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Sur l'existence et l'unicité de solutions pour quelques problèmes différentiels fractionnaires au sens de Hilfer-Hadamard

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DEDICATIONS

This work is dedicated to my parents,

to all my family from near and far

Thank you.

Fatima Si Bachir

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Publications

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Abstract

The main aim of this thesis is to discuss some results on the global convergence of successive approximations and uniqueness of the solution for some classes of initial value problems involving the implicit Caputo q-difference equations, Caputo-Fabrizio, random coupled Hilfer, ψ-Hilfer and hybrid Caputo fractional differential equations. We also worked on the existence and attractivity of the solution for a class of nonlinear ψ-Hilfer hybrid and Hilfer-Hadamard fractional differential equations, the existence results are based on the Schauder fixed point theorem. We give a result on the global convergence of successive approximations towards the unique solution for some problems. Some examples are given to illustrate the application of the given main results.

Key words and phrases : Fractional q-difference equation, implicit, random solution, ψ-Hilfer Cauchy-type problem, ψ-Hilfer fractional derivative, Hybrid Caputo fractional differential equation, Caputo-Fabrizio fractional derivative, Random coupled Hilfer fractional differential system, Hilfer-Hadamard fractional derivative, uniformly locally attractive, global convergence, successive approximations, Schauder fixed-point Theorem.

AMS Subject Classification : 26A33, 34A08, 65L05.

Table of contents

Introduction

Fractional calculus is the calculation of integrals and derivatives of any real or complex order, and has become of great importance these last three decades. The question which arises is on the appearance of this concept and the answer is by a simple question of intelligence that the marquis L'Hopital asked Gottfried Wilhelm Leibniz if $\frac{d^n y}{dx^n}$, and $n = \frac{1}{2}$ [77].

It was September 30, 1695 the birth of the fractional calculus. The works were done on this over the years by Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), Lagrange (1772), Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grünwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy (1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964). The first book was published on fractional calculus by Oldham and Spanier in 1974. There was also the monograph by Samko, Kilbas and Marichev which was published in Russian in 1987 and in English in 1993 [92]. Other works have been done on fractional differential equations were Miller's and Ross (1993) [87], Podlubny (1999)[89], Kilbas et al. (2006) [77], Diethelm (2010) [51], Oldham et al.[88], Abbas et al. [5, 8, 14, 15], Benchohra et al. [35, 36, 37], Zhou [116, 117] and Zhou *et al.* [118, 119]. Fractional differential equations were of great importance to physics, mathematics, engineering, biology, image processing, electricity, control theory, economics, biophysics, and mechanics, etc, (see[27, 59, 63, 68, 85, 86, 107, 108]). For a few fundamental results in the theory of fractional differential equations (see [4, 13, 50, 69, 61, 77, 73, 74, 76, 79, 80, 81, 84, 90, 92, 93, 105, 106]) also some work on the fractional by Benbachir *et al.* (see[30, 34, 38, 39, 40]).

The fixed point theory is a powerful tool for mathematics, this is a very important area of research for many mathematicians. The origins of this theory go back to the nineteenth century from the successive approximations to demonstrate the existence and uniqueness of the solutions of differential equations, considered by Peano and Picard.

Equations with q-fractional differences began in the 19th century [19, 43] and has had a great interest in the last few years, some results about initial and boundary value problems of q-difference and also fractional q-difference equations are in [20, 21, 54, 56].

The functional differential equations with random effects have been very interesting in the theory of random dynamic systems [1, 2, 18, 64, 70]. The theory of random operators are used in the modeling of phenomena in the physical, biological and sciences systems .

Using new kinds of Caputo-Fabrizio derivative (see [33, 83]), several results were obtained for various fractional differential equations (see [29, 31, 32, 52, 53]).

Recently, the global convergence of successive approximations has been considered by Abbas *et al.* [3, 9]. Some results about the global convergence of successive approximations for abstract semilinear differential equations are obtained in [3], and other results concerning the successive approximations for the Darboux problem for implicit partial differential equations are mentioned in [9]. Attractivity results for various classes of fractional differential equations are considered in [6, 7, 8, 10, 16].

In this thesis we are concerned with the global convergence of successive approximation to the unique solutions of various classes of fractional differential equations. We are also interested to the attractivity of solutions of some fractional differential equations by means of the fixed point approach.

Now we give the outline of this thesis which is as follows :
In **Chapter** 1, we devote the notation to preliminary results, theorems, lemmas and other necessary results.

In **Chapter** 2, we discuss the global convergence of successive approximations for the following implicit fractional q-difference equation :

$$({}^cD_q^\alpha \rho)(t) = f(t, \rho(t), ({}^cD_q^\alpha \rho)(t)),\ t \in I := [0, T],$$

with the initial condition

$$\rho(0) = \phi \in \mathbb{R},$$

where $q \in (0, 1),\ \alpha \in (0, 1],\ T > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α. The main results of the problem considered before are in the article [95].
In **Chapter 3**, we study the uniform convergence of successive approximations for the initial value problem for hybrid Caputo fractional differential equation :

$${}^cD_{0^+}^\alpha \left[\frac{\rho(t)}{f(t, \rho(t))}\right] = g(t, \rho(t)),\ t \in I = [0, 1],$$

with initial condition

$$\rho(0) = \phi,$$

where $\alpha \in (0, 1), {}^C D_{0^+}^\alpha$ is the Caputo fractional derivative, $f \in C(I \times \mathbb{R}, \mathbb{R}^*)$, $g \in C(I \times \mathbb{R}, \mathbb{R})$. The main results of the problem considered before are in the article [100].

In **Chapter 4**, we discuss the uniform convergence of successive approximations for the coupled random Hilfer fractional differential system :

$$\begin{cases} \left(D_0^{\alpha_1,\beta_1}\rho\right)(t, w) = f_1(t, \rho(t, w), \varrho(t, w), w) \\ \left(D_0^{\alpha_2,\beta_2}\varrho\right)(t, w) = f_2(t, \rho(t, w), \varrho(t, w), w) \end{cases} ; t \in I := [0, T], w \in \Omega,$$

with the initial conditions

$$\begin{cases} \left(I_0^{1-\gamma_1}\rho\right)(0, w) = \phi_1(w) \\ \left(I_0^{1-\gamma_2}\varrho\right)(0, w) = \phi_2(w) \end{cases} ; w \in \Omega,$$

where $T > 0,\ \alpha_i \in (0, 1), \beta_i \in [0, 1], (\Omega, \mathcal{A})$ is a measurable space, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $\phi_i : \Omega \to \mathbb{R}^m,\ f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m; i = 1, 2,$ are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann-Liouville integral of order $1 - \gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized

Riemann–Liouville derivative (Hilfer) operator of order α_i and type $\beta_i : i = 1, 2$. The main results of the problem considered before are in the published article [96].

In **Chapter 5**, we study the uniform convergence of successive approximations for the ψ-Hilfer Cauchy-type problem :

$$D_{a^+}^{\alpha,\beta;\psi}\rho(t) = g\left(t, \rho(t), D_{a^+}^{\alpha,\beta;\psi}\rho(t)\right);\ 0 < \alpha < 1, 0 \leq \beta \leq 1,\ 0 \leq a < t \leq b.$$

$$I_{a^+}^{1-\gamma;\psi}\rho(a) = \rho_a, \qquad \rho_a \in \mathbb{R}, \quad \gamma = \alpha + \beta - \alpha\beta,$$

where $D_{a^+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative, $I_{a^+}^{1-\gamma;\psi}$ is ψ-Riemann-Liouville fractional integral, $g : (a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function and ρ_a is a constant. The main results of the problem considered before are in the published article [97].
In **Chapter 6**, we are interested in the existence and attractivity of solutions for the following problem :

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{\rho(t)}{\varrho(t,\rho(t))} = w\left(t, \rho(t)\right);\ a.e. \quad t \in \mathbb{R}_+, \\ \left(\psi(t) - \psi(0)\right)^{1-\varsigma}\rho(t)\mid_{t=0} = \rho_0; \qquad \rho_0 \in \mathbb{R}, \end{cases}$$

where $\mathbb{R}_+ := [0, +\infty),\ 0 < \lambda < 1,\ 0 \leq \sigma \leq 1,\ \varsigma = \lambda + \sigma(1 - \lambda),\ {}^H D_{0^+}^{\lambda,\sigma;\psi}$ is the ψ -Hilfer fractional derivative of order λ and type σ, $\varrho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^*$ and $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, are given functions. The main results of the problem considered before are in the published article [101].

In **Chapter 7**, we study the global convergence of successive approximations for Caputo-Fabrizio fractional differential equation(CFFDE) :

$$\begin{cases} ({}^{CF}D_0^s\rho)(t) = \varphi(t, \rho(t)); \quad t \in \Upsilon := [0, \lambda], \\ \rho(0) = \rho_0. \end{cases}$$

Here ${}^{\mathrm{CF}}D_t^s$ is for the CFFDE, $0 < s < 1$, $\varphi : [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\rho_0 \in \mathbb{R}$. The main results of the problem considered before are in the article [98].

In **Chapter 8**, we discuss the existence and attractivity of solutions of the following problem :

$$\begin{cases} ({}^H D_{c^+}^{\tau,\theta} i)(t) = \chi(t, i(t)); \qquad t \in [c, +\infty),\ c > 0, \\ ({}^H I_{c^+}^{1-\varrho} i)(c) = d, \quad \varrho = \tau + \theta(1 - \tau), \end{cases}$$

where $d \in \mathbb{R}$, $\chi : [c, +\infty) \times \mathbb{R} \to \mathbb{R}$, ${}^H I_{c^+}^{1-\varrho}$ is the left-sided Hadamard fractional integral of order $\tau > 0$ and ${}^H D_{c^+}^{\tau,\theta}$ is the Hilfer-Hadamard derivative operator of order τ $(0 < \tau < 1)$ and type θ $(0 \leq \theta \leq 1)$. The main results of the problem considered before are in the published article [99].

Finally we give a conclusion and some perspectives.

Chapter 1

Preliminaries

In this chapter, we introduce the necessary mathematical tools, notations and concepts that are useful for the following chapters.

1.1 Notations and Definitions

Let $I = [0, T]$ and $T > 0$. We take into account the Banach space $C(I) := C(I, \mathbb{R})$ of continuous functions from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|\rho\|_\infty := \sup_{t \in I} |\rho(t)|.$$

$L^1(I)$ denotes the space of measurable functions $\rho : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|\rho\|_1 = \int_0^T |\rho(t)| dt.$$

$AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$.
We denote by C the Banach space of all continuous functions from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\| \cdot \|_\infty$. As usual, $AC(I, \mathbb{R}^m)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$. By $L^1(I, \mathbb{R}^m)$, we denote the space of Lebesgue-integrable functions $\rho : I \to \mathbb{R}^m$ with the norm :

$$\|\rho\|_1 = \int_0^T \|\rho(t)\| dt.$$

By $C_\gamma(I)$ and $C^1_\gamma(I)$, we denote the weighted spaces of continuous functions defined by :

$$C_\gamma(I) = \left\{ \rho : (0, T] \to \mathbb{R}^m : t^{1-\gamma} \rho(t) \in C \right\},$$

with the norm :

$$\|\rho\|_{C_\gamma} := \sup_{t \in I} \left\| t^{1-\gamma} \rho(t) \right\|.$$

And

$$C^1_\gamma(I) = \left\{ \rho \in C : \frac{d\rho}{dt} \in C_\gamma \right\},$$

with the norm :

$$\|\rho\|_{C^1_\gamma} := \|\rho\|_\infty + \|\rho'\|_{C_\gamma}.$$

Let $\psi : [a, e] \to \mathbb{R}$ be an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, e], (-\infty \leq a < e \leq +\infty)$.

Define on $[a, e]$, $(0 < a < e < \infty)$ the weighted space

$$C_{\varsigma;\psi}[a, e] = \{\rho : (a, e] \to \mathbb{R} : (\psi(t) - \psi(d))^{\varsigma}\rho(t) \in C[a, e]\}, \quad 0 \leq \varsigma < 1,$$

with the norm

$$\|\rho\|_{C_{\varsigma;\psi}[a,e]} = \|(\psi(t) - \psi(a))^{\varsigma}\rho(t)\|_{C[a,e]} = \max\left\{|(\psi(t) - \psi(a))^{\varsigma}\rho(t)| : t \in [a, e]\right\},$$

and

$$C^n_{\gamma;\psi}[a, e] = \{\rho : (a, e] \to \mathbb{R} : \rho(t) \in C^{n-1}[a, e]; \rho^{(n)}(t) \in C_{\gamma;\psi[a,e]}\}, \quad 0 \leq \gamma < 1, n \in \mathbb{N},$$

with the norm

$$\|\rho\|_{C^n_{\gamma;\psi}[a,e]} = \sum_{k=0}^{n-1} \left\|\rho^{(k)}\right\|_{C[a,e]} + \left\|\rho^{(n)}\right\|_{C_{\gamma;\psi}[a,e]},$$

where $C([a, e])$ denotes the Banach space of all real continuous functions on $[a, e]$ (see [112]).

1.2 Special Functions

1.2.1 Gamma Function

The Euler's *Gamma* function $\Gamma(z)$, is a basic function of the fractional calculus generalizing the factorial $n!$ for values of n real and complex.

Definition 1.2.1 *([89]) The function Gamma is defined as follows :*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\mathrm{d}t,$$

which converges for $Re(z) > 0$.
It has the following property :

$$\Gamma(z + 1) = z\Gamma(z),$$

so for the particular case positive integer values n, we have $\Gamma(n) = (n - 1)!$.

1.2.2 Beta Function

In some cases, using the beta function is convenient.

Definition 1.2.2 *([89]) The definition of the Beta function is as follows*

$$\mathrm{B}(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\mathrm{d}t, \quad p, q > 0.$$

The next formula gives the relation between the Beta function and the Gamma function :

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}, \quad p, q > 0.$$

1.2.3 q-Gamma function

Let $q \in \mathbb{R}_+ - \{1\}$. For $d \in \mathbb{R}$, we have

$$[d]_q = \frac{1-q^d}{1-q}.$$

The q-analogue of the power $(d-e)^n$ is

$$(d-e)^{(0)} = 1,\ (d-e)^{(n)} = \Pi_{k=0}^{n-1}(d-eq^k);\ d,e \in \mathbb{R},\ n \in \mathbb{N}.$$

In a general way,

$$(d-e)^{(\alpha)} = d^\alpha \Pi_{k=0}^{\infty}\left(\frac{d-eq^k}{d-eq^{k+\alpha}}\right);\ d,e,\alpha \in \mathbb{R}.$$

(See [71]).

