3 (\Lambda \mathcal{R}_1)^{p-1} + \sum_{j=1}^m \frac{\Lambda^{p-1}}{4} m \mathcal{R}_1^{p-1-\ell_j^2} \mathcal{R}_1^{\ell_j^2}, \\
\leqq{}& (\Lambda 4 \mathcal{R}_1)^{p-1},
\end{aligned}$$

$$\begin{aligned}
|\, \mathcal{L}u(t) \,| \leqq{}& |\, u_0 \,| + |\, \lambda \left| \frac{\lambda}{\Gamma(\gamma)} \int_a^T \psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1} \eta(s) \, |u(s)|^{p-1} \, \mathrm{d}s \right|^{p^\star - 1} + \sum_{j=1}^m |\, I_j^1 u(t_j) \,| \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1} \phi_{p^\star}\left(|\mathfrak{F}\mathcal{N}u(s)|\right) \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqq | u_0 | + | \lambda | \frac{\| u \| M_4^{p^\star - 1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^\gamma + \sum_{j=1}^{m} r_j^1 + e_j^1 \| u \|^{\ell_j^1} \\
&\qquad\qquad\qquad + \frac{1}{\Gamma(\alpha+1)} (\psi_1(T) - \psi_1(a))^\alpha \Lambda \mathcal{R}_1 \\
&\leqq | u_0 | + \sum_{j=1}^{m} r_j^1 + | \lambda | \frac{\mathcal{R}_1 M_4^{p^\star - 1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^\gamma + \sum_{j=1}^{m} e_j^1 \mathcal{R}_1^{\ell_j^1} \\
&\qquad\qquad\qquad + \frac{1}{\Gamma(\alpha+1)} (\psi_1(T) - \psi_1(a))^\alpha \Lambda \mathcal{R}_1 \\
&= C^5 + \frac{C^6}{4} \mathcal{R}_1 + \sum_{j=1}^{m} \frac{C_j^7}{4m} \mathcal{R}_1^{\ell_j^1} + \frac{1}{4} \mathcal{R}_1 \\
&\leqq \frac{\mathcal{R}_1}{4} + \frac{\mathcal{R}_1}{4} + \sum_{j=1}^{m} \frac{1}{4m} \mathcal{R}_1^{1-\ell_j^1} \mathcal{R}_1^{\ell_j^1} + \frac{1}{4} \mathcal{R}_1 \doteq \mathcal{R}_1.
\end{aligned}
$$

Thus,

$$\| \mathcal{L} u(t) \| \leqq \mathcal{R}_1$$

which implies that $\mathcal{L}(\Omega) \subseteq \Omega$ for every $u \in \Omega_1$. Hence, from Lemma 4.3.4 the integral operator $\mathcal{L} : \Omega \to \Omega$ is completely continuous. According to the fixed point Theorem 1.5.8, $\mathcal{L}$ has at least one fixed point which is a solution of the problem (4.1.1). □

Remark 4.3.7. Assume that the conditions($\mathbf{H}_3^\star$) and ($\mathbf{H}_4^\star$) hold. If

$$C^2, \;\; C^3, \;\; \sum_{j=1}^{m} C_j^4, \; C^6, \; \sum_{j=1}^{m} C_j^7 \leqq 1. \tag{4.3.19}$$

Then, the problem (4.1.1) also has at least one solution, which can be shown by use of the same process of the proof of Theorem 4.3.6.

Theorem 4.3.8. *Assume that $f(t,u)$ continuous function on $J \times \mathbb{R}$ and $I_k^i(u)$ are continuous functions on $\mathbb{R}$, let $\lim_{u\to 0} \frac{f(t,u)}{r+e|u|^\ell} = 0$ and $\lim_{u\to 0} \frac{I_k^i(u)}{r_k^i + e_k^i |u|^{\ell_k^i}} = 0$, for $i = 1,2$, $k = 1,2,\cdots,m$, where r, e, r_k^i, e_k^i are nonegative constants and $0 \leqq \ell, \ell_k^2 < p-1$, $0 \leqq \ell_k^1 < 1$. Then, the problem (4.1.1) has at least one solution.*

Proof. Now, in view of $\lim_{u\to 0} \frac{f(t,u)}{r+e|u|^\ell} = 0$ and $\lim_{u\to 0} \frac{I_k^i(u)}{r_k^i + e_k^i |u|^{\ell_k^i}} = 0$, for $i = 1,2$, $k = 1,2,\cdots,m$, there exists a constant $\mathcal{R}_1 > 0$ such that $| f(t,u) | \leqq \epsilon \left(r + e \mid u \mid^\ell \right)$, $| I_k^i(u) | \leqq \epsilon^i \left(r_k^i + e_k^i \mid u \mid^{\ell_k^i} \right)$ for $0 < | u | < \mathcal{R}_1$, where $\epsilon, \epsilon^i > 0$.