Definition 1.2.3 *[71, 102] We define the q-Gamma function as follows*

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}};\ \xi \in \mathbb{R} - \{0,-1,-2,\ldots\}.$$

Definition 1.2.4 *[71, 102] We define the q-derivative of order $n \in \mathbb{N}$ of a function $\rho : I \to \mathbb{R}$ as follows $(D_q^0\rho)(t) = \rho(t)$,*

$$(D_q\rho)(t) := (D_q^1\rho)(t) = \frac{\rho(t)-\rho(qt)}{(1-q)t};\ t \neq 0,\quad (D_q\rho)(0) = \lim_{t\to 0}(D_q\rho)(t),$$

and

$$(D_q^n\rho)(t) = (D_q D_q^{n-1}\rho)(t);\ t \in I,\ n \in \{1,2,\ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 1.2.5 *[71] We define the q-integral of a function $\rho : I_t \to \mathbb{R}$ as follows*

$$(I_q\rho)(t) = \int_0^t \rho(s)d_qs = \sum_{n=0}^{\infty} t(1-q)q^n\rho(tq^n),$$

provided that the series converges.

We have $(D_qI_q\rho)(t) = \rho(t)$, while if ρ is continuous at 0, then

$$(I_qD_q\rho)(t) = \rho(t) - \rho(0).$$

1.3 Elements From Fractional Calculus Theory

In this section, we present some definitions of fractional integral and fractional differential operators and also some properties, lemmas and the fixed point theorem useful in this thesis.

1.3.1 Fractional Integrals

Now, we give some essential definitions and lemmas of fractional calculus theory in this section.

Definition 1.3.1 *[77] Let $\alpha > 0$, for a function $\rho : [0, \infty) \to \mathbb{R}$. We define the Riemann-Liouville fractional integral of order α of ρ as follows*

$$I_{0^+}^{\alpha}\rho(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 1.3.2 *[102] We define the Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $\rho : I \to \mathbb{R}$ as follows $(I_q^0\rho)(t) = \rho(t)$, and*

$$(I_q^{\alpha}\rho)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\rho(s)d_qs;\ t \in I.$$

Definition 1.3.3 *[23, 77] We define the left-sided ψ-Riemann-Liouville fractional integral of order $\alpha(n-1 < \alpha < n)$ for an integrable function $\rho : [a, e] \to \mathbb{R}$ with respect to another function $\psi : [a, e] \to \mathbb{R}$, which is increasing and differentiable such that $\psi'(t) \neq 0$, for all $t \in [a, e], (-\infty \leq a < e \leq +\infty)$, as follows :*

$$I_{a^+}^{\alpha;\psi}\rho(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\rho(s)ds.$$

Definition 1.3.4 *[77]. Let (c, e) $(0 \leq c < e \leq \infty)$ and $\tau > 0$. The Hadamard left-sided fractional integral ${}^HI_{c^+}^{\tau}\rho$ of order τ is defined by*

$$\left({}^HI_{c^+}^{\tau}\rho\right)(t) := \frac{1}{\Gamma(\tau)}\int_c^t \left(\log\frac{t}{s}\right)^{\tau-1}\frac{\rho(s)ds}{s}, \quad c < t < e.$$

When $\tau = 0$, we set

$${}^HI_{c^+}^{0}\rho = \rho.$$

Definition 1.3.5 *[42] We define the Caputo-Fabrizio fractional integral of order $0 < s < 1$ for a function $\rho \in L^1(I)$ as follows*

$${}^{CF}I^s\rho(\tau) = \frac{2(1-s)}{M(s)(2-s)}\rho(\tau) + \frac{2s}{M(s)(2-s)}\int_0^{\tau}\rho(\eta)d\eta, \quad \tau \geq 0.$$

Where $M(s)$ is normalization constant, which depends on s. For $M(s) = \frac{2}{2-s}$, we get

$${}^{CF}I^s\rho(\tau) = (1-s)\rho(\tau) + s\int_0^{\tau}\rho(\eta)d\eta, \quad \tau \geq 0.$$

1.3.2 Fractional Derivatives

Definition 1.3.6 *[77] Let $\alpha > 0$. We define the Caputo fractional derivative of order α of a function $\rho : (0, \infty) \to \mathbb{R}$ as follows*

$$^{C}D_{0^+}^{\alpha}\rho(t) = I_{0^+}^{n-\alpha}\rho^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\rho^{(n)}(s)ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of the real number α.

Definition 1.3.7 *[102] We define the Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $\rho : I \to \mathbb{R}$ as follows $(D_q^0\rho)(t) = \rho(t)$, and*

$$(D_q^{\alpha}\rho)(t) = (D_q^{[\alpha]}I_q^{[\alpha]-\alpha}\rho)(t); \; t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 1.3.8 *[91] We define the Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $\rho : I \to \mathbb{R}$ as follows $(^{C}D_q^0\rho)(t) = \rho(t)$, and*

$$(^{C}D_q^{\alpha}\rho)(t) = (I_q^{[\alpha]-\alpha}D_q^{[\alpha]}\rho)(t); \; t \in I.$$

Definition 1.3.9 *[42] We define the Caputo-Fabrizio fractional derivative of order $0 < s < 1$ for a function $\rho \in AC(I)$ as follows*

$$^{CF}D^s\rho(\tau) = \frac{(2-s)M(s)}{2(1-s)}\int_0^{\tau}\exp(-\frac{s}{1-s}(\tau-\eta))\rho'(\eta)d\eta; \; \tau \in I.$$

We will note that $(^{CF}D^s)(\rho) = 0$ if and only if ρ is a constant function.
If $M(s) = \frac{2}{2-s}$, we have

$$^{CF}D^s\rho(\tau) = \frac{1}{1-s}\int_0^{\tau}\exp(-\frac{s}{1-s}(\tau-\eta))\rho'(\eta)d\eta; \; \tau \in I.$$

Definition 1.3.10 *[77] The left-sided Hadamard fractional derivative of order $\tau (0 \leq \tau < 1)$ on (c, e) is defined by*

$$\left(^{H}D_{c^+}^{\tau}\rho\right)(t) = \frac{1}{\Gamma(1-\tau)}\left(t\frac{d}{dt}\right)\int_c^t\left(\log\frac{t}{s}\right)^{-\tau}\frac{\rho(s)ds}{s}, \quad c < t < e.$$

In particular, when $\tau = 0$ we have

$$^{H}D_{c^+}^{0}\rho = \rho.$$

Definition 1.3.11 *[68] (Hilfer derivative). Let $\alpha \in (0, 1), \beta \in [0, 1], \rho \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)}\rho \in AC(I)$. We define the Hilfer fractional derivative of order α and type β of ρ as follows*

$$\left(D_0^{\alpha,\beta}\rho\right)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}\rho\right)(t); \; \textit{for a.e.} t \in I.$$

Definition 1.3.12 *[72] (Hilfer-Hadamard fractional derivative) We define the left sided fractional derivative of order τ $(0 < \tau < 1)$ and type $0 \leq \theta \leq 1$ with respect to t as follows*

$$\left(^{H}D_{c^+}^{\tau,\theta}\rho\right)(t) = \left(^{H}I_{c^+}^{\theta(1-\tau)} \; ^{H}D_{c^+}^{\tau+\theta-\tau\theta}\rho\right)(t).$$

Definition 1.3.13 *[77] We define the left-sided ψ-Riemann-Liouville fractional derivative of order $\alpha(n-1<\alpha<n)$ for an integrable function $\rho:[a,e]\to\mathbb{R}$ with respect to a different function $\psi:[a,e]\to\mathbb{R}$, which is increasing and differentiable, with $\psi'(t)\neq 0$, for all $t\in[a,e],(-\infty\leq a<e\leq+\infty)$, as follows :*

$$\begin{aligned} D_{a+}^{\alpha;\psi}\rho(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}\rho(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}\rho(s)ds. \end{aligned}$$

Definition 1.3.14 *[112] Let $n-1<\alpha<n, n\in\mathbb{N}$, with $[a,e],-\infty\leq a<e\leq+\infty$, and $\psi\in C^n([a,e],\mathbb{R})$ a function such that $\psi(t)$ is increasing and $\psi'(t)\neq 0$, for all $t\in[a,e]$. The ψ-Hilfer fractional derivative (left-sided) of function $\rho\in C^n([a,e],\mathbb{R})$ of order α and type $\beta\in[0,1]$ is defined as*

$$D_{a^+}^{\alpha,\beta;\psi}\rho(t) = I_{a^+}^{\beta(n-\alpha);\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}\rho(t), t>a.$$

In other way

$$D_{a^+}^{\alpha,\beta;\psi}\rho(t) = I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi}\rho(t), t>a,$$

where

$$D_{a^+}^{\gamma;\psi}\rho(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}\rho(t).$$

Especially, the ψ-Hilfer fractional derivative of order $\alpha\in[0,1]$ and type $\beta\in[0,1]$, can be reformulated in the following form

$$\begin{aligned} D_{a^+}^{\alpha,\beta;\psi}\rho(t) &= \frac{1}{\Gamma(\gamma-\alpha)}\int_a^t (\psi(t)-\psi(s))^{\gamma-\alpha-1} D_{a^+}^{\gamma;\psi}\rho(s)ds \\ &= I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi}\rho(t), \end{aligned}$$

where $\gamma=\alpha+\beta-\alpha\beta$, and

$$D_{a^+}^{\gamma;\psi}\rho(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right] I_{a^+}^{1-\gamma;\psi}\rho(t).$$

1.3.3 Necessary Lemmas, Definitions, Theorems and Properties

Definition 1.3.15 *[55] Let $\mathcal{P}(L)$ be the family of all nonempty subsets of L and H be a mapping from Ω into $\mathcal{P}(L)$. A mapping $T:\{(w,x):w\in\Omega,x\in H(w)\}\to L$ is called a random operator with stochastic domain H if H is measurable (i.e., for all closed $N\subset L,\{w\in\Omega,H(w)\cap N\neq\varnothing\}$ is measurable), and for all open $G\subset L$ and all $x\in L,\{w\in\Omega:x\in H(w),T(w,x)\in G\}$ is measurable. T will be continuous if every $T(w)$ is continuous.*

Let $\beta_{\mathbb{R}^m}$ be the Borel σ-algebra. A mapping $\xi:\Omega\to\mathbb{R}^m$ is said to be measurable if for any $C\in\beta_{\mathbb{R}^m}$; one has,

$$\xi^{-1}(C)=\{w\in\Omega:\xi(w)\in C\}\subset\mathcal{A}.$$

Definition 1.3.16 *[28] Let $\mathcal{A} \times \beta_{\mathbb{R}^m}$ be the direct product of the σ -algebras $\mathcal{A}$ and $\beta_{\mathbb{R}^m}$ those defined in Ω and $\mathbb{R}^m$, respectively. A mapping $L : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if for any $D \in \beta_{\mathbb{R}^m}$, one has :*

$$L^{-1}(D) = \{(w, v) \in \Omega \times E : L(w, v) \in D\} \subset \mathcal{A} \times \beta_{\mathbb{R}^m}.$$

Definition 1.3.17 *[28] A function $L : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $L(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $L(w, \cdot)$ is continuous for all : $w \in \Omega$.*

Definition 1.3.18 *[1] A function $\chi : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is called random Carathéodory in the case of the following conditions being satisfied :*
(i) The map $(t, w) \to \chi(t, \rho, \varrho, w)$ is jointly measurable for all $\rho, \varrho \in \mathbb{R}^m$; and
(ii) The map $(\rho, \varrho) \to \chi(t, \rho, \varrho, w)$ is continuous for a.e. $t \in I$ and $w \in \Omega$.

Theorem 1.3.1 (Arzelà-Ascoli) *[82] Let $H = \{x_n\}_{n\in\mathbb{N}} \subset C[d, e]$. If H is equicontinuous and uniformly bounded, then H is relatively compact (precompact) in $C[d, e]$.*

Lemma 1.3.1 (Corduneanu criteria) *[45] Let $A \subset BC([0, \infty), \mathbb{R})$. Then A is relatively compact if*
(i) A is bounded
(ii) A is equicontinous on all compact subset of $[0, \infty)$
(iii) A is equiconvergent, that is for any $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|y(t) - y(+\infty)| < \epsilon$ for any $t \geq T(\epsilon)$ and $y \in A$.

Lemma 1.3.2 *[77] Let $\alpha, \beta \geq 0$, and $\rho \in L^1([0, 1])$. Then,*

$$I^{\alpha}_{0^+} I^{\beta}_{0^+} \rho(t) = I^{\alpha+\beta}_{0^+} \rho(t),$$

and

$${}^{C}D^{\alpha}_{0^+} I^{\alpha}_{0^+} \rho(t) = \rho(t),$$

for all $t \in [0, 1]$.

Lemma 1.3.3 *[77] Let $\alpha > 0, n = [\alpha] + 1$, then*

$$I^{\alpha}_{0^+} \, {}^{C}D^{\alpha}_{0^+} \rho(t) = \rho(t) - \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

Lemma 1.3.4 *[91] Let $\alpha \in \mathbb{R}_+$. one has :*

$$(I^{\alpha}_{q} \, {}^{C}D^{\alpha}_{q} \rho)(t) = \rho(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1 + k)} (D^k_q \rho)(0).$$

If $\alpha \in (0, 1)$, then

$$(I^{\alpha}_{q} \, {}^{C}D^{\alpha}_{q} \rho)(t) = \rho(t) - \rho(0).$$

Lemma 1.3.5 *[60] Let $\alpha > 0$, $\beta > 0$ and $\rho \in L^1(a, e)$. Then*

$$I^{\alpha;\psi}_{a^+} I^{\beta;\psi}_{a^+} \rho(t) = I^{\alpha+\beta;\psi}_{a^+} \rho(t), \ \text{a.e. } t \in [a, e].$$

In particular
(i) if $\rho \in C_{\gamma;\psi}[a, e]$, then $I^{\alpha;\psi}_{a+} I^{\beta;\psi}_{a+} \rho(t) = I^{\alpha+\beta;\psi}_{a+} \rho(t), t \in (a, e]$.
(ii) If $\rho \in C[a, e]$, then $I^{\alpha;\psi}_{a^+} I^{\beta;\psi}_{a^+} \rho(t) = I^{\alpha+\beta;\psi}_{a^+} \rho(t), t \in [a, e]$.

Lemma 1.3.6 *[112] Let $\alpha > 0, \beta > 0$. If $\rho \in C_{\gamma;\psi}[a,e]$, then*

$$D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} \rho(t) = \rho(t), t \in (a,e].$$

If $\rho \in C^1[a,e]$ then

$$D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} \rho(t) = \rho(t), t \in [a,e].$$

Lemma 1.3.7 *[112] Let $\alpha > 0, 0 \leq \gamma < 1$ and $\rho \in C_{\gamma;\psi}[a,e]$. If $\alpha > \gamma$, then $I_{a^+}^{\alpha;\psi} \rho \in C[a,e]$ and*

$$I_{a^+}^{\alpha;\psi} \rho(a) = \lim_{t \to a^+} I_{a^+}^{\alpha;\psi} \rho(t) = 0.$$

Theorem 1.3.2 *[112] If $\rho \in C^n[a,e], n-1 < \alpha < n, 0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta - \alpha\beta$. Then for all $t \in (a,e]$*

$$I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} \rho(t) = \rho(t) - \sum_{k=1}^{n} \frac{[\psi(t) - \psi(a)]^{\gamma-k}}{\Gamma(\gamma - k + 1)} \rho_{\psi}^{(n-k)} I_{a^+}^{(1-\beta)(n-\alpha);\psi} \rho(a).$$

In particular, if $0 < \alpha < 1$, we have

$$I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} \rho(t) = \rho(t) - \frac{[\psi(t) - \psi(a)]^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{(1-\beta)(1-\alpha);\psi} \rho(a).$$

Additionally, if $\rho \in C_{1-\gamma;\psi}[a,e]$ and $I_{a^+}^{1-\gamma;\psi} \rho \in C^1_{1-\gamma;\psi}[a,e]$ such that $0 < \gamma < 1$. Then for all $t \in (a,e]$

$$I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} \rho(t) = \rho(t) - \frac{[\psi(t) - \psi(a)]^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{1-\gamma;\psi} \rho(a).$$

Properties 1.3.1 *[1] Let $\alpha \in (0,1), \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta$, and $\rho \in L^1(I)$.*
1. The operator $\left(D_0^{\alpha,\beta} \rho\right)(t)$ could be written as :

$$\left(D_0^{\alpha,\beta} \rho\right)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} \rho\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} \rho\right)(t); \text{ for a.e. } t \in I.$$

2. If $D_0^{\gamma} \rho$ exists and is in $L^1(I)$, then

$$\left(I_0^{\alpha} D_0^{\alpha,\beta} \rho\right)(t) = \left(I_0^{\gamma} D_0^{\gamma} \rho\right)(t) = \rho(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \text{ for a.e. } t \in I.$$

Let $BC := BC(\mathbb{R}_+)$ denotes the space of continuous and bounded functions $\iota : \mathbb{R}_+ \to \mathbb{R}$.

Lemma 1.3.8 *[10, 45] Let $N \subset BC$. Then N is relatively compact in BC if the next conditions are satisfied :*

(a) N is uniformly bounded in BC ;
(b) the functions belonging to N are almost equicontinuous in $\mathbb{R}_+$, i.e., equicontinuous on every compact set in $\mathbb{R}_+$;
(c) the functions from N are equiconvergent, i.e., provided $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that

$$\left| w(t) - \lim_{t \to \infty} w(t) \right| < \varepsilon,$$

for any $t \geq L(\varepsilon)$ and $w \in N$.

Let $\varnothing \neq \Lambda \subset BC$ and let $K : \Lambda \to \Lambda$.

$$(K\rho)(t) = \rho(t). \tag{1.1}$$

We present the concept of attractivity of solutions for equation (1.1).

Definition 1.3.19 *[49] Solutions of equation (1.1) are locally attractive if there exists a ball $B(\rho_0, \mu)$ in the space BC such that, for any solutions $\tau = \tau(t)$ and $\xi = \xi(t)$ of equations (1.1) that belong to $B(\rho_0, \mu) \cap \Lambda$, we can write*

$$\lim_{t\to\infty}(\tau(t) - \xi(t)) = 0. \tag{1.2}$$

If the limit (1.2) is uniform with respect to $B(\rho_0, \mu) \cap \Lambda$, then the solutions of equation (1.1) are uniformly locally attractive (or, equivalently, that the solutions of (1.1) are locally asymptotically stable).

1.4 Fixed Point Theorem

Theorem 1.4.1 *(Schauder Fixed-Point Theorem [25, 65]). Let L be a Banach space, let D be a nonempty bounded convex and closed subset of L, and let $P : D \to D$ be a compact and continuous map. Then P has at least one fixed point in D.*

Chapter 2

Successive Approximations for Implicit Fractional q-Difference Equations

2.1 Introduction

This chapter discusses the global convergence of successive approximations and the uniqueness of solutions for a class of implicit Caputo q-difference equations. We provide a theorem on the global convergence of successive approximations for the unique solution of our problem. An illustrative example is given in the last section.

More precisely, in this chapter, we focus on the global convergence of successive approximations for the following implicit fractional q-difference equation

$$({}^cD_q^\alpha \rho)(t) = f(t, \rho(t), ({}^cD_q^\alpha \rho)(t)), \ t \in I := [0, T], \tag{2.1}$$

with the initial condition

$$\rho(0) = \phi \in \mathbb{R}, \tag{2.2}$$

where $q \in (0, 1), \ \alpha \in (0, 1], \ T > 0, \ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

2.2 Successive Approximations and Uniqueness Results

In this section, we are concerned with the main result for the global convergence of successive approximations to the unique solution of the problem (2.1)-(2.2).

Lemma 2.2.1 *[12] Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the problem (2.1)-(2.2) is equivalent to the following integral equation*

$$\rho(t) = \phi + (I_q^\alpha f(\cdot, \rho(\cdot), ({}^CD_q^\alpha \rho)(\cdot))(t). \tag{2.3}$$

Define the space $G := G(I, \mathbb{R})$ as the following :

$$G := \{\rho \in C(I) : \ {}^CD_q^\alpha \rho \ \textit{exists and} \ {}^CD_q^\alpha \rho \in C(I)\}.$$

For $\rho \in G$, denote

$$\|\rho(t)\|_1 = |\rho(t)| + |^C D_q^\alpha \rho(t)|.$$

In the space G we define the norm

$$\|\rho\|_G = \sup_{t \in I} \|\rho(t)\|_1.$$

Remark 2.2.1 $(G, \|\cdot\|_G)$ *is a Banach space.*

Definition 2.2.1 *By a solution of the problem (2.1)-(2.2), we mean a function $\rho \in G$ that satisfies the equation (2.1) on I and the initial condition (2.2).*

Set $I_\varsigma := [0, \varsigma T]$; for any $\varsigma \in [0,1]$. Let us start with the following hypotheses :

(H_1) The function $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exist a constant $\mu > 0$ and a continuous function $w : I \times [0,\mu] \times [0,\mu] \to \mathbb{R}_+$ such that $w(t,\cdot,\cdot)$ is nondecreasing for all $t \in I$, and the inequality

$$|f(t,\rho,\varrho) - f(t,\overline{\rho},\overline{\varrho})| \leq w(t, |\rho - \overline{\rho}|, |\varrho - \overline{\varrho}|) \tag{2.4}$$

holds for all $t \in I$ and $\rho, \varrho, \overline{\rho}, \overline{\varrho} \in \mathbb{R}$ with $|\rho - \overline{\rho}| \leq \mu$ and $|\varrho - \overline{\varrho}| \leq \mu$,

(H_3) $v \equiv 0$ is the only function in $G(I_\gamma, [0,\mu])$ which satisfies the integral inequality

$$v(t) \leq \int_0^{\gamma T} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s, v(s), {}^C D_q^\alpha v(s)) d_q s, \tag{2.5}$$

with $\varsigma \leq \gamma \leq 1$.

Define the successive approximations of the problem (2.3) as follows :

$$\rho_0(t) = \phi; \ t \in I,$$

$$\rho_{n+1}(t) = \phi + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, \rho_n(s), ({}^C D_q^\alpha \rho_n)(s)) d_q s; \ t \in I.$$

Theorem 2.2.1 *Assume that the hypotheses $(H_1) - (H_3)$ hold. Then the successive approximations ρ_n; $n \in \mathbb{N}$ are well defined and converge to the unique solution of the problem (2.1)-(2.2) uniformly on I.*

Proof. Differentiating the two sides of the successive approximations ρ_n; $n \in \mathbb{N}$ by using the Caputo q-fractional derivative, we have

$$({}^C D_q^\alpha \rho_0)(t) = 0; \ t \in I,$$

and

$$({}^C D_q^\alpha \rho_{n+1})(t) = f(t, \rho_n(t), ({}^C D_q^\alpha \rho_n)(t)); \ t \in I.$$

There exist $\mu_1, \ \mu_2 > 0$ such that

$$\|\rho_n\|_\infty \leq \mu_1, \ \|^C D_q^\alpha \rho_n\|_\infty \leq \mu_2.$$

For each $t_1, t_2 \in I$ with $t_1 < t_2$, and for all $t \in I$, we have

$$\begin{aligned}|\rho_n(t_2) - \rho_n(t_1)| &\leq \Bigg| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \rho_{n-1}(s), {}^C D_q^\alpha \rho_{n-1}(s))| d_q s \\ &- \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, \rho_{n-1}(s), {}^C D_q^\alpha \rho_{n-1}(s)) d_q s \Bigg| \\ &\leq \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s, \rho_{n-1}(s), {}^C D_q^\alpha \rho_{n-1}(s))| d_q s \\ &+ \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s, \rho_{n-1}(s), {}^C D_q^\alpha \rho_{n-1}(s))| d_q s.\end{aligned}$$

Then

$$\begin{aligned}|\rho_n(t_2) - \rho_n(t_1)| &\leq \sup_{(t,\rho,\varrho) \in I \times [0,\mu_1] \times [0,\mu_2]} |f(t, \rho, \varrho)| \int_0^{t_1} \frac{|(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\ &+ \sup_{(t,\rho,\varrho) \in I \times [0,\mu_1] \times [0,\mu_2]} |f(t, \rho, \varrho)| \int_{t_1}^{t_2} \frac{|(t_2 - qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\ &\longrightarrow 0, \ \ as \ t_1 \to t_2.\end{aligned}$$

We get

$$\begin{aligned}&|({}^C D_q^\alpha \rho_n)(t_2) - ({}^C D_q^\alpha \rho_n)(t_1)| \\ &\leq |f(t_2, \rho_{n-1}(t_2), {}^C D_q^\alpha \rho_{n-1}(t_2)) - f(t_1, \rho_{n-1}(t_1), {}^C D_q^\alpha \rho_{n-1}(t_1))| \\ &\longrightarrow 0, \ \ as \ t_1 \to t_2.\end{aligned}$$

Thus

$$\|\rho_n(t_2) - \rho_n(t_1)\|_1 \longrightarrow 0, \quad as \quad t_1 \to t_2.$$

Then, the sequence $\{\rho_n(t);\ n \in \mathbb{N}\}$ is equi-continuous on I.

Let

$$\lambda := \sup\{\varsigma \in [0,1] : \{\rho_n(t)\} \text{ converges uniformly on } I_\varsigma\}.$$

If $\lambda = 1$, we obtain the global convergence of successive approximations. Suppose that $\lambda < 1$, then the sequence $\{\rho_n(t)\}$ converges uniformly on I_λ. As this sequence is equicontinuous, and converges uniformly to a continuous function $\tilde{\rho}(t)$. In the case that we prove that there exists $\gamma \in (\lambda, 1]$ such that $\{\rho_n(t)\}$ converges uniformly on I_γ, this will in a contradiction.

Put $\rho(t) = \tilde{\rho}(t)$; for $t \in I_\lambda$. From (H_2), there exist a constant $\mu > 0$ and a continuous function $w : I \times [0, \mu] \times [0, \mu] \to \mathbb{R}_+$ ensuring inequality (2.4). Also, there exist $\gamma \in [\lambda, 1]$ and $n_0 \in \mathbb{N}$, such that for all $t \in I_\gamma$ and $n, m > n_0$, we have

$$|\rho_n(t) - \rho_m(t)| \leq \mu,$$

and

$$|({}^C D_q^\alpha \rho_n)(t) - ({}^C D_q^\alpha \rho_m)(t)| \leq \mu.$$

For any $t \in I_\gamma$, put

$$v^{(n,m)}(t) = |\rho_n(t) - \rho_m(t)|,$$

$$v_k(t) = \sup_{n,m \geq k} v^{(n,m)}(t),$$

$$^{C}D_q^{\alpha}v^{(n,m)}(t)=|^{C}D_q^{\alpha}\rho_n(t)-^{C}D_q^{\alpha}\rho_m(t)|,$$

and

$$(^{C}D_q^{\alpha}v_k)(t)=\sup_{n,m\geq k}(^{C}D_q^{\alpha}v^{(n,m)})(t).$$

Then the sequence $v_k(t)$ is non-increasing, converging to a function $v(t)$ for each $t\in I_\gamma$. From the equi-continuity of $\{v_k(t)\}$ it follows that $\lim_{k\to\infty}v_k(t)=v(t)$ uniformly on I_γ. Additionally, for $t\in I_\gamma$ and $n,m\geq k$, we get

$$\begin{aligned}
v^{(n,m)}(t)&=|\rho_n(t)-\rho_m(t)|\\
&\leq\sup_{s\in[0,t]}|\rho_n(s)-\rho_m(s)|\\
&\leq\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,\rho_{n-1}(s),^{C}D_q^{\alpha}\rho_{n-1}(s))-f(s,\rho_{m-1}(s),^{C}D_q^{\alpha}\rho_{m-1}(s))|d_qs\\
&\leq\int_0^{\gamma T}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,\rho_{n-1}(s),^{C}D_q^{\alpha}\rho_{n-1}(s))-f(s,\rho_{m-1}(s),^{C}D_q^{\alpha}\rho_{m-1}(s))|d_qs.
\end{aligned}$$

Thus, by (2.4) we have

$$\begin{aligned}
v^{(n,m)}(t)&\leq\int_0^{\gamma T}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}w(s,|\rho_{n-1}(s)-\rho_{m-1}(s)|,|^{C}D_q^{\alpha}\rho_{n-1}(s)-^{C}D_q^{\alpha}\rho_{m-1}(s)|)d_qs\\
&=\int_0^{\gamma T}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}w(s,v^{(n-1,m-1)}(s),^{C}D_q^{\alpha}v^{(n-1,m-1)}(s))d_qs.
\end{aligned}$$

Hence

$$v_k(t)\leq\int_0^{\gamma T}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}w(s,v_{k-1}(s),^{C}D_q^{\alpha}v_{k-1}(s))d_qs.$$

By the Lebesgue dominated convergence theorem we have

$$v(t)\leq\int_0^{\gamma T}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}w(s,v(s),^{C}D_q^{\alpha}v(s))d_qs.$$

Moreover, by (H_1) and (H_3) we have $v\equiv0$ on I_γ, which yields that $\lim_{k\to\infty}v_k(t)=0$ uniformly on I_γ. Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on I_γ. In consequence, $\{\rho_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on I_γ that gives us the contradiction.

Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ converges uniformly on I to a continuous function $\rho_*(t)$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,\rho_k(s),^{C}D_q^{\alpha}\rho_k(s))d_qs\\
&=\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,\rho_*(s),^{C}D_q^{\alpha}\rho_*(s))d_qs,
\end{aligned}$$

for each $t\in I$. This means that ρ_* is a solution of the problem (2.1)-(2.2).

Lastly, we prove the uniqueness of solutions of the problem (2.1)-(2.2). Let ρ_1 and ρ_2 be two solutions of (2.3). As previously, put

$$\overline{\lambda}:=\sup\{\varsigma\in[0,1]:\rho_1(t)=\rho_2(t)\ for\ t\in I_\varsigma\},$$

and assuming that $\overline{\lambda} < 1$. There exist a constant $\mu > 0$ and a comparison function $w : I_{\overline{\lambda} \times [0,\mu] \times [0,\mu] \to \mathbb{R}_+}$ satisfying inequality (2.4). We take $\gamma \in (\varsigma, 1)$ such that

$$|\rho_1(t) - \rho_2(t)| \leq \mu \textit{ and } \|({}^C D_q^\alpha \rho_1)(t) - ({}^C D_q^\alpha \rho_2)(t)| \leq \mu;$$

for $t \in I_\gamma$. Then for all $t \in I_\gamma$, we get

$$\begin{aligned} |\rho_1(t) - \rho_2(t)| &\leq \int_0^{\gamma T} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \rho_0(s), {}^C D_q^\alpha \rho_0(s)) - f(s, \rho_1(s), {}^C D_q^\alpha \rho_1(s))| d_q s \\ &\leq \int_0^{\gamma T} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s, |\rho_0(s) - \rho_1(s)|, |{}^C D_q^\alpha \rho_0(s) - {}^C D_q^\alpha \rho_1(s)|) d_q s. \end{aligned}$$

Again, by (H_1) and (H_3) we get $\rho_1 - \rho_2 \equiv 0$ on I_γ. This results $\rho_1 = \rho_2$ on I_γ, which makes a contradiction. Consequently, $\overline{\lambda} = 1$ and the solution of the problem (2.1)-(2.2) is unique on I.