As $f(t,u)$ continuous function on $J \times \mathbb{R}$ and $I_k^i(u)$ are continuous functions on $\mathbb{R}$, we have that, the conditions ($\mathbf{H}_3$) and ($\mathbf{H}_4$) hold. Then the proof is analogous to that of Theorem 4.3.6. □

Remark 4.3.9. Assume that $f(t,u)$ continuous function on $J \times \mathbb{R}$ and $I_k^i(u)$ are continuous functions on $\mathbb{R}$, for $i = 1,2$, $k = 1,2,\cdots,m$, let $\lim_{u\to 0} \frac{f(t,u)}{r+e|u|^{p-1}} = 0$, $\lim_{u\to 0} \frac{I_k^1(u)}{r_k^1 + e_k^1 |u|} = 0$ and $\lim_{u\to 0} \frac{I_k^2(u)}{r_k^2 + e_k^1 |u|^{p-1}} =$

0, wher r, e, r_k^i, e_k^i are nonegative constants. Then, the problem (4.1.1) has at least one solution. Then, the problem (4.1.1) also has at least one solution, which can be shown by use of the same process of the proof of Theorem 4.3.8.

In the remainder of the document, we will give the existence and uniqueness results to the problem (4.1.1), for that we will use the Banach contraction mapping principle to prove them. For the sake of convenience, denote $Fu =^{\psi_3}{}_t\mathcal{I}^{\gamma}_{a^+}\eta(t) \mid u(t)|^{p-1}_{t=T}$

Theorem 4.3.10. *Suppose that, there exists a positive constants $\Theta_1,\ \Theta_2,\ \Theta_3,\ \Theta_4,\ \Theta_5$ and Θ_6 such that*

$$\Theta_1 \leqq \mid \mathfrak{F}\mathcal{N}u(t) \mid \leqq \Theta_2; \tag{4.3.20}$$

$$\Theta_3 \leqq \| u \| \leqq \Theta_4; \tag{4.3.21}$$

$$\Theta_5 \leqq \mid Fu \mid \leqq \Theta_6; \tag{4.3.22}$$

$\forall t \in J, u \in PC(J, \mathbb{R})$. If $\boldsymbol{H}_5$ and $\boldsymbol{H}_6$ hold, then the problem (4.1.1) has a unique solution.

Proof. Assume that $\mathbf{H}_5$ and $\mathbf{H}_6$. we only consider the case $1 < p < 2$ as the other case $p \geqq 2$ is straigtforward. If $1 < p \leqq 2$, we have $p^\star \geqq 2$ from that $\frac{1}{p} + \frac{1}{p^\star} = 1$, applied (4.3.20)-(4.3.22) and Lemma 4.2.1 for every $t \in J$, $u \in PC(J, \mathbb{R})$, we obtain

$$\begin{aligned}
|\phi_{p^\star}(\mathfrak{F}\mathcal{N}u(t)) - \phi_{p^\star}(\mathfrak{F}\mathcal{N}v(t))| &\leqq (p^\star - 1)\Theta_2^{p^\star-2}|\mathfrak{F}\mathcal{N}u(t) - \mathfrak{F}\mathcal{N}v(t)| \\
&= (p^\star - 1)\Theta_2^{p^\star-2}\frac{1}{|\rho(t)|}\left|\frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}\times\right. \\
&\quad (f(\tau, u(\tau) - f(\tau, v(\tau))\,\mathrm{d}\tau - \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}s(\tau)\times \\
&\quad \left.(\phi_p(u(\tau)) - \phi_p(u(\tau)))\mathrm{d}\tau\right|, \\
&\leqq (p^\star - 1)\Theta_2^{p^\star-2}M_3\left(\frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}\times\right. \\
&\quad \mid f(\tau, u(\tau) - f(\tau, v(\tau) \mid \mathrm{d}\tau + \frac{1}{\Gamma(\beta)}\int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1}s(\tau)\times \\
&\quad \left.\mid \phi_p(u(\tau)) - \phi_p(u(\tau)) \mid \mathrm{d}\tau + \sum_{j=1}^{k} \mid I_j^2(u(t_j) - I_j^2(u(t_j) \mid\right) \\
&\leqq (p^\star - 1)\Theta_2^{p^\star-2}M_3\left(\frac{L + (p-1)\Theta_1^{p-2}}{\Gamma(\beta+1)}(\psi_2(\tau) - \psi_2(a))^\beta + \sum_{j=1}^{m} L_j^2\right) \| u - v \|
\end{aligned}$$

and

$$\Big|\mid Fu\mid^{p^\star-1}-\mid Fv\mid^{p^\star-1}\Big| = \Big|\Big|\frac{1}{\Gamma(\gamma)}\int_a^T(\psi_3'(s)(\psi_3(t)-\psi_3(s))^{\gamma-1}\eta(s)\mid u\mid^{p-1}\mathrm{d}s\Big|^{p^\star-1}$$
$$-\Big|\frac{1}{\Gamma(\gamma)}\int_a^T(\psi_3'(s)(\psi_3(t)-\psi_3(s))^{\gamma-1}\eta(s)\mid u\mid^{p-1}\mathrm{d}s\Big|^{p^\star-1}\Big|$$
$$\leqq (p^\star-1)\Theta_6^{p^\star-2}\Big|\frac{1}{\Gamma(\gamma)}\int_a^T(\psi_3'(s)(\psi_3(t)-\psi_3(s))^{\gamma-1}\eta(s)\times$$
$$\left[\phi_p\left(u(s)\right)-\phi_p\left(v(s)\right)\right]\mathrm{d}s\Big|$$
$$\leqq (p^\star-1)\Theta_6^{p^\star-2}\left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma}\right)\parallel u-v\parallel.$$