2.3 An Example

Consider the following implicit fractional $\frac{1}{4}$-difference equation

$$\begin{cases} ({}^c D_{\frac{1}{4}}^{\frac{1}{2}} \rho)(t) = \frac{te^{t-3}}{1+|\rho(t)|+|({}^c D_{\frac{1}{4}}^{\frac{1}{2}} \rho)(t)|}; \ t \in [0,1], \\ \rho(0) = 2. \end{cases} \tag{2.6}$$

For each $\rho, \ \varrho, \ \overline{\rho}, \ \overline{\varrho} \in \mathbb{R}, \ p \in \mathbb{N}^*$ and $t \in [0,1]$, we have

$$|f(t, \rho, \varrho) - f(t, \overline{\rho}, \overline{\varrho})| \leq te^t(|\rho - \overline{\rho}| + |\varrho - \overline{\varrho}|).$$

This implies that condition (2.4) holds with any $t \in [0,1], \ \mu > 0$ and the comparison function $w : [0,1] \times [0,\mu] \times [0,\mu] \to [0,\infty)$ given by

$$w(t, \rho, \varrho) = te^t(\rho + \varrho).$$

Consequently, Theorem 2.2.1 means that the successive approximations $\rho_n; \ n \in \mathbb{N}$, defined by

$$\rho_0(t) = 2,$$

$$\rho_{n+1}(t) = 2 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, \rho_n(s), ({}^C D_q^\alpha \rho_n)(s)) d_q s; \ t \in [0,1],$$

converges uniformly on $[0,1]$ to the unique solution of the problem (2.6).

Chapter 3

Uniform Convergence of Successive Approximations for Hybrid Caputo Fractional Differential Equations

3.1 Introduction

This chapter investigates the global convergence of successive approximations and the uniqueness of solutions for a class of hybrid Caputo fractional differential equations. We provide proof for the theorem on the global convergence of successive approximations to the unique solution of our problem. In the final section, an illustrative example is given.

We are mainly concerned, by study the uniformly convergence of successive approximations for the initial value problem for hybrid Caputo fractional differential equation :

$$^{c}D_{0^+}^{\alpha}\left[\frac{\rho(t)}{f(t,\rho(t))}\right] = g(t,\rho(t)),\ t \in I = [0,1], \tag{3.1}$$

with the initial condition

$$\rho(0) = \phi, \tag{3.2}$$

where $\alpha \in (0,1), ^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $f \in C(I \times \mathbb{R}, \mathbb{R}^*)$, $g \in C(I \times \mathbb{R}, \mathbb{R})$.

3.2 Successive Approximations and Uniqueness Results

In this section, we investigate the result of the global convergence of successive approximation towards a unique solution of our problem.

Lemma 3.2.1 *[66] Let $f : I \times \mathbb{R} \to \mathbb{R}^*$, $g : I \times \mathbb{R} \to \mathbb{R}$ be continuous functions. The problem* $(3.1) - (3.2)$ *is equivalent to the integral equation*

$$\rho(t) = f(t,\rho(t))\left\{\frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,\rho(s))ds\right\}. \tag{3.3}$$

Definition 3.2.1 *By a solution of the problem* $(3.1)-(3.2)$*, we mean a function* $\rho \in C(I)$ *that satisfies the equation* (3.1) *on* I *and initial condition* (3.2).

Set $I_\eta := [0, \eta T]$; for any $\eta \in [0,1]$. Let us present the following hypotheses :

(H_1) The functions $f : I \times \mathbb{R} \to \mathbb{R}^*$ and $g : I \times \mathbb{R} \to \mathbb{R}$ are continuous,

(H_2) There exist a constant $\mu > 0$ and a continuous function $w : I \times [0,\mu] \to \mathbb{R}_+$ such that $w(t,\cdot)$ is nondecreasing for all $t \in I$, and the inequality

$$|g(t,\rho) - g(t,\overline{\rho})| \leq w(t, |\rho - \overline{\rho}|), \tag{3.4}$$

holds for all $t \in I$ and $\rho, \overline{\rho} \in \mathbb{R}$ such that $|\rho - \overline{\rho}| \leq \mu$,

(H_3) $V \equiv 0$ is the only function in $C\left(I_\beta, [0,\mu]\right)$ which satisfies the integral inequality

$$V(t) \leq 2 \sup_{(t,\rho)\in I_\beta \times [0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta \times [0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right)$$

$$+ \sup_{(t,\rho)\in I_\beta \times [0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V(s))ds, \tag{3.5}$$

with $\eta \leq \beta \leq 1$.

Define the successive approximations of the problem (3.3) as follows :

$$\rho_0(t) = \phi; \;\; t \in I,$$

$$\rho_{n+1}(t) = f(t, \rho_n(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, \rho_n(s))ds \right\}; \;\; t \in I.$$

Theorem 3.2.1 *Assume* $(H_1)-(H_3)$ *hold. Then the successive approximations* $\rho_n; n \in \mathbb{N}$ *are well defined and converge to the unique solution of the problem* $(3.1)-(3.2)$ *uniformly on I.*

Proof. There exist $\delta > 0$ such that

$$\|\rho_n\|_\infty \leq \delta.$$

Now, for each $t_1, t_2 \in I$ with $t_1 < t_2$, and for all $t \in I$,

$$
\begin{aligned}
|\rho_n(t_2)-\rho_n(t_1)| \le \Bigg| & f(t_2,\rho_{n-1}(t_2))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\} \\
& - f(t_1,\rho_{n-1}(t_1))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\}\Bigg| \\
\le \Bigg| & f(t_2,\rho_{n-1}(t_2))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\} \\
& - f(t_1,\rho_{n-1}(t_1))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\} \\
& + f(t_1,\rho_{n-1}(t_1))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\} \\
& - f(t_1,\rho_{n-1}(t_1))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right\}\Bigg| \\
\le \Big| & f(t_2,\rho_{n-1}(t_2))-f(t_1,\rho_{n-1}(t_1))\Big|\left|\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\right| \\
& + |f(t_1,\rho_{n-1}(t_1))|\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds \\
& + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds \\
& - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\Bigg|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\rho_n(t_2)-\rho_n(t_1)| \le \Big| & f(t_2,\rho_{n-1}(t_2))-f(t_1,\rho_{n-1}(t_1))\Big|\Bigg|\frac{\phi}{f(0,\phi)} \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\Bigg| \\
& + \sup_{(t,\rho)\in I\times[0,\delta]}|f(t,\rho)|\frac{1}{\Gamma(\alpha)}\Bigg|\int_0^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)g(s,\rho_{n-1}(s))ds \\
& + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}g(s,\rho_{n-1}(s))ds\Bigg| \\
\le \Big| & f(t_2,\rho_{n-1}(t_2))-f(t_1,\rho_{n-1}(t_1))\Big|\left(\left|\frac{\phi}{f(0,\phi)}\right|+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}|g(s,\rho_{n-1}(s))|ds\right) \\
& + \sup_{(t,\rho)\in I\times[0,\delta]}|f(t,\rho)|\frac{1}{\Gamma(\alpha)}\Bigg(\int_0^{t_1}\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big||g(s,\rho_{n-1}(s))|ds \\
& + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|g(s,\rho_{n-1}(s))|ds\Bigg) \\
\le \Big| & f(t_2,\rho_{n-1}(t_2))-f(t_1,\rho_{n-1}(t_1))\Big|\left(\left|\frac{\phi}{f(0,\phi)}\right|+\sup_{(t,\rho)\in I\times[0,\delta]}|g(t,\rho)|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right) \\
& + \sup_{(t,\rho)\in I\times[0,\delta]}|f(t,\rho)| \\
& \times \sup_{(t,\rho)\in I\times[0,\delta]}|g(t,\rho)|\left(\int_0^{t_1}\frac{\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big|}{\Gamma(\alpha)}ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right).
\end{aligned}
$$

From the continuity of the fonction f, we obtain

$$\begin{aligned}|\rho_n(t_2)-\rho_n(t_1)| &\le \Big|f(t_2,\rho_{n-1}(t_2))-f(t_1,\rho_{n-1}(t_1))\Big| \\ &\times \left(\left|\frac{\phi}{f(0,\phi)}\right| + \sup_{(t,\rho)\in I\times[0,\delta]}|g(t,\rho)|\frac{t_2^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &+ \sup_{(t,\rho)\in I\times[0,\delta]}|f(t,\rho)| \\ &\times \sup_{(t,\rho)\in I\times[0,\delta]}|g(t,\rho)|\left(\int_0^{t_1}\frac{\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big|}{\Gamma(\alpha)}ds \right. \\ &\left. + \frac{(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\longrightarrow 0,\ as\ t_1\to t_2.\end{aligned}$$

Hence

$$|\rho_n(t_2)-\rho_n(t_1)| \longrightarrow 0, \quad as \quad t_1\to t_2.$$

The sequence $\{\rho_n(t);\ n\in\mathbb{N}\}$ is equi-continuous on I.

Let

$$\varsigma := \sup\{\eta\in[0,1] : \{\rho_n(t)\} \text{ converges uniformly on } I_\eta\}.$$

If $\varsigma = 1$, we obtain the global convergence of successive approximations. Assuming that $\varsigma < 1$, then this sequence being equicontinuous, so it converges uniformly to a continuous function $\tilde{\rho}(t)$. Proving that there exists $\beta\in(\varsigma,1]$ such that $\{\rho_n(t)\}$ converges uniformly on I_β, will eventually leads to a contradiction.

Put $\rho(t)=\tilde{\rho}(t)$; for $t\in I_\varsigma$. From (H_2), there exist a constant $\mu>0$ and a continuous function $w : I\times[0,\mu]\to\mathbb{R}_+$ satisfying inequality (3.4). Also, there exist $\beta\in[\varsigma,1]$ and $n_0\in\mathbb{N}$ such that for all $t\in I_\beta$ and $n,m>n_0$, we have

$$|\rho_n(t)-\rho_m(t)|\le\mu.$$

For any $t\in I_\beta$, put

$$V^{(n,m)}(t) = |\rho_n(t)-\rho_m(t)|,$$

$$V_k(t) = \sup_{n,m\ge k} V^{(n,m)}(t).$$

Since the sequence $V_k(t)$ is non-increasing, converging to a function $V(t)$ for each $t\in I_\beta$. From the equi-continuity of $\{V_k(t)\}$ it follows that $\lim\limits_{k\to\infty} V_k(t) = V(t)$ uniformly on I_β. Furthermore, for $t\in I_\beta$ and $n,m\ge k$, we get

$$\begin{aligned}&V^{(n,m)}(t) = |\rho_n(t)-\rho_m(t)| \\ &\le \sup_{s\in[0,t]}|\rho_n(s)-\rho_m(s)| \\ &\le \left|f(t,\rho_{n-1}(t))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,\rho_{n-1}(s)ds\right\}\right. \\ &\left. - f(t,\rho_{m-1}(t))\left\{\frac{\phi}{f(0,\phi)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,\rho_{m-1}(s)ds\right\}\right|.\end{aligned}$$

Then,

$$\begin{aligned}
V^{(n,m)}(t) &= |\rho_n(t) - \rho_m(t)| \\
&\leq \Bigg| f(t,\rho_{n-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{n-1}(s) ds \right\} \\
&- f(t,\rho_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{n-1}(s) ds \right\} \\
&+ f(t,\rho_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{n-1}(s) ds \right\} \\
&- f(t,\rho_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{m-1}(s) ds \right\} \Bigg| \\
&\leq \Big| f(t,\rho_{n-1}(t)) - f(t,\rho_{m-1}(t) \Big| \left| \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{n-1}(s) ds \right| \\
&+ |f(t,\rho_{m-1}(t))| \\
&\times \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{n-1}(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\rho_{m-1}(s) ds \right|.
\end{aligned}$$

Thus

$$\begin{aligned}
V^{(n,m)}(t) &= |\rho_n(t) - \rho_m(t)| \\
&\leq \Big| f(t,\rho_{n-1}(t)) - f(t,\rho_{m-1}(t) \Big| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s,\rho_{n-1}(s)| ds \right) \\
&+ |f(t,\rho_{m-1}(t))| \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right| g(s,\rho_{n-1}(s) - g(s,\rho_{m-1}(s) \Big| ds \\
&\leq \Big| f(t,\rho_{n-1}(t)) - f(t,\rho_{m-1}(t) \Big| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\
&+ \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s,\rho_{n-1}(s) - g(s,\rho_{m-1}(s) \Big| ds.
\end{aligned}$$

This gives

$$\begin{aligned}
V^{(n,m)}(t) &= |\rho_n(t) - \rho_m(t)| \\
&\leq 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\
&+ \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s,\rho_{n-1}(s) - g(s,\rho_{m-1}(s) \Big| ds.
\end{aligned}$$

Next, by (3.4) we get

$$\begin{aligned}
V^{(n,m)}(t) &\leq 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&+ \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, |\rho_{n-1} - \rho_{m-1}|) ds \\
&\leq 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&+ \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V^{(n-1,m-1)}(s)) ds.
\end{aligned}$$

Hence

$$V_k(t) \le 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right) + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V_{k-1}(s))ds.$$

By the Lebesgue dominated convergence theorem we have

$$V(t) \le 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right) + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V(s))ds.$$

Moreover, by (H_1) and (H_3) we have $V \equiv 0$ on I_β, which yields that $\lim_{k\to\infty} V_k(t) = 0$ uniformly on I_β. Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on I_β. Consequently $\{\rho_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on I_β that gives us the contradiction.

Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ converges uniformly on I to a continuous function $\rho_*(t)$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} f(t,\rho_k(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, \rho_k(s)ds \right\} = f(t,\rho_*(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, \rho_*(s)ds \right\},$$

for each $t \in I$. This means that ρ_* is a solution of the problem (3.1)-(3.2).