So, for every $t\in J$, $u,v\in PC(J,\mathbb{R})$, we obtain

$$\mid \mathcal{L}u(t)-\mathcal{L}v(t)\mid \leqq\mid\lambda\mid(p^\star-1)\Theta_6^{p^\star-2}\left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma}\right)\parallel u-v\parallel$$
$$+(p^\star-1)\Theta_2^{p^\star-2}M_3\left(\frac{L+(p-1)\Theta_3^{p-2}}{\Gamma(\beta+1)}(\psi_2(\tau)-\psi_2(a))^{\beta}+\sum_{j=1}^m L_j^2\right)\parallel u-v\parallel$$
$$+\sum_{j=1}^m L_j^1\parallel u-v\parallel$$
$$=\left[\mid\lambda\mid(p^\star-1)\Theta_6^{p^\star-2}\left(\frac{M_4(p-1)\Theta_3^{p-2}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma}\right)+\sum_{j=1}^m L_j^1\right.$$
$$\left.+(p^\star-1)\Theta_2^{p^\star-2}M_3\left(\frac{L+(p-1)\Theta_3^{p-2}}{\Gamma(\beta+1)}(\psi_2(\tau)-\psi_2(a))^{\beta}+\sum_{j=1}^m L_j^2\right)\right]\parallel u-v\parallel,$$
$$=:\Lambda^\star\parallel u-v\parallel.$$

Hence, for each $u,v\in PC(J,\mathbb{R})$

$$\|\mathcal{L}u-\mathcal{L}v\|\leqq\Lambda^\star\parallel u-v\parallel,$$

otherwise,

$$\|\mathcal{L}u-\mathcal{L}v\|\leqq\Lambda^{\star\star}\parallel u-v\parallel,$$

for each $u,v\in PC(J,\mathbb{R})$, where

$$\Lambda^{\star\star}=\left[\mid\lambda\mid(p^\star-1)\Theta_5^{p^\star-2}\left(\frac{M_4(p-1)\Theta_4^{p-2}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma}\right)+\sum_{j=1}^m L_j^1\right.$$
$$\left.+(p^\star-1)\Theta_1^{p^\star-2}M_3\left(\frac{L+(p-1)\Theta_4^{p-2}}{\Gamma(\beta+1)}(\psi_2(\tau)-\psi_2(a))^{\beta}+\sum_{j=1}^m L_j^2\right)\right].$$

If $0 < \Lambda^{\star}, \Lambda^{\star\star} < 1$, then, $\mathcal{L} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is a contraction mapping. By Banach contraction mapping principle, $\mathcal{L}$ has a unique fixed point in $PC(J, \mathbb{R})$ which is a solution of problem (4.1.1) □

Theorem 4.3.11. *Suppose that* $\mathbf{H}_5^{\star}$ *and* $\mathbf{H}_6^{\star}$ *hold. If*

$$\Lambda^{\star},\ C^8,\ \Lambda^{\star\star} < 1, \tag{4.3.23}$$

where

$$\Theta_1 =: M_3 \rho(T) u_1^{p-1};\ \Theta_2 := M_3 \left(\rho(T) u_1^{p-1} + \frac{\max_{t \in J}(\Phi((t))}{\Gamma(\beta+1)} (\psi_2(T) - \psi_2(a))^{\beta} + \sum_{j=1}^{m} \Psi_j^2(t_j) \right);$$

$$\Theta_3 =: u_0;\ \ \Theta_4 =: \frac{u_0 + \sum_{j=1}^{m} \Psi_j^1(t_j) + \frac{1}{\Gamma(\alpha+1)} (\psi_1(T) - \psi_1(a))^{\alpha} \Theta_2^{p^{\star}-1}}{1 - C^8};$$

$$\Theta_5 =: \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{\gamma};\ \Theta_6 =: \frac{M_6 \Theta_3^{p-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{\gamma};$$

$$C^8 =: \sum_{j=1}^{m} e_j^1 + \lambda \frac{M_4^{p^{\star}-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{\gamma(p^{\star}-1)},$$

then the problem (4.1.1) *has a unique solution.*

Proof. Assume that $\mathbf{H}_5^{\star}$ and $\mathbf{H}_6^{\star}$ hold. Let $u \in PC(J, \mathbb{R})$, then for every $t \in [a, t_1]$, we obtain