Finally, we focus on the uniqueness of solutions of the problem (3.1)-(3.2). Let ρ_1 and ρ_2 be two solutions of (3.3). Put

$$\bar{\varsigma} := \sup\{\eta \in [0,1] : \rho_1(t) = \rho_2(t) \ for \ t \in I_\eta\},$$

and assuming that $\bar{\varsigma} < 1$. There exist a constant $\mu > 0$ and a comparison function $w : I_{\bar{\varsigma}\times[0,\mu]\to\mathbb{R}_+}$ satisfying inequality (3.3). We take $\beta \in (\eta, 1)$ such that

$$|\rho_1(t) - \rho_2(t)| \le \mu \ ;$$

for $t \in I_\beta$. Then for all $t \in I_\beta$, we obtain

$$\begin{aligned} |\rho_1(t) - \rho_2(t)| &\le 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| \right. \\ &\quad \left. + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\ &\quad + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s,\rho_0(s) - g(s,\rho_1(s) \Big| ds \\ &\le 2 \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \left(\left| \frac{\phi}{f(0,\phi)} \right| \right. \\ &\quad \left. + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |g(t,\rho)| \frac{t^\alpha - (t-\beta t)^\alpha}{\Gamma(\alpha+1)} \right) \\ &\quad + \sup_{(t,\rho)\in I_\beta\times[0,\delta]} |f(t,\rho)| \int_0^{\beta t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, |\rho_0 - \rho_1|)ds. \end{aligned}$$

Again, by (H_1) and (H_3) we have $\rho_1 - \rho_2 \equiv 0$ on I_β. This gives $\rho_1 = \rho_2$ on I_β, which yields a contradiction. So, $\overline{\varsigma} = 1$ and the solution of the problem (3.1)-(3.2) is unique on I.

3.3 An Example

Consider the following hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD_{0^+}^{\frac{1}{2}}\left[\frac{\rho(t)}{\sqrt{1+|\rho(t)|}}\right] = \frac{te^t}{1+|\rho(t)|};\ t \in [0,1], \\ \rho(0) = 3. \end{cases} \tag{3.6}$$

For each $\rho,\ \overline{\rho} \in \mathbb{R}$ and $t \in [0,1]$ we have

$$|g(t,\rho) - g(t,\overline{\rho})| \leq te^t(|\rho - \overline{\rho}|).$$

This implies that condition (3.4) holds with any $t \in [0,1],\ \mu > 0$ and the comparison function $w : [0,1] \times [0,\mu] \to [0,\infty)$ given by

$$w(t,\rho) = te^t|\rho|.$$

Consequently, Theorem 3.2.1 shows that the successive approximations $\rho_n;\ n \in \mathbb{N}$, defined by

$$\rho_0(t) = 3;\ t \in [0,1],$$

$$\rho_{n+1}(t) = f(t,\rho_n(t))\left\{\frac{3}{f(0,3)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,\rho_n(s)ds\right\};\ t \in [0,1],$$

converges uniformly on $[0,1]$ to the unique solution of the problem (3.6).

Chapter 4

Successive Approximations for Random Coupled Hilfer Fractional Differential Systems

4.1 Introduction

In this chapter, we study the global convergence of the successive approximations as well as the uniqueness of the random solution of a coupled random Hilfer fractional differential system. Our main result is a theorem on the global convergence of successive approximations to the unique solution of our problem. In the last section of this chapter, we present an illustrative example. Recently, a great attention has been given to the existence of solutions of fractional differential equations with Hilfer fractional derivative [60, 62, 68, 109, 110].

We are interested by the uniform convergence of successive approximations for the coupled random Hilfer fractional differential system

$$\begin{cases} \left(D_0^{\alpha_1,\beta_1}\rho\right)(t,w) = f_1(t,\rho(t,w),\varrho(t,w),w) \\ \left(D_0^{\alpha_2,\beta_2}\varrho\right)(t,w) = f_2(t,\rho(t,w),\varrho(t,w),w) \end{cases} ; t\in I:=[0,T], w\in\Omega, \qquad (4.1)$$

with the initial conditions,

$$\begin{cases} \left(I_0^{1-\gamma_1}\rho\right)(0,w) = \phi_1(w) \\ \left(I_0^{1-\gamma_2}\varrho\right)(0,w) = \phi_2(w) \end{cases} ; w\in\Omega, \qquad (4.2)$$

where $T>0,\ \alpha_i\in(0,1), \beta_i\in[0,1], (\Omega,\mathcal{A})$ is a measurable space and $\mathcal{A}$ is the σ-algebras, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i,\ \phi_i:\Omega\to\mathbb{R}^m,\ f_i: I\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m; i=1,2$, are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann-Liouville derivative (Hilfer) operator of order α_i and type $\beta_i : i=1,2$.

4.2 Successive Approximations and Uniqueness Results

In this section, we discuss the main result of the global convergence of approximations of the problem $(4.1)-(4.2)$. We define, by $\mathcal{C}:=C_{\gamma_1}\times C_{\gamma_2}$, we denote the product weighted space with the norm :

$$\|(\rho,\varrho)\|_{\mathcal{C}} = \|\rho\|_{C_{\gamma_1}} + \|\varrho\|_{C_{\gamma_2}}.$$

Lemma 4.2.1 *[1] Let $\chi \in C_\gamma(I)$. Then, the Cauchy problem*

$$\begin{cases} \left(D_0^{\alpha,\beta}\rho\right)(t) = \chi(t); \ t \in I \\ \left(I_0^{1-\gamma}\rho\right)(t)\Big|_{t=0} = \varphi, \end{cases}$$

has the unique solution

$$\rho(t) = \frac{\varphi}{\Gamma(\gamma)}t^{\gamma-1} + \left(I_0^\alpha \chi\right)(t). \tag{4.3}$$

Definition 4.2.1 *By a solution of the problem* (4.1) − (4.2) *we mean coupled functions* $(\rho,\varrho) \in C_{\gamma_1} \times C_{\gamma_2}$ *that satisfy the system* (4.1) *on* I *and the system* (4.2).

Set $I_\eta := [0, \eta T]$; for any $\eta \in [0,1]$. Assume the following hypothesis :

(H_1) The functions $f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m; i = 1,2$, are random Carathéodory,

(H_2) There exist a constant $\mu > 0$ and continuous functions $g_i : I \times [0,\mu]^m \times [0,\mu]^m \times \Omega \to \mathbb{R}_+$; $i = 1,2$, such that $g_i(t,\cdot,\cdot,w)$ is nondecreasing for any $w \in \Omega$ and each $t \in I$, and

$$\left\|f_i(t,\rho,\varrho,w) - f_i(t,\overline{\rho},\overline{\varrho},w)\right\| \le g_i(t, \|\rho-\overline{\rho}\|_{C_{\gamma_1}}, \|\varrho-\overline{\varrho}\|_{C_{\gamma_2}}, w); \ i = 1,2. \tag{4.4}$$

For any $w \in \Omega$ and each $t \in I$, $\rho,\overline{\rho} \in C_{\gamma_1}$, and $\varrho,\overline{\varrho} \in C_{\gamma_2}$, such that $\|\rho-\overline{\rho}\|_{C_{\gamma_1}} \le \mu$, and $\|\varrho-\overline{\varrho}\|_{C_{\gamma_2}} \le \mu$,

(H_3) $(V,W) \equiv (0,0)$ is the only coupled functions in $\Omega \times C_{\gamma_1}(I_\varsigma,[0,\mu]) \times C_{\gamma_2}(I_\varsigma,[0,\mu])$ which satisfies the integral inequalities,

$$V(t,w) \le \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} g_1(s,V(s,w),W(s,w),w)(t-s)^{\alpha_1-1}ds, \tag{4.5}$$

and

$$W(t,w) \le \frac{1}{\Gamma(\alpha_2)}\int_0^{\varsigma T} g_2(s,V(s,w),W(s,w),w)(t-s)^{\alpha_2-1}ds, \tag{4.6}$$

with $\eta \le \varsigma \le 1$.

Remark 4.2.1 *From (4.4), for any $w \in \Omega$ and each $t \in I$, $\rho \in C_{\gamma_1}$, $\varrho \in C_{\gamma_2}$, and $i = 1,2$, we get*

$$\begin{aligned}\|f_i(t,\rho,\varrho,w)\| &\le \|f_i(t,0,0,w)\| + g_i(t,\|\rho\|_{C_{\gamma_1}},\|\varrho\|_{C_{\gamma_2}},w)\\ &\le f_i^*(w) + g_i^*(w),\end{aligned}$$

where $f_i^*(w) := \sup_{t\in I}\|f_i(t,0,0,w)\|$, *and* $g_i^*(w) := \sup_{(t,x,y)\in I\times[0,\mu]\times[0,\mu]} g_i(t,x,y,w); \ i = 1,2.$

Define the operators $L_1 : \mathcal{C}\times\Omega \to C_{\gamma_1}$, and $L_2 : \mathcal{C}\times\Omega \to C_{\gamma_2}$ by :

$$\left(L_1(\rho,\varrho)\right)(t,w) = \frac{\phi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{f_1(s,\rho(s,w),\varrho(s,w),w)}{\Gamma(\alpha_1)}ds,$$

and

$$\left(L_2(\rho,\varrho)\right)(t,w) = \frac{\phi_2(w)}{\Gamma(\gamma_2)}t^{\gamma_2-1} + \int_0^t (t-s)^{\alpha_2-1}\frac{f_2(s,\rho(s,w),\varrho(s,w),w)}{\Gamma(\alpha_2)}ds.$$

Consider the operator $L : \mathcal{C}\times\ \Omega \to \mathcal{C}$,

$$(L(\rho,\varrho))(t,w) = \left(\left(L_1(\rho,\varrho)\right)(t,w), \left(L_2(\rho,\varrho)\right)(t,w) \right).$$

For any $w \in \Omega$, we define the successive approximations of the problem (4.3) as follows :

$$(\rho_0(t,w), \varrho_0(t,w)) = \Bigg(\phi_1(w), \phi_2(w)\Bigg);\ t \in I$$

$$(\rho_{n+1}(t,w), \varrho_{n+1}(t,w)) = \Bigg(\left(L_1(\rho_n,\varrho_n)\right)(t,w), \left(L_2(\rho_n,\varrho_n)\right)(t,w)\Bigg);\ t \in I.$$

Theorem 4.2.1 *Assume $(H_1)-(H_3)$ hold. Then the successive approximations $((\rho_n)_{n\in\mathbb{N}}, (\varrho_n)_{n\in\mathbb{N}})$ are defined and converge uniformly on I to the unique random solution of problem (4.1)-(4.2).*

Proof. There exist $\theta_1, \theta_2 > 0$ such that $\|\rho\|_{C_{\gamma_1}} \leq \theta_1$, $\|\varrho\|_{C_{\gamma_1}} \leq \theta_2$. Next, for any $w \in \Omega$, and each $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\|t_2^{1-\gamma_1}\rho_n(t_2,w) - t_1^{1-\gamma_1}\rho_n(t_1,w)\| \\
&= \Bigg\| t_2^{1-\gamma_1}\left(\frac{\phi_1(w)}{\Gamma(\gamma_1)}t_2^{\gamma_1-1} + \int_0^{t_2}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\right) \\
&- t_1^{1-\gamma_1}\left(\frac{\phi_1(w)}{\Gamma(\gamma_1)}t_1^{\gamma_1-1} + \int_0^{t_1}(t_1-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\right)\Bigg\| \\
&= \Bigg\| t_2^{1-\gamma_1}\int_0^{t_2}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds \\
&- t_1^{1-\gamma_1}\int_0^{t_1}(t_1-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\Bigg)\Bigg\| \\
&= \Bigg\| t_2^{1-\gamma_1}\int_0^{t_1}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds \\
&+ t_2^{1-\gamma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds \\
&- t_1^{1-\gamma_1}\int_0^{t_1}(t_1-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\Bigg\| \\
&\leq T^{1-\gamma_1}\Bigg\|\int_0^{t_1}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds \\
&+ \int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds \\
&- \int_0^{t_1}(t_1-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\Bigg\| \\
&\leq \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)}\Bigg\|\int_0^{t_1}\Bigg((t_2-s)^{\alpha_1-1} - (t_1-s)^{\alpha_1-1}\Bigg)f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)ds \\
&+ \int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)ds\Bigg\|.
\end{aligned}$$

Then, from Remark 4.2.1, we get

$$\|t_2^{1-\gamma_1}\rho_n(t_2,w)-t_1^{1-\gamma_1}\rho_n(t_1,w)\| \leq \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)}(f_1^*(w)+g_1^*(w))$$
$$\times\left(\int_0^{t_1}|(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}|ds+\int_{t_1}^{t_2}|(t_2-s)^{\alpha_1-1}|ds\right)$$
$$\longrightarrow 0,\ as\ t_1\to t_2.$$

Thus,

$$\|t_2^{1-\gamma_1}\rho_n(t_2,w)-t_1^{1-\gamma_1}\rho_n(t_1,w)\| \longrightarrow 0,\ as\ t_1\to t_2.$$

Also, we obtain that

$$\|t_2^{1-\gamma_2}\varrho_n(t_2,w)-t_1^{1-\gamma_2}\varrho_n(t_1,w)\| \longrightarrow 0,\ as\ t_1\to t_2.$$

Hence

$$\|t_2^{1-\gamma_1}\rho_n(t_2,w)-t_1^{1-\gamma_1}\rho_n(t_1,w)\|+\|t_2^{1-\gamma_2}\varrho_n(t_2,w)-t_1^{1-\gamma_2}\varrho_n(t_1,w)\| \longrightarrow 0,\ as\ t_1\to t_2.$$

So ; the sequence $\{(\rho_n,\varrho_n); n\in\mathbb{N}\}$ is equi-continuous on I, for any $w\in\Omega$.

Let

$$\beta := \sup\left\{\eta\in[0,1] : \{(\rho_n,\varrho_n)\} \text{ converges uniformly on } I_\eta, \text{ for any } w\in\Omega\right\}.$$

If $\beta = 1$, one has the global convergence of successive approximations. Supposing that $\beta < 1$, then the sequence $\{(\rho_n,\varrho_n)\}$ converges uniformly on I_β. As this sequence is equi-continuous, and converges uniformly to a continuous function $(\tilde{\rho}(t),\tilde{\varrho}(t))$. If we prove that there exists $\varsigma\in(\beta,1]$ such that $\{(\rho_n,\varrho_n)\}$ converges uniformly on I_ς, for any $w\in\Omega$. Which is a contradiction.