$$\begin{aligned} 0 < \mathfrak{F}\mathcal{N} u(t) &\leqq M_3 \left(\rho(T) \phi_p(u_1) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(s) (\psi_2(s) - \psi_2(t))^{\beta-1} \left(f(\tau, u(\tau) - s(\tau) \phi_p(u(\tau)) \right) d\tau \right. \\ &\quad \left. + \sum_{j=1}^{m} I_k^2 u(t_k) \right), \\ &\leqq M_3 \left(\rho(T) u_1^{p-1} + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(s) (\psi_2(s) - \psi_2(t))^{\beta-1} \Phi(\tau) d\tau + \sum_{j=1}^{m} \Psi_j^2(t_j) \right), \\ &\leqq M_3 \left(\rho(T) u_1^{p-1} + \frac{\max_{t \in J} \{\Phi(t)\}}{\Gamma(\beta+1)} (\psi_2(T) - \psi_2(a))^{\beta} + \sum_{j=1}^{m} \Psi_j^2(t_j) \right) =: \Theta_2, \end{aligned}$$

from (4.3.15) we have

$$\begin{aligned} 0 < u(t) \leqq & u_0 + \lambda \left| \frac{1}{\Gamma(\gamma)} \int_a^T \psi_3'(s) (\psi_3(T) - \psi_3(s))^{\gamma-1} \eta(s) \left| u(s) \right|^{p-1} ds \right|^{p^{\star}-1} \\ & + \sum_{j=1}^{m} I_j^1 u(t_j) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s) (\psi_1(t) - \psi_1(s))^{\alpha-1} \phi_{p^{\star}} (\mathfrak{F}\mathcal{N} u(s)) ds \end{aligned}$$

$$\begin{aligned}
\leqq & u_0+\lambda \frac{\| u \| M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma(p^\star-1)}+\sum_{j=1}^m \Psi_j^1(t_j)+e_j^1 \| u \| \\
& +\frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^\alpha \Theta_2^{p^\star-1}, \\
= & u_0+C^8 \| u \|+\sum_{j=1}^m \Psi_j^1(t_j)+\frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^\alpha \Theta_2^{p^\star-1},
\end{aligned}$$

as $C^8<1$, then

$$\| u \| \leqq \frac{u_0+\sum_{j=1}^m \Psi_j^1(t_j)+\frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^\alpha \Theta_2^{p^\star-1}}{1-C^8}=: \Theta_4,$$

and

$$\begin{aligned}
Fu & =\left|\frac{1}{\Gamma(\gamma)} \int_a^T(\psi_3'(s)(\psi_3(t)-\psi_3(s))^{\gamma-1} \eta(s) \phi_p(u(s)) \mathrm{d}s\right| \\
& \leqq \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^\gamma=: \Theta_6 .
\end{aligned}$$

On the other hand, from the posivety of $\mathfrak{F}\mathcal{N}u(t)$, λ, u_0 and I_k^i for $i=1,2$, $k=1,2,\cdots,m$ for $\forall t \in[a, t_1], u \in PC(J, \mathbb{R})$, we have the following inequalities:

$$\begin{aligned}
u(t) & \geqq u_0=: \Theta_3, \\
\mathfrak{F}\mathcal{N}u(t) & \geqq M_3 \rho(T) \phi_p(u_1)=: \Theta_1, \\
Fu & \geqq \frac{M_6 \Theta_4^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^\gamma=: \Theta_5,
\end{aligned}$$

by use of the same process above, we get

$$\begin{aligned}
& \Theta_1 \leqq| \mathfrak{F}\mathcal{N}u(t) | \leqq \Theta_2 \\
& \Theta_3 \leqq\| u \| \leqq \Theta_4 \\
& \Theta_5 \leqq| Fu | \leqq \Theta_6
\end{aligned}$$

for any $t \in J_k$, $u \in PC(J, \mathbb{R})$, $k=1,\ 2,\ \cdots,\ m$, hence, by applied Theorem 4.3.10 one can deduce that the problem (4.1.1) has a unique solution. □

Theorem 4.3.12. *Suppose that $\mathbf{H}_5^{\star\star}$ and $\mathbf{H}_6^{\star\star}$ hold. If*

$$\Lambda^\star,\ C^9,\ \Lambda^{\star\star}<1, \tag{4.3.24}$$

where

$$M_7 =: M_3\left(\rho(T)+\frac{e+M_5}{\Gamma(\beta+1)}(\psi_2(T)-\psi_2(a))^{\beta}+\sum_{j=1}^{m}e_j^2(t_j)\right),$$

$$\Theta_1 =: M_3\rho(T)u_1^{p-1};\ \Theta_2 := M_7\Theta_4^{p-1};$$

$$\Theta_3 =: u_0;\ \ \Theta_4 =: \frac{u_0+\sum_{j=1}^{m}\Psi_j^1(t_j)}{1-C^9};$$

$$\Theta_5 =: \frac{M_6\Theta_3^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma};\ \Theta_6 =: \frac{M_5\Theta_4^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma};$$

$$C^9 =: \sum_{j=1}^{m}e_j^1+\lambda\frac{M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{(p^\star-1)\gamma}\frac{M_7^{p^\star-1}}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^{\alpha},$$