Put $\rho(t,w)=\tilde{\rho}(t,w)$ and $\varrho(t,w)=\tilde{\varrho}(t,w)$; for each $t\in I_\beta$ and any $w\in\Omega$.
From (H_3) , there exists a constant $\mu>0$ and a function$g_i : I\times[0,\mu]^m\times[0,\mu]^m\times\Omega\to\mathbb{R}_+$ ensuring inequality (4.4). Also, there exist $\varsigma\in[\beta,1]$ and $n_0\in\mathbb{N}$, such that, for all $t\in I_\varsigma$ and any $w\in\Omega$, and $n,m>n_0$, we have

$$\|\rho_n(\cdot,w)-\rho_m(\cdot,w)\|_{C_{\gamma_1}}\leq\mu,$$

,

$$\|\varrho_n(\cdot,w)-\varrho_m(\cdot,w)\|_{C_{\gamma_2}}\leq\mu.$$

For each $t\in I_\varsigma$, and any $w\in\Omega$, we put

$$V^{(n,m)}(\cdot,w)=\|\rho_n(\cdot,w)-\rho_m(\cdot,w)\|_{C_{\gamma_1}},$$
$$V_k(t,w)=\sup_{n,m\geq k}V^{(n,m)}(t,w),$$
$$W^{(n,m)}(\cdot,w)=\|\varrho_n(t)-\varrho_m(\cdot,w)\|_{C_{\gamma_2}},$$
$$W_k(t,w)=\sup_{n,m\geq k}W^{(n,m)}(t,w).$$

Since the sequence $(V_k(t,w),W_k(t,w))$ is non-increasing, it is convergent to $(V(t,w),W(t,w))$ for each $t\in I_\varsigma$, and for any $w\in\Omega$. From the equicontinuity of $\{(V_k(t,w),W_k(t,w))\}$, it follows that $\lim_{k\to\infty} V_k(t,w) = V(t,w)$ and $\lim_{k\to\infty} W_k(t,w) = W(t,w)$ uniformly on I_ς. Additionally, for each $t \in I_\varsigma$, and any $w \in \Omega$, and for $n, m \geq k$, we get

$$
\begin{aligned}
&V^{(n,m)}(\cdot,w) = \|\rho_n(\cdot,w) - \rho_m(\cdot,w)\|_{C_{\gamma_1}} = \|t^{1-\gamma_1}\big(\rho_n(t,w) - \rho_m(t,w)\big)\| \\
&\leq \sup_{s\in[0,t]} \|t^{1-\gamma_1}\big(\rho_n(s,w) - \rho_m(s,w)\big)\| \\
&\leq \Bigg\| t^{1-\gamma_1}\Bigg[\Bigg(\frac{\phi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w)}{\Gamma(\alpha_1)}ds\Bigg) \\
&- \Bigg(\frac{\phi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1} + \int_0^t (t-s)^{\alpha_1-1}\frac{f_1(s,\rho_{m-1}(s,w),\varrho_{m-1}(s,w),w)}{\Gamma(\alpha_1)}ds\Bigg)\Bigg]\Bigg\| \\
&\leq \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\Bigg\| f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w) \\
&- f_1(s,\rho_{m-1}(s,w),\varrho_{m-1}(s,w),w)\Bigg\| ds \\
&\leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} (t-s)^{\alpha_1-1}s^{1-\gamma_1}\Bigg\| f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w) \\
&- f_1(s,\rho_{m-1}(s,w),\varrho_{m-1}(s,w),w)\Bigg\| ds \\
&\leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} \Bigg\| f_1(s,\rho_{n-1}(s,w),\varrho_{n-1}(s,w),w) - f_1(s,\rho_{m-1}(s,w),\varrho_{m-1}(s,w),w)\Bigg\|_{C_{\gamma_1}} \\
&\times (t-s)^{\alpha_1-1}ds.
\end{aligned}
$$

Thus, from (4.4) we get

$$
\begin{aligned}
&V^{(n,m)}(t,w) \leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} g_1(s, \|\rho_{n-1}(s,w) - \rho_{m-1}(s,w)\|_{C_{\gamma_1}}, \|\varrho_{n-1}(s,w) - \varrho_{m-1}(s,w)\|_{C_{\gamma_2}}, w) \\
&(t-s)^{\alpha_1-1}ds \\
&= \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} g_1(s, V^{(n-1,m-1)}(s,w), W^{(n-1,m-1)}(s,w), w).(t-s)^{\alpha_1-1}ds.
\end{aligned}
$$

Hence

$$V_k(t,w) \leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} g_1(s, V_{k-1}(s,w), W_{k-1}(s,w), w)(t-s)^{\alpha_1-1}ds.$$

By the Lebesgue dominated convergence theorem we have

$$V(t,w) \leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T} g_1(s, V(s,w), W(s,w), w)(t-s)^{\alpha_1-1}ds.$$

Also, we find that

$$W(t,w) \leq \frac{1}{\Gamma(\alpha_2)}\int_0^{\varsigma T} g_2(s, V(s,w), W(s,w), w)(t-s)^{\alpha_2-1}ds.$$

Then, from (H_1) and (H_3) we get $V \equiv 0$ and $W \equiv 0$ on $I_\varsigma \times \Omega$, which yields that $\lim_{k\to\infty}(V_k(t,w), W_k(t,w)) = (0,0)$ uniformly on $I_\varsigma \times \Omega$. Thus $\{(\rho_k(t,w), \varrho_k(t,w))\}_{k=1}^\infty$ is a Cauchy sequence on $I_\varsigma \times \Omega$. Consequently $\{(\rho_k(t,w), \varrho_k(t,w))\}_{k=1}^\infty$ is uniformly convergent on I_ς that gives us the contradiction.

Thus $\{(\rho_k(t,w),\varrho_k(t,w))\}_{k=1}^{\infty}$ converges uniformly on I for any $w\in\Omega$ to a continuous function $(\rho_*(t,w),\varrho_*(t,w))$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty}\frac{\phi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1}+\int_0^t(t-s)^{\alpha_1-1}\frac{f(s,\rho_k(s,w),\varrho_k(s,w),w)}{\Gamma(\alpha_1)}ds$$

$$=\frac{\phi_1(w)}{\Gamma(\gamma_1)}t^{\gamma_1-1}+\int_0^t(t-s)^{\alpha_1-1}\frac{f(s,\rho_*(s,w),\varrho_*(s,w),w)}{\Gamma(\alpha_1)}ds,$$

and

$$\lim_{k\to\infty}\frac{\phi_2(w)}{\Gamma(\gamma_2)}t^{\gamma_2-1}+\int_0^t(t-s)^{\alpha_2-1}\frac{f(s,\rho_k(s,w),\varrho_k(s,w),w)}{\Gamma(\alpha_2)}ds$$

$$=\frac{\phi_2(w)}{\Gamma(\gamma_2)}t^{\gamma_2-1}+\int_0^t(t-s)^{\alpha_2-1}\frac{f(s,\rho_*(s,w),\varrho_*(s,w),w)}{\Gamma(\alpha_2)}ds,$$

for each $t\in I$. This means that (ρ_*,ϱ_*) is a solution of the problem (4.1)-(4.2).

In the last step, we discuss the uniqueness of solutions of the problem (4.1)-(4.2). Let (ρ_1,ϱ_1) and (ρ_2,ϱ_2) be two solutions. As previously, we put

$$\overline{\beta}:=\sup\{\eta\in[0,1]:\rho_1(t,w)=\rho_2(t,w),\ \varrho_1(t,w)=\varrho_2(t,w)\text{ for }t\in I_\eta,\text{ and }w\in\Omega\},$$

and assuming that $\overline{\beta}<1$. There exist a constant $\mu>0$ and a comparison function $g_i:I_{\overline{\beta}\times[0,\mu]^m\times[0,\mu]^m\times\Omega\to\mathbb{R}_+}$; $i=1,2$, satisfying inequality (4.4). We take $\varsigma\in(\eta,1)$ such that

$$\|\rho_1(\cdot,w)-\rho_2(\cdot,w)\|_{C_{\gamma_1}}\le\mu\ and\ \|\varrho_1(\cdot,w)-\varrho_2(\cdot,w)\|_{C_{\gamma_2}}\le\mu;$$

$$\begin{aligned}&\|\rho_1(\cdot,w)-\rho_2(\cdot,w)\|_{C_{\gamma_1}}\\&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T}\left\|f_1(s,\rho_0(s,w),\varrho_0(s,w),w)-f_1(s,\rho_1(s,w),\varrho_1(s,w),w)\right\|_{C_{\gamma_1}}(t-s)^{\alpha_1-1}ds\\&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{\varsigma T}g_1(s,\|\rho_0(s,w)-\rho_1(s,w)\|_{C_{\gamma_1}},\|\varrho_0(s,w)-\varrho_1(s,w)\|_{C_{\gamma_2}},w)(t-s)^{\alpha_1-1}ds,\end{aligned}$$

and

$$\begin{aligned}&\|\varrho_1(\cdot,w)-\varrho_2(\cdot,w)\|_{C_{\gamma_2}}\\&\le\frac{1}{\Gamma(\alpha_2)}\int_0^{\varsigma T}\left\|f_2(s,\rho_0(s,w),\varrho_0(s,w),w)-f_2(s,\rho_1(s,w),\varrho_1(s,w),w)\right\|_{C_{\gamma_1}}(t-s)^{\alpha_2-1}ds\\&\le\frac{1}{\Gamma(\alpha_2)}\int_0^{\varsigma T}g_2(s,\|\rho_0(s,w)-\rho_1(s,w)\|_{C_{\gamma_1}},\|\varrho_0(s,w)-\varrho_1(s,w)\|_{C_{\gamma_2}},w)(t-s)^{\alpha_2-1}ds.\end{aligned}$$

Again, by (H_1) and (H_3) we get $\rho_1-\rho_2\equiv0$ and $\varrho_1-\varrho_2\equiv0$ on $I_\varsigma\times\Omega$. This results $\rho_1=\rho_2$ and $\varrho_1=\varrho_2$ on $I_\varsigma\times\Omega$, which makes a contradiction. Consequently, $\overline{\beta}=1$ and the solution of the problem (4.1)-(4.2) is unique.

4.3 An Example

We equip the space $\mathbb{R}_-^*:=(-\infty,0)$ with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the following random coupled Hilfer fractional differential system :

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}}\rho)(t,w) = f_1(t,\rho(t,w),\varrho(t,w),w) \\ (D_0^{\frac{1}{2},\frac{1}{2}}\varrho)(t) = f_2(t,\rho(t,w),\varrho(t,w),w)) \\ (I_0^{\frac{1}{4}}\rho)(0,w) = 2\sin w \\ (I_0^{\frac{1}{4}}\varrho)(0,w) = 2\cos w, \end{cases} \quad ; \ t \in [0,1], \ w \in \mathbb{R}_-^*, \tag{4.7}$$

where

$$f_1(t,\rho,\varrho,w) = \frac{w^2 \sin t}{(2+w^2)(1+|\rho|+|\varrho|)}; t \in [0,1], \ w \in \mathbb{R}_-^*,$$

$$f_2(t,\rho,\varrho,w) = \frac{w^2 \cos t}{(2+w^2)(1+|\rho|+|\varrho|)}; t \in [0,1], \ w \in \mathbb{R}_-^*,$$

$$\text{with } \alpha_i = \beta_i = \frac{1}{2}; i = 1,2, \text{ and } \gamma_i = \frac{3}{4}; i = 1,2.$$

For each $\rho, \ \varrho, \ \overline{\rho}, \ \overline{\varrho} \in \mathbb{R}, \ p \in \mathbb{N}^*$ and $t \in [0,1]$, we have,

$$\begin{aligned} &\left\| f_1(s,\rho,\varrho,w) - f_1(s,\overline{\rho},\overline{\varrho},w) \right\|_{C_{\frac{3}{4}}} \\ &= \left\| t^{\frac{1}{4}} \left(\frac{w^2 \sin t}{(2+w^2)(1+|\rho|+|\varrho|)} - \frac{w^2 \sin t}{(2+w^2)(1+|\overline{\rho}|+|\overline{\varrho}|)} \right) \right\| \\ &\le \left\| \frac{t^{\frac{1}{4}} w^2 \sin t}{(2+w^2)} \left(\frac{1}{1+|\rho|+|\varrho|} - \frac{1}{1+|\overline{\rho}|+|\overline{\varrho}|} \right) \right\| \\ &\le \frac{w^2 t^{\frac{1}{4}}}{(2+w^2)} \left\| \frac{((|\overline{\rho}|-|\rho|)+(|\overline{\varrho}|-|\varrho|)}{(1+|\rho|+|\varrho|)(1+|\overline{\rho}|+|\overline{\varrho}|)} \right\| \\ &\le \frac{w^2}{2+w^2} \left(\|\rho-\overline{\rho}\|_{C_{\frac{1}{4}}} + \|\rho-\overline{\rho}\|_{C_{\frac{1}{4}}} \right). \end{aligned}$$

Also, we obtain

$$\left\| f_2(s,\rho,\varrho,w) - f_2(s,\overline{\rho},\overline{\varrho},w) \right\|_{C_{\frac{3}{4}}} \le \frac{w^2}{2+w^2} \left(\|\rho-\overline{\rho}\|_{C_{\frac{1}{4}}} + \|\rho-\overline{\rho}\|_{C_{\frac{1}{4}}} \right).$$

This implies that condition (4.4) holds for $t \in [0,1], \ \mu > 0$ and the comparison functions $g_i : [0,1] \times [0,\mu] \to [0,\infty); \ i = 1,2$ given by

$$g_i(t,\rho,\varrho,w) = \frac{w^2}{2+w^2}(\rho+\varrho); \ i = 1,2.$$

Consequently, Theorem 4.2.1 implies that the successive approximations $(\rho_n, \varrho_n); \ n \in \mathbb{N}$, defined by

$$(\rho_0(t), \varrho_0(t)) = \left(2\sin w, 2\cos w \right); t \in I,$$

$$(\rho_{n+1}(t), \varrho_{n+1}(t)) = \left(\left(L_1(\rho_n,\varrho_n) \right)(t,w), \left(L_2(\rho_n,\varrho_n) \right)(t,w) \right); t \in I,$$

where

$$\left(L_1(\rho,\varrho)\right)(t,w)=\frac{2\sin w}{\Gamma\left(\frac{3}{4}\right)}t^{\frac{3}{4}-1}+\int_0^t (t-s)^{\frac{1}{2}-1}\frac{f_1(s,\rho(s,w),\varrho(s,w),w)}{\Gamma\left(\frac{1}{2}\right)}ds,$$

and

$$\left(L_2(\rho,\varrho)\right)(t,w)=\frac{2\cos w}{\Gamma\left(\frac{3}{4}\right)}t^{\frac{3}{4}-1}+\int_0^t (t-s)^{\frac{1}{2}-1}\frac{f_2(s,\rho(s,w),\varrho(s,w),w)}{\Gamma\left(\frac{1}{2}\right)}ds,$$

converge uniformly on $[0,1]$ to the unique solution of the problem (4.7).

Chapter 5

Successive Approximations for Nonlinear ψ-Hilfer Implicit Fractional Differential Equations

5.1 Introduction

In this chapter, we discuss the global convergence of successive approximations and the uniqueness of the solution for a class of nonlinear ψ-Hilfer differential equations. We provide a theorem on the global convergence of successive approximations to the unique solution of our problem.

In the following, we investigate the uniform convergence of successive approximations for the ψ-Hilfer Cauchy-type problem

$$D_{a^+}^{\alpha,\beta;\psi}\rho(t) = g\left(t, \rho(t), D_{a^+}^{\alpha,\beta;\psi}\rho(t)\right);\ 0 < \alpha < 1, 0 \leq \beta \leq 1,\ 0 \leq a < t \leq b. \tag{5.1}$$

$$I_{a^+}^{1-\gamma;\psi}\rho(a) = \rho_a, \qquad \rho_a \in \mathbb{R}, \quad \gamma = \alpha + \beta - \alpha\beta, \tag{5.2}$$

where $D_{a^+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative, $I_{a^+}^{1-\gamma;\psi}$ is $\psi-$ Riemann-Liouville fractional integral, $g : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function and ρ_a is a constant.

5.2 Successive Approximations and Uniqueness Results

In this section, we are interested on the main result for the global convergence of successive approximations to the unique solution of the problem (5.1) – (5.2).