then the problem (4.1.1) *has a unique solution.*

Proof. Assume that $\mathbf{H}_5^{\star\star}$ and $\mathbf{H}_6^{\star\star}$ hold. Let $u\in PC(J,\mathbb{R})$, then for every $t\in[a,t_1]$, then from we obtain

$$\begin{aligned}0<u_0<u(t)\leqq& u_0+\lambda\left|\frac{1}{\Gamma(\gamma)}\int_a^T\psi_3'(s)(\psi_3(T)-\psi_3(s))^{\gamma-1}\eta(s)\,|u(s)|^{p-1}\,\mathrm{d}s\right|^{p^\star-1}\\&+\sum_{j=1}^{m}I_j^1u(t_j)+\frac{1}{\Gamma(\alpha)}\int_a^t\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}\phi_{p^\star}\left(\mathfrak{F}\mathcal{N}u(s)\right)\mathrm{d}s\\ \leqq& u_0+\lambda\frac{\|u\|\,M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma}+\sum_{j=1}^{m}\Psi_j^1(t_j)+e_j^1\,\|u\|\\&+\frac{1}{\Gamma(\alpha)}\int_a^t\psi_1'(s)(\psi_1(t)-\psi_1(s))^{\alpha-1}\phi_{p^\star}\left(\mathfrak{F}\mathcal{N}u(s)\right)\mathrm{d}s,\end{aligned}\tag{4.3.25}$$

$$\begin{aligned}0<\mathfrak{F}\mathcal{N}u(t)\leqq& M_3\Bigg(\rho(T)\phi_p\left(u_1\right)+\frac{1}{\Gamma(\beta)}\int_t^T\psi_2'(s)(\psi_2(s)-\psi_2(t))^{\beta-1}\Big[|f(\tau,u(\tau)|\\&+s(\tau)\phi_p\left(|u(\tau)|\right)\Big]\mathrm{d}\tau+\sum_{j=1}^{m}I_k^2u(t_k)\Bigg)\\ \leqq& M_3\left(\rho(T)u_0^{p-1}+\frac{\|u\|^{p-1}}{\Gamma(\beta)}\int_t^T\psi_2'(s)(\psi_2(s)-\psi_2(t))^{\beta-1}\left(e+s(\tau)\right)\mathrm{d}\tau\right.\\&\left.+\|u\|^{p-1}\sum_{j=1}^{m}e_j^2(t_j)\right),\end{aligned}$$

$$\leqq \| u \|^{p-1} M_3 \left(\rho(T) + \frac{e + M_5}{\Gamma(\beta+1)} (\psi_2(T) - \psi_2(a))^{\beta} + \sum_{j=1}^{m} e_j^2(t_j) \right)$$

$$:= M_7 \| u \|^{p-1}, \tag{4.3.26}$$

from(4.3.25) and (4.3.26) we have

$$u_0 \leqq u(t) \leqq u_0 + \lambda \frac{\| u \| M_4^{p^\star-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{(p^\star-1)\gamma} + \sum_{j=1}^{m} \Psi_j^1(t_j) + e_j^1 \| u \|$$

$$+ \frac{\| u \| M_7^{p^\star-1}}{\Gamma(\alpha+1)} (\psi_1(T) - \psi_1(a))^{\alpha}$$

$$= u_0 + C^9 \| u \| + \sum_{j=1}^{m} \Psi_j^1(t_j),$$

as $C^9 < 1$, then

$$\| u \| \leqq \frac{u_0 + \sum_{j=1}^{m} \Psi_j^1(t_j)}{1 - C^9} =: \Theta_4$$

and

$$Fu = \left| \frac{1}{\Gamma(\gamma)} \int_a^T (\psi_3'(s)(\psi_3(T) - \psi_3(s))^{\gamma-1} \eta(s) \phi_p(u(s)) \, ds \right| \leqq \frac{M_5 \Theta_4^{p-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{\gamma} =: \Theta_6.$$

On the other hand, from the positiveness of $\mathfrak{F}\mathcal{N}u(t)$, λ, u_0 and I_k^i for $i = 1, 2$, $k = 1, 2, \cdots, m$ for $\forall t \in [a, t_1], u \in PC(J, \mathbb{R})$, we have the following inequalities:

$$u(t) \geqq u_0 =: \Theta_3,$$
$$\mathfrak{F}\mathcal{N}u(t) \geqq M_3 \rho(T) \phi_p(u_1) =: \Theta_1,$$
$$Fu \geqq \frac{M_6 \Theta_3^{p-1}}{\Gamma(\gamma+1)} (\psi_3(T) - \psi_3(a))^{\gamma} =: \Theta_5,$$

by use of the same process above, we get

$$\Theta_1 \leqq | \mathfrak{F}\mathcal{N}u(t) | \leqq \Theta_2,$$
$$\Theta_3 \leqq \| u \| \leqq \Theta_4,$$
$$\Theta_5 \leqq | Fu | \leqq \Theta_6,$$

for any $t \in J_k$, $u \in PC(J, \mathbb{R})$, $k = 1, 2, \cdots, m$, hence, by applied Theorem 4.3.10 one can deduce that the problem (4.1.1) has a unique solution. □

Theorem 4.3.13. *Suppose that $\boldsymbol{H}_5^{\star\star\star}$ and $\boldsymbol{H}_6^{\star\star\star}$ hold. If*

$$\Lambda^\star, \; C^{10}, \; \Lambda^{\star\star} < 1, \tag{4.3.27}$$

where

$$\Theta_1 =: -M_3\rho(T)\phi_p(u_1);\ \Theta_2 := M_3\left(-\rho(T)\phi_p(u_1) + \frac{\max_{t\in J}(\Pi((t))}{\Gamma(\beta+1)}(\psi_2(T)-\psi_2(a))^{\beta} + \sum_{j=1}^{m}\chi_j^2(t_j)\right);$$

$$\Theta_3 =: -u_0;\ \ \Theta_4 =: \frac{-u_0 + \sum_{j=1}^{m}\chi_j^1(t_j) + \frac{1}{\Gamma(\alpha+1)}(\psi_1(T)-\psi_1(a))^{\alpha}\Theta_2^{p^\star-1}}{1-C^{10}};$$

$$\Theta_5 =: \frac{M_5\Theta_4^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma};\ \Theta_6 =: \frac{M_6\Theta_3^{p-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{\gamma};$$

$$C^{10} =: \sum_{j=1}^{m} e_j^1 - \lambda\frac{M_4^{p^\star-1}}{\Gamma(\gamma+1)}(\psi_3(T)-\psi_3(a))^{(p^\star-1))\gamma},$$

then the problem (4.1.1) *has a unique solution.*

Proof. It is sufficient to refer to the proof of Theorem 4.3.10 □

4.4 Application

In this section, by given tow examples we illustrate the results.

Example 4.4.1. Consider the boundary value problem of impulsive differential equation

$$\begin{cases} {}^{C-H}\mathcal{D}_{e^{2-}}^{\frac{3}{4}}\left(\sqrt{t}\phi_p\left({}^{C-H}\mathcal{D}_{e^{1+}}^{\frac{3}{4}}u\right)\right)(t) + \ln(t)\phi_p\left(u(t)\right) = \dfrac{\sin(t)+u(t)}{10\left(e^{|u(t)|}+\mid u(t)\mid\right)} \quad e^1 < t < e^2, \\ \Delta\left(u\left(\tau\right)\right) = \mid u\left(\tau\right)\mid^{1/2}\sin\left(u(\tau)\right), \\ \Delta\phi_p\left({}^{C-H}\mathcal{D}_{e^{1+}}^{\frac{3}{4}}u\right)(\tau) = \mid u\left(\tau\right)\mid^{1/2}\cos\left(u(\tau)\right), \\ u(e^1) = 1, {}^{C-H}\mathcal{D}_{e^{1+}}^{\frac{3}{4}}u(e^2) = 0. \end{cases} \tag{4.4.1}$$

Here $\tau \in (e^1, e^2)$

$$\begin{array}{cccc} \psi_1(t) = \psi_2(t) = s(t) = \ln(t), & \rho(t) = \sqrt{t}, & \alpha = \beta = \frac{3}{4}, & \lambda = 0, \\ \tau \in (e^1, e^2), & p = 3, & p^\star = \frac{3}{2}, & u_1 = 0, \\ u_0 = 1, & t_1 = \tau, & m = 1, & \end{array}$$

${}^{C-H}\mathcal{D}_{e^{1+}}^{\frac{3}{4}}$ and ${}^{C-H}\mathcal{D}_{e^{2-}}^{\frac{3}{4}}$ are the left and right-sided Caputo-Hadamard fractional derivatives.
It is easy to show that (4.4.1) is a form of (4.1.1).

Set

$$f(t,u(t)) = \frac{\sin(t)+u(t)}{10\left(e^{|u(t)|}+\mid u(t)\mid\right)}, \quad (t,u) \in [e^1, e^2] \times \mathbb{R},$$

$$\mid f(t,u(t))\mid \leqq \frac{1}{10} + \frac{1}{10}\mid u(t)\mid, \quad (t,u) \in [e^1, e^2] \times \mathbb{R}.$$

Set

$$I^1(u(t) =| u(t)|^{1/2} \sin(u(t)), \; I^2(u(t) =| u(t)|^{1/2} \cos(u(t))$$

Also,

$$| I^1(u(t) |\leqq| u(t)|^{1/2}, \; | I^2(u(t) |\leqq| u(t)|^{1/2} \quad (t,u) \in [e^1, e^2] \times \mathbb{R}.$$

- It is not difficult to see that all the assumptions in Theorem (4.3.5) are satisfied. Thus, the problem (4.4.1) has a solution in $PC([e^1, e^2], \mathbb{R})$.
- It is obvious that all the assumptions in Theorem (4.3.6) are satisfied. Thus, the problem (4.4.1) has a solution in $PC([e^1, e^2], \mathbb{R})$.

Example 4.4.2.

$$\begin{cases} {}^{\sin(x/2);C}_{\ \ t}\mathcal{D}^{\frac{4}{5}}_{1^-}\left((18+2e^t)\phi_{\frac{3}{2}}\left({}^{\ln(x+1);C}_{\ \ t}\mathcal{D}^{\frac{4}{5}}_{0^+}u\right)\right)(t) = f(t,u(t)) - \frac{t^2\sin(t+1)}{10}\phi_{\frac{3}{2}}(u(t)) \quad 0<t<1, \\ \Delta\left(u\left(\frac{1}{2}\right)\right) = \frac{1}{20}\left(|\sin(u(\tau))| + \exp(-\frac{1}{2}\mid u\left(\frac{1}{2}\right)\mid)\right), \\ \Delta\phi_{\frac{3}{2}}\left({}^{\ln(x+1);C}_{\ \ t}\mathcal{D}^{\frac{4}{5}}_{0^+}u\right)\left(\frac{1}{2}\right) = \frac{1}{10}\exp(-\frac{1}{2}\mid u\left(\frac{1}{2}\right)\mid), \\ u(0) = \frac{1}{2} + \lambda\left|{}^{(x+1)^2}_{\ \ t}\mathcal{I}^{\gamma}_{a^+}\frac{t^2+1}{10}\sqrt{|u(t)|}\right|^2_{t=1}, \; {}^{\ln(x+1);C}\mathcal{D}^{\frac{4}{5}}_{0^+}u(1) = \frac{1}{5}, \end{cases} \tag{4.4.2}$$

where

$$f(t,u) = \frac{u^2}{(19+e^t)(1+u^2)} + \frac{|\sin(u)|\,t^2}{10} + \frac{e^{-t}\mid u\mid}{(18+2e^{-t})(u^2+1)} + \frac{t^2\sin(t+1)}{10}\phi_{\frac{3}{2}}(u),$$
$$I^1(u) = \frac{1}{20}\left(|\sin(u(\tau))| + \exp(-\frac{1}{2}\mid u\left(\frac{1}{2}\right)\mid)\right), \; I^2(u) = \frac{1}{10}\exp(-\frac{1}{2}\mid u\left(\frac{1}{2}\right)\mid).$$

Here

$$\begin{array}{ccc} \psi_1 = \ln(x+1), & \psi_2 = \sin(\frac{x}{2}), & \psi_3 = (x+1)^2, \\ \rho(t) = 18+2e^t, & s(t) = \frac{t^2\sin(t+1)}{10}, & \eta(t) = \frac{t^2+1}{10}, \\ \alpha = \beta = \frac{4}{5}, & p = \frac{3}{2}, p^\star = 3, & t_1 = \frac{1}{2}, \; m = 1, \\ u_0 = \frac{1}{4}, & u_1 = \frac{1}{100}, & \lambda = \frac{1}{10}. \end{array}$$

It is easy to show that (4.4.1) is a form of (4.1.3). Further, there exists a function $\Phi(t) = \frac{1}{19+e^t} + \frac{t^2}{5} + \frac{1}{20}$ such that $|f(t,u(t)) - s(t)\phi_{\frac{3}{2}}(u(t))| \leqq \Phi(t)$.

One can see that the solution $u(t)$ of the boundary value problem (4.4.1) which is given by the integral equation (4.3.15) is well defined and satisfy:

$$\Theta_1 \leqq| \mathfrak{F}\mathcal{N}u(t) |\leqq \Theta_2,$$
$$\Theta_3 \leqq\| u \|\leqq \Theta_4,$$
$$\Theta_5 \leqq| Fu |\leqq \Theta_6,$$

where

$$\Theta_1 = \phi_{\frac{3}{2}}\left(\frac{1}{100}\right),\ \Theta_2 = \frac{(19+2e)\sqrt{\pi}+8\sin(\frac{1}{2})^{\frac{1}{2}}}{200\sqrt{\pi}},$$

$$\Theta_3 = \frac{1}{4},\ \Theta_4 = \frac{7\sqrt{\pi}+40\Theta_2^2(\ln(2))^{\frac{1}{2}}}{20\sqrt{\pi}(1-C^8)},\ C^8 = \frac{6\lambda}{25\pi},$$

$$\Theta_5 = \frac{1}{5}\sqrt{\frac{3\Theta_3}{\pi}},\ \Theta_6 = \frac{2}{5}\sqrt{\frac{3\Theta_4}{\pi}}.$$

We can easily show that $f(t,u)$, $I^1(u)$ and $I^2(u)$ satisfies:

$$\begin{aligned}
|f(t,u)-f(t,v)| &= \left|\frac{1}{19+e^t}\left(\frac{u^2}{1+u^2}-\frac{v^2}{1+v^2}\right)+\frac{t^2}{10}\left(|\sin(u)|-|\sin(v)|\right)\right.\\
&\left.+\frac{e^{-t}}{18+2e^{-t}}\left(\frac{|u|}{v^2+1}-\frac{|u|}{v^2+1}\right)+\frac{t^2\sin(t+1)}{10}\left(\phi_{\frac{3}{2}}(u)-\phi_{\frac{3}{2}}(v)\right)\right|\\
&\leqq \frac{1}{20}\left|\frac{u^2}{1+u^2}-\frac{v^2}{1+v^2}\right|+\frac{1}{10}\left|\sin(u)-\sin(v)\right|\\
&+\frac{1}{20}\left|\frac{u}{v^2+1}-\frac{u}{v^2+1}\right|+\frac{\sin(2)}{10}\left|\phi_{\frac{3}{2}}(u)-\phi_{\frac{3}{2}}(v)\right|\\
&\leqq \frac{1}{20}\left|\frac{u^2}{1+u^2}-\frac{v^2}{1+v^2}\right|+\frac{1}{10}\left|\sin(u)-\sin(v)\right|\\
&+\frac{1}{20}\left|\frac{u}{v^2+1}-\frac{u}{v^2+1}\right|+\frac{\sin(2)}{10}\left|\phi_{\frac{3}{2}}(u)-\phi_{\frac{3}{2}}(v)\right|\\
&\leqq \frac{1}{10}|u-v|+\frac{1}{10}|u-v|+\frac{1}{10}|u-v|+\frac{\sqrt{2}\sin(2)}{20}|u-v|\\
&= \frac{6+\sqrt{2}\sin(2)}{20}|u-v|, \quad t\in[0,1],\ u,v\in PC([0,1],\mathbb{R}),
\end{aligned}$$

$$\begin{aligned}
|\,I^1(u)-I^1(v)\,| &= \frac{1}{20}\left|\left(|\sin(u)|-|\sin(u)|+\exp(-\frac{1}{20}\,|\,u\left(\frac{1}{2}\right)|)-\exp(-\frac{1}{2}\,|\,v\left(\frac{1}{2}\right)|)\right)\right|\\
&\leqq \frac{1}{10}\,|\,u-v\,|, \quad (t,u)\in[0,1]\times\mathbb{R},
\end{aligned}$$

and

$$|\,I^2(u)-I^2(v)\,| = \frac{1}{10}|\exp(-\frac{1}{2}\,|\,u\left(\frac{1}{2}\right)|)-\exp(-\frac{1}{2}\,|\,v\left(\frac{1}{2}\right)|)| \leqq \frac{1}{10}\,|\,u-v\,|, \quad (t,u)\in[0,1]\times\mathbb{R}.$$

Also, we have

$$|\,I^1(u(t)\,| \leqq \frac{1}{10},\ |\,I^2(u(t)\,| \leqq \frac{1}{10} \quad (t,u)\in[0,1]\times\mathbb{R}.$$

That is to say, the conditions ($\mathbf{H}_5^\star$), ($\mathbf{H}_6^\star$) hold, where $L^1 = L^2 = \frac{1}{10}$, $L = \frac{6+\sqrt{2}\sin(2)}{20}$. Through some calculation, we get that $\Lambda^\star \approx 0.1134991 < 1$.

Obviously,(4.4.2) satisfies all the assumptions of Theorem 4.3.13. Hence (4.4.2) has a unique solution.

General conclusion and perspectives

In this thesis, we have successfully investigated the existence of at least one positive solution, the existence of multiple positive solutions, nonexistence and uniqueness of the solutions for various classes of boundary value problems for nonlinear fractional differential equations with p-Laplacien operator in chosen Banach Spaces. The existence of solutions is provided by using some fixed point theoremes, whereas the uniqueness result is achieved by Banach's fixed point theorem. After that, we have presented an illustrative examples to support our main results.

In future works, many results can be established when one takes a more generalized operator. Precisely, it will be of interest to study the current problemes in this work for other fractional and p–Laplacien operator.

Abstract

The fractional differential equations with p-Laplacien operator (**FDEswP-L**) appear as a natural description of observed evolution phenomena in various scientific areas such as physics, engineering, medicine, electrochemistry, control theory, etc. The efficiency of these equations in the modeling of many real-world problems motivated a lot of researchers to investigate their quantitative and qualitative aspects. The aim of this thesis is to contribute and give some existence results for various classes of boundary value problems for nonlinear fractional differential equations with p-Laplacien operator in chosen Banach Spaces. For this purpose, the technique used is to reduce the study of our problem to the research of a fixed point of an integral operator. The obtained results are based on some standard fixed point theorems. We have also provided a illustrative example to each of our considered problems to show the validity of conditions and justify the efficiency of our established results. Here we investigate two types of such equations in Banach spaces.
Key words and phrases: p-Laplacian boundary value problems, fractional differential equations, impulsive fractional differential equations, Caputo fractional derivative, Caputo-Katugampola fractional derivative, $\psi-$ Caputo ($\psi-$ C) fractional derivative, Caputo-Hadamard fractional derivative, impulsive boundary value problem, differential equations, Banach space, fixed-point, existence, uniqueness, Banach, Sheafer, Schawder, Gua-Kranoselskii and Leggett-Williams fixed point theoremes, positive solutions, Cone .

AMS Subject Classification : 26A33, 34A08, 34B15, 34K37, 34K45.

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