Lemma 5.2.1 *[113] Let $g : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot, \rho, \varrho) \in C_{1-\gamma;\psi}[a,b]$ for any $\rho, \varrho \in C_{1-\gamma;\psi}[a,b]$. Then the problem (5.1)-(5.2) is equivalent to the following integral equation*

$$\rho(t) = \frac{\rho_a}{\Gamma(\gamma)}(\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g\left(s, \rho_s, D_{a^+}^{\alpha,\beta;\psi}\rho_s\right) ds. \tag{5.3}$$

Define the space $G := G((a,b],\mathbb{R})$ as the following :

$$G := \{\rho \in C_{1-\gamma;\psi[a,b]} : \ D^{\alpha,\beta;\psi}_{a^+}\rho \ exists \ and \ D^{\alpha,\beta;\psi}_{a^+}\rho \in C_{1-\gamma;\psi[a,b]}\}.$$

For $\rho \in G$, in the space G we define the norm

$$\|\rho\|_G = \|\rho\|_{C_{1-\gamma;\psi}}[a,b] + \|D^{\alpha,\beta;\psi}_{a^+}\rho\|_{C_{1-\gamma;\psi}}[a,b].$$

Remark 5.2.1 *$(G, \|\cdot\|_G)$ is a Banach space.*

Definition 5.2.1 *By a solution of the problem* $(5.1) - (5.2)$*, we mean a function* $\rho \in G$ *that satisfies the equation (5.1) on* $(a.b]$ *and the initial condition (5.2).*

Set $I_\xi := (a, \xi b]$; for any $\xi \in (0,1]$. Let us innumerate the following hypotheses :

(H_1) $g : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot,\rho,\varrho) \in C_{1-\gamma;\psi}[a,b]$ for any $\rho, \varrho \in C_{1-\gamma;\psi}[a,b]$.

(H_2) There exist a constant $\mu > 0$ and a function $w : (a,b] \times (a,\mu] \times (a,\mu] \to \mathbb{R}_+$ such that $w(\cdot,\rho,\varrho) \in C_{1-\gamma;\psi[a,b]}$ for any $\rho, \varrho \in C_{1-\gamma;\psi[a,b]}$,and $w(t,\cdot,\cdot)$ is nondecreasing for all $t \in (a,b]$, the inequality

$$\|g(t,\rho,\varrho) - g(t,\overline{\rho},\overline{\varrho})\|_{C_{1-\gamma;\psi}[a,b]} \leq w(t, \|\rho - \overline{\rho}\|_{C_{1-\gamma;\psi}[a,b]}, \|\varrho - \overline{\varrho}\|_{C_{1-\gamma;\psi}[a,b]}), \quad (5.4)$$

holds for all $t \in (a,b]$ and $\rho, \varrho, \overline{\rho}, \overline{\varrho} \in C_{1-\gamma;\psi}[a,b]$ that $\|\rho - \overline{\rho}\|_{C_{1-\gamma;\psi}[a,b]} \leq \mu$ and $\|\varrho - \overline{\varrho}\|_{C_{1-\gamma;\psi}[a,b]} \leq \mu$,

(H_3) $v \equiv 0$ is the only function in $G(I_\nu, (a,\mu])$ that satisfies the integral inequality

$$v(t) \leq \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} w(s, v(s), D^{\alpha,\beta;\psi}_{a^+}v(s))\psi'(s)(\psi((t)) - \psi(s))^{\alpha-1}ds, \quad (5.5)$$

with $\xi \leq \nu \leq 1$.

Define the successive approximations of the problem (5.3) as follows :

$\rho_0(t) = \rho_a, \quad \rho_{n+1}(t) = \frac{\rho_a}{\Gamma(\gamma)}(\psi(t) - \psi(a))^{\gamma-1}$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g\left(s, \rho_n(s), D^{\alpha,\beta;\psi}_{a^+}\rho_n(s)\right) ds.$$

Theorem 5.2.1 *Assume* $(H_1)-(H_3)$ *hold. Then the successive approximations* $\rho_n;\ n \in \mathbb{N}$ *are well defined and converge to the unique solution of the problem (5.1)-(5.2) uniformly on* $(a.b]$

Proof. There exist $\mu_1,\ \mu_2 > 0$ such that

$$\|\rho_n\|_{C_{1-\gamma;\psi}[a,e]} \leq \mu_1, \ \|D^{\alpha,\beta;\psi}_{a^+}\rho_n\|_{C_{1-\gamma;\psi}[a,e]} \leq \mu_2.$$

For each $t_1, t_2 \in (a,b]$ with $t_1 < t_2$, and for all $t \in (a,b]$, we have

$$
\begin{aligned}
&\left|[(\psi(t_2)-\psi(a)]^{1-\gamma}\rho_n(t_2)-[(\psi(t_1)-\psi(a)]^{1-\gamma}\rho_n(t_1)\right|\\
&=\left|[\psi(t_2)-\psi(a)]^{1-\gamma}\left(\frac{\rho_a}{\Gamma(\gamma)}(\psi(t_2)-\psi(a))^{\gamma-1}\right.\right.\\
&+\left.\frac{1}{\Gamma(\alpha)}\int_a^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)ds\right)\\
&-[\psi(t_1)-\psi(a)]^{1-\gamma}\left(\frac{\rho_a}{\Gamma(\gamma)}(\psi((t_1))-\psi(a))^{\gamma-1}\right.\\
&+\left.\left.\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)ds\right)\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left|[\psi(t_2)-\psi(a)]^{1-\gamma}\right.\\
&\times\left(\int_a^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)ds\right.\\
&+\left.\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)ds\right)\\
&-\left.[(\psi(t_1)-\psi(a)]^{1-\gamma}\int_a^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)ds\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|[\psi(t_2)-\psi(a)]^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\right.\\
&-\left.[\psi(t_1)-\psi(a)]^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\right||\psi'(s)||g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)|ds\\
&+\frac{1}{\Gamma(\alpha)}[\psi(t_2)-\psi(a)]^{1-\gamma}\\
&\times\int_{t_1}^{t_2}|\psi'(s)||(\psi(t_2)-\psi(s))^{\alpha-1}|\left|g\left(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s)\right)\right|ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|[\psi(t_2)-\psi(a)]^{1-\gamma}\rho_n(t_2)-[\psi(t_1)-\psi(a)]^{1-\gamma}\rho_n(t_1)\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\sup_{(t,\rho,\varrho)\in I\times(a,\mu_1]\times(a,\mu_2]}|g(t,\rho,\varrho)|\int_a^{t_1}\left|[\psi(t_2)-\psi(a)]^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\right.\\
&-\left.|[\psi(t_1)-\psi(a)]^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\right||\psi'(s)|ds\\
&+\frac{1}{\Gamma(\alpha)}\sup_{(t,\rho,\varrho)\in I\times(a,\mu_1]\times(a,\mu_2]}|g(t,\rho,\varrho)|[\psi(t_2)-\psi(a)]^{1-\gamma}\int_{t_1}^{t_2}|\psi'(s)||\psi(t_2)-\psi(s))^{\alpha-1}|ds\\
&\leq\frac{1}{\Gamma(\alpha)}\sup_{(t,\rho,\varrho)\in I\times(0,\mu_1]\times(0,\mu_2]}|g(t,\rho,\varrho)|\int_a^{t_1}\left|[\psi(t_2)-\psi(a)]^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\right.\\
&-\left.|[\psi(t_1)-\psi(a)]^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\right||\psi'(s)|ds\\
&+\frac{1}{\Gamma(\alpha+1)}\sup_{(t,\rho,\varrho)\in I\times(a,\mu_1]\times(a,\mu_2]}|g(t,\rho,\varrho)|[\psi(t_2)-\psi(a)]^{1-\gamma}(\psi(t_2)-\psi(t_1))^{\alpha}|\\
&\longrightarrow 0,\ as\ t_1\to t_2.
\end{aligned}
$$

$$\|[(\psi(t_2)-\psi(a)]^{1-\gamma}\rho_n(t_2)-[\psi(t_1)-\psi(a)]^{1-\gamma}\rho_n(t_1)\|_{C[a,e]} \longrightarrow 0,\ as\ t_1\to t_2.$$

On the other hand, and since $g(\cdot,\rho,\varrho)\in C_{1-\gamma;\psi}[a,b]$, then by using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&|[(\psi(t_2)-\psi(a)]^{1-\gamma}(D^{\alpha,\beta;\psi}_{a^+}\rho_n)(t_2)-[(\psi(t_1)-\psi(a)]^{1-\gamma}(D^{\alpha,\beta;\psi}_{a^+}\rho_n)(t_1)|\\
&\le |[\psi(t_2)-\psi(a)]^{1-\gamma}g(t_2,\rho_{n-1}(t_2),D^{\alpha,\beta;\psi}_{a^+}\rho_{n-1}(t_2))\\
&-[(\psi(t_1)-\psi(a)]^{1-\gamma}g(t_1,\rho_{n-1}(t_1),D^{\alpha,\beta;\psi}_{a^+}\rho_{n-1}(t_1))|\\
&\longrightarrow 0,\ as\ t_1\to t_2.
\end{aligned}$$

Thus

$$\|[\psi(t_2)-\psi(a)]^{1-\gamma}(D^{\alpha,\beta;\psi}_{a^+}\rho_n)(t_2)-[(\psi(t_1)-\psi(a)]^{1-\gamma}(D^{\alpha,\beta;\psi}_{a^+}\rho_n)(t_1)\|_{C[a,b]}$$
$$\longrightarrow 0,\ as\ t_1\to t_2.$$

So, we get

$$\|\rho_n(t_2)-\rho_n(t_1)\|_G \longrightarrow 0,\quad as\quad t_1\to t_2.$$

Then, the sequence $\{\rho_n(t);\ n\in\mathbb{N}\}$ is equicontinuous on $(a,b]$.

Let

$$\iota := \sup\{\xi\in[0,1] : \{\rho_n(t)\} \text{ converges uniformly on } I_\xi\}.$$

If $\iota=1$ and then we will obtain the global convergence of successive approximations. Supposing that $\iota<1$, so the sequence $\{\rho_n(t)\}$ converges uniformly on I_ι. This sequence is equicontinuous, which means it converges uniformly to a continuous function $\tilde{\rho}(t)$. Proving that there exists $\nu\in(\iota,1]$ such that $\{\rho_n(t)\}$ converges uniformly on I_ν, this leads to a contradiction.

Put $\rho(t)=\tilde{\rho}(t)$; for $t\in I_\iota$. From (H_2), there exist a constant $\mu>0$ and a continuous function $w: I\times(a,\mu]\times(a,\mu]\to\mathbb{R}_+$ satisfying inequality (5.4). Also, there exist $\nu\in[\iota,1]$ and $n_0\in\mathbb{N}$ such that, for all $t\in I_\nu$ and $n,m>n_0$, we have

$$\|[\psi(t)-\psi(a)]^{1-\gamma}[\rho_n(t)-\rho_m(t)]\|_{C[a,b]}\le\mu,$$

and

$$|[\psi(t)-\psi(a)]^{1-\gamma}[(D^{\alpha,\beta;\psi}_{a^+}\rho_n)(t)-(D^{\alpha,\beta;\psi}_{a^+}\rho_m)(t)]\|_{C[a,b]}\le\mu.$$

For any $t\in I_\nu$, put

$$v^{(n,m)}(t)=\|[\psi(t)-\psi(a)]^{1-\gamma}[\rho_n(t)-\rho_m(t)]\|_{C[a,b]},$$

$$v_k(t)=\sup_{n,m\ge k} v^{(n,m)}(t),$$

$$D^{\alpha,\beta;\psi}_{a^+}v^{(n,m)}(t)=\|[\psi(t)-\psi(a)]^{1-\gamma}[D^{\alpha,\beta;\psi}_{a^+}\rho_n(t)-D^{\alpha,\beta;\psi}_{a^+}\rho_m(t))]\|_{C[a,b]},$$

and

$$(D^{\alpha,\beta;\psi}_{a^+}v_k)(t)=\sup_{n,m\ge k} D^{\alpha,\beta;\psi}_{a^+}v^{(n,m)})(t).$$

Since the sequence $v_k(t)$ is non-increasing, converging to a function $v(t)$ for each $t\in I_\nu$. From the equi-continuity of $\{v_k(t)\}$ it follows that $\lim\limits_{k\to\infty} v_k(t)=v(t)$ uniformly on I_ν. Furthermore, for $t\in I_\nu$ and $n,m\ge k$, we have

$$v^{(n,m)}(t) = \|(\psi(t)-\psi(a)]^{1-\gamma}[\rho_n(t)-\rho_m(t)]\|_{C[a,b]}$$

$$\left|[\psi(t)-\psi(a)]^{1-\gamma}\bigg(\rho_n(t)-\rho_m(t)\bigg)\right| \leq \bigg|[\psi(t)-\psi(a)]^{1-\gamma}\bigg[\frac{\rho_a}{\Gamma(\gamma)}(\psi((t))-\psi(a))^{\gamma-1}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}g(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))ds$$
$$-\frac{\rho_a}{\Gamma(\gamma)}(\psi((t))-\psi(a))^{\gamma-1}$$
$$-\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}g\left(s,\rho_{m-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s)\right)ds\bigg]\bigg|$$
$$\leq |[(\psi(t)-\psi(a)]^{1-\gamma}|\frac{1}{\Gamma(\alpha)}\int_a^t |\psi'(s)(\psi((t))-\psi(s))^{\alpha-1}|$$
$$\left|g(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))-g(s,\rho_{m-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s))\right|ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}\bigg|[(\psi(s)-\psi(a)]^{1-\gamma}$$
$$[g(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))-g(s,\rho_{m-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s))]\bigg|ds$$

$$v^{(n,m)}(t) \leq \frac{1}{\Gamma(\alpha)}\int_a^t \bigg\|[(\psi(s)-\psi(a)]^{1-\gamma}$$
$$[g(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))-g(s,\rho_{m-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s))]\bigg\|_{C[a,e]}$$
$$\times \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} \bigg\|[(\psi(s)-\psi(a)]^{1-\gamma}$$
$$[g(s,\rho_{n-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))-g(s,\rho_{m-1}(s),D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s))]\bigg\|_{C[a,e]}$$
$$\times \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds.$$

Thus, by (5.4) we get

$$v^{(n,m)}(t) \leq \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} w(s,\|\rho_{n-1}(s)-\rho_{m-1}(s)\|_{C_{1-\gamma;\psi}[a,e]},$$
$$\|D_{a^+}^{\alpha,\beta;\psi}\rho_{n-1}(s))-D_{a^+}^{\alpha,\beta;\psi}\rho_{m-1}(s))\|_{C_{1-\gamma;\psi}[a,b]})\psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^{\nu t} w(s,v^{(n-1,m-1)}(s),D_{a^+}^{\alpha,\beta;\psi}v^{(n-1,m-1)}(s))\psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds.$$

Hence

$$v_k(t) \leq \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} w(s,v_{k-1}(s),D_{a^+}^{\alpha,\beta;\psi}v_{k-1}(s))\psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds.$$

By the Lebesgue dominated convergence theorem we have

$$v(t) \leq \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} w(s,v(s),D_{a^+}^{\alpha,\beta;\psi}v(s))\psi'(s)(\psi((t))-\psi(s))^{\alpha-1}ds.$$

Moreover, by (H_1) and (H_3) we have $v \equiv 0$ on I_ν, which yields that $\lim_{k\to\infty} v_k(t) = 0$ uniformly on I_ν. Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on I_ν. Consequently $\{\rho_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on I_ν that gives us a contradiction.

Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ converges uniformly on I to a continuous function $\rho_*(t)$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \frac{\rho_a}{\Gamma(\gamma)}(\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g\left(s, \rho_k(s), D_{a^+}^{\alpha,\beta;\psi}\rho_k(s)\right) ds$$

$$= \frac{\rho_a}{\Gamma(\gamma)}(\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g\left(s, \rho_*(s), D_{a^+}^{\alpha,\beta;\psi}\rho_*(s)\right) ds,$$

for each $t \in I$. This means that ρ_* is a solution of the problem (5.1)-(5.2).

Finally, we prove the uniqueness of solutions of the problem (5.1)-(5.2). Let ρ_1 and ρ_2 be two solutions of (5.3). As above, put

$$\bar{\iota} := \sup\{\xi \in [0,1] : \rho_1(t) = \rho_2(t) \ for \ t \in I_\xi\},$$

and assum that $\bar{\iota} < 1$. There exist a constant $\mu > 0$ and a comparison function $w : I_{\bar{\iota}\times[0,\mu]\times[0,\mu]\to\mathbb{R}_+}$ satisfying inequality (5.4). We take $\nu \in (\xi, 1)$ such that

$$\|\rho_1(t)-\rho_2(t)\|_{C_{1-\gamma;\psi}[a,e]} \le \mu \ and \ \|(D_{a^+}^{\alpha,\beta;\psi}\rho_1)(t) - (D_{a^+}^{\alpha,\beta;\psi}\rho_2)(t)\|_{C_{1-\gamma;\psi}[a,e]} \le \mu;$$

for $t \in I_\nu$. Then for all $t \in I_\nu$, we obtain

$$\|\rho_1(t)-\rho_2(t)\|_{C_{1-\gamma;\psi}[a,b]} \le \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} \psi'(s)(\psi((t))-\psi(s))^{\alpha-1} \Big\| [(\psi(s)-\psi(a)]^{1-\gamma}$$
$$[g(s,\rho_0(s), D_{a^+}^{\alpha,\beta;\psi}\rho_0(s)) - g(s,\rho_1(s), D_{a^+}^{\alpha,\beta;\psi}\rho_1(s))]\Big\|_{C[a,b]} ds$$
$$\le \frac{1}{\Gamma(\alpha)}\int_a^{\nu t} \psi'(s)(\psi((t))-\psi(s))^{\alpha-1}$$
$$w(s, \|\rho_0(s)-\rho_1(s)\|_{C_{1-\gamma;\psi}[a,b]}, \|D_{a^+}^{\alpha,\beta;\psi}\rho_0(s) - D_{a^+}^{\alpha,\beta;\psi}\rho_1(s)\|_{C_{1-\gamma;\psi}[a,b]}) ds.$$

Again, by (H_1) and (H_3) we have $\rho_1 - \rho_2 \equiv 0$ on I_ν. This gives $\rho_1 = \rho_2$ on I_ν, which yields a contradiction. So, $\bar{\iota} = 1$ and the solution of the problem (5.1)-(5.2) is unique on $(a, b]$.

5.3 An Example

Consider the following ψ-Hilfer Cauchy-type problem

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{3},\psi} u(t) = \dfrac{2}{7e^t\left(1+|\rho(t)|+\left|D_{0^+}^{\frac{1}{2},\frac{1}{3},\psi}\rho(t)\right|\right)}; \ 0 < t \le 1, \\ I_{0^+}^{\frac{1}{3},\psi}\rho(0) = \rho_0, \quad \rho_0 \in \mathbb{R}. \end{cases} \tag{5.6}$$

where the function $\psi : [0,1] \to \mathbb{R}$ is defined by $\psi(t) = \sqrt{t+2}$.
We have $g\left(t, \rho(t), D_{0^+}^{\frac{1}{2},\frac{1}{3},\psi}\rho(t)\right) \in C_{1-\frac{2}{3},\psi}[0,1]$, $t \in (0,1]$, and

$$
\begin{aligned}
&\|g(t,\rho,\varrho) - g(t,\overline{\rho},\overline{\varrho})\|_{C_{1-\frac{2}{3},\psi}[0,1]} = \max_{t\in[0,1]}[\psi(t)-\psi(0)]^{\frac{1}{3}}|g(t,\rho,\varrho) - g(t,\overline{\rho},\overline{\varrho}| \\
&= \max_{t\in[0,1]}[\psi(t)-\psi(0)]^{\frac{1}{3}}\left|\frac{2}{7e^t(1+|\rho|+|\varrho|)} - \frac{2}{7e^t\,(1+|\overline{\rho}|+|\overline{\varrho}|)}\right| \\
&\le \frac{2}{7}\max_{t\in[0,1]}[\psi(t)-\psi(0)]^{\frac{1}{3}}\left|\frac{|\overline{\rho}|+|\overline{\varrho}|-|\rho|-|\varrho|}{(1+|\rho|+|\varrho|)\,(1+|\overline{\rho}|+|\overline{\varrho}|)}\right| \\
&\le \frac{2}{7}\max_{t\in[0,1]}[\psi(t)-\psi(0)]^{\frac{1}{3}}\,[|\rho-\overline{\rho}|+|\varrho-\overline{\varrho}|] \\
&= \frac{2}{7}\Bigg(\,\|\rho-\overline{\rho}\|_{C_{1-\frac{2}{3},\psi}[0,1]} + \|\varrho-\overline{\varrho}\|_{C_{1-\frac{2}{3},\psi}[0,1]}\Bigg).
\end{aligned}
$$

The condition(5.4) holds with any $t \in (0,1]$, μ and the function $w : [0,1]\times[0,\mu] \to [0,\infty)$ given by

$$w(t,\rho,\varrho) = \frac{2}{7}(\rho+\varrho).$$

Consequently, Theorem 5.2.1 means that the successive approximations ρ_n; $n \in \mathbb{N}$, defined by

$$\rho_0(t) = u_0\,;\ t \in (0,1],$$

$$\rho_{n+1}(t) = \frac{\rho_0}{\Gamma(\frac{2}{3})}(\psi(t)-\psi(a))^{\frac{2}{3}-1}$$

$$+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\frac{1}{2}-1} f\left(s, \rho_n(s), D_{a^+}^{\frac{1}{2},\frac{1}{3};\psi}\rho_n(s)\right) ds;\ t \in (0,1],$$

converge uniformly on $(0,1]$ to the unique solution of problem (5.6).

Chapter 6

Existence and Attractivity Results for ψ-Hilfer Hybrid Fractional Differential Equations

6.1 Introduction

In this chapter, we focus on the existence of attractive solutions of fractional differential equations of the ψ-Hilfer Hybrid type. The results on the existence of solutions are applied to the Schauder fixed point Theorem. Then, we prove that all solutions are uniformly locally attractive.

Recently hybrid differential equations see[48] and hybrid fractional differential equations attracted the attention of a large wide a researchers [22, 47, 58, 66, 67, 104, 115].

Functional $\psi-$ fractional differential equations has a great importance in applied mathematics and other sciences, see [24, 75, 78, 103, 111, 112, 114].

In this work, we are concerned with the existence and attractivity of solutions for the following problem

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi} \frac{\rho(t)}{\varrho(t,\rho(t))} = w\left(t,\rho(t)\right); \ a.e. \quad t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma}\rho(t) \mid_{t=0} = \rho_0; \qquad \rho_0 \in \mathbb{R}, \end{cases} \tag{6.1}$$

where $\mathbb{R}_+ := [0,+\infty),\ 0<\lambda<1,\ 0\leq\sigma\leq 1,\ \varsigma=\lambda+\sigma(1-\lambda),\ {}^{H}D_{0^+}^{\lambda,\sigma;\psi}$ is the ψ-Hilfer fractional derivative of order λ and type σ, $\varrho:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}^*$ and $w:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, are given functions.

Special cases :

• For $\sigma=0,\psi(t)=t,\rho_0=0$, we will get nonlinear hybrid FDEs of the form

$$\begin{cases} {}^{RL}D_{0^+}^{\lambda}\left[\frac{\rho(t)}{\varrho(t,\rho(t))}\right] = w(t,\rho(t)), \text{ a.e. } t\in\mathbb{R}_+, \\ \rho(0)=0. \end{cases}$$

• For $\lambda=1,\sigma=1,\psi(t)=t$, we obtain nonlinear integer order hybrid differential equations of the form

$$\begin{cases} \frac{d}{dt}\left[\frac{\rho(t)}{\varrho(t,\rho(t))}\right] = w(t,\rho(t)), \text{ a.e. } t\in\mathbb{R}_+, \\ \rho(0)=\rho_0\in\mathbb{R}. \end{cases}$$

• For $\varrho = 1$, we acquire nonlinear ψ-Hilfer FDEs of the form

$$\begin{cases} {}^{H}D_{0^+}^{\lambda,\sigma;\psi}\rho(t) = w(t,\rho(t)), \text{ a.e. } t \in \mathbb{R}_+, \\ (\psi(t)-\psi(0))^{1-\varsigma}\rho(t)\big|_{t=0} = \rho_0 \in \mathbb{R}. \end{cases}$$

• For $\varrho = 1, \sigma = 0$ (in this case $\varsigma = \lambda$), $\psi(t) = t$, we get nonlinear FDEs involving Riemann-Liouville fractional derivative

$${}^{RL}D_{0^+}^{\lambda}\rho(t) = w(t,\rho(t)), \text{ a.e. } t \in \mathbb{R}_+.$$

6.2 Existence and Attractivity Results

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. By $BC_\varsigma := BC_\varsigma(\mathbb{R}_+)$, we denote the weighted space of all bounded and continuous functions defined by $BC_\varsigma = \{\phi : \mathbb{R}_+ \to \mathbb{R} : (\psi(t)-\psi(0))^{1-\varsigma}\phi(t) \in BC\}$, with the norm

$$\|\phi\|_{BC_\varsigma} := \sup_{t\in\mathbb{R}_+}\left|(\psi(t)-\psi(0))^{1-\varsigma}\phi(t)\right|.$$

Lemma 6.2.1 *[78] Let $v \in C(\Upsilon\times\mathbb{R},\mathbb{R}^*)$; $\Upsilon := [0,d]$, $d > 0$, $\kappa \in C_{1-\zeta,\psi}(\Upsilon)$. The problem*

$$\begin{cases} D_{0^+}^{\lambda,\sigma;\psi}\frac{\rho(t)}{\varrho(t,\rho(t))\text{Ł}} = \kappa(t), a.e. \quad t \in \Upsilon. \\ (\psi(t)-\psi(0))^{1-\varsigma}\rho(t) \mid_{t=0} = \rho_0, \qquad \rho_0 \in \mathbb{R}, \end{cases}$$

has a unique solution given by

$$\rho(t) = \varrho(t,\rho(t))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}(\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi}\kappa(t)\right\}.$$

Lemma 6.2.2 *[78] Let $\varrho \in C(\Upsilon\times\mathbb{R},\mathbb{R}^*)$, $w : \Upsilon\times\mathbb{R} \to \mathbb{R}$ be a function such that $w(\cdot,\rho(\cdot)) \in BC_\varsigma$ for any $\rho \in BC_\varsigma$. So the problem (6.1) is equivalent to the integral equation*

$$\rho(t) = \varrho(t,\rho(t))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}(\psi(t)-\psi(0))^{\varsigma-1} + I_{0^+}^{\lambda;\psi}w(\cdot,\rho(\cdot))(t)\right\}.$$

Definition 6.2.1 *A function $\rho \in BC_\varsigma$ is a solution of problem(6.1), if it verifies the initial condition $(\psi(t)-\psi(0))^{1-\varsigma}\rho(t) \mid_{t=0} = \rho_0$ and the equation $D_{0^+}^{\lambda,\sigma;\psi}\frac{\rho(t)}{\varrho(t,\rho(t))} = w(t,\rho(t))$ on $\mathbb{R}_+$.*

We provide the following hypotheses :

(H_1) The function $t \mapsto w(t,\rho)$ is measurable on $\mathbb{R}_+$ for each $\rho \in BC_\varsigma$, the function $\rho \mapsto w(t,\rho)$ is continuous on BC_ς for a.e. $t \in \mathbb{R}_+$, and the function ϱ is bounded such that $\rho \mapsto \varrho(t,\rho)$ is continuous.

(H_2) There exists a continuous function $T : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$ and each $\rho \in \mathbb{R}$,

$$|w(t,\rho)| \le \frac{T(t)}{1+|\rho|},$$

and

$$\lim_{t\to\infty}(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t) = 0.$$

Set

$$T^* = \sup_{t\in\mathbb{R}_+} (\psi(t)-\psi(0))^{1-\varsigma} \left(I_{0^+}^{\lambda;\psi} T\right)(t) < \infty.$$

Now, we introduce a theorem on the existence and attractivity of solutions of the problem (6.1).

Theorem 6.2.1 *Assume (H_1) and (H_2) hold. Then the problem (6.1) has at least one solution defined on $\mathbb{R}_+$ and the solutions of problem (6.1) are uniformly locally attractive.*

Proof. Consider the operator K such that, for any $\rho \in BC_\varsigma$,

$$\begin{aligned}(K\rho)(t) = \varrho(t,\rho(t))\bigg\{ &\frac{\rho_0}{\varrho(0,\rho(0))}(\psi(t)-\psi(0))^{\varsigma-1} \\ &+ \frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} w(s,\rho(s))ds\bigg\}.\end{aligned}$$

Let L be a bound of the function ϱ. For any $\rho \in BC_\varsigma$, and for each $t\in\mathbb{R}_+$, we get

$$\begin{aligned}\bigg|\Big((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)\bigg| &\le |\varrho(t,\rho(t))|\bigg\{\bigg|\frac{\rho_0}{\varrho(0,\rho(0))}\bigg| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \\ &\quad \times \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\bigg\} \\ &\le |\varrho(t,\rho(t))|\bigg\{\bigg|\frac{\rho_0}{\varrho(0,\rho(0))}\bigg| \\ &\quad + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\bigg\} \\ &\le L\bigg\{\bigg|\frac{\rho_0}{\varrho(0,\rho(0))}\bigg| + T^*\bigg\} \\ &:= R_*.\end{aligned}$$

So

$$\|K(\rho)\|_{BC_\varsigma} \le R_*. \tag{6.2}$$

Consequently, $K(\rho)\in BC_\varsigma$. Since, the map $K(\rho)$ is continuous on $\mathbb{R}_+$; for any $\rho\in BC_\varsigma$, and $K(BC_\varsigma)\subset BC_\varsigma$, so the operator K maps BC_ς into itself. Meaning that, equation (6.2) shows that the operator K transforms the ball

$$B_{R_*} := B(0,R_*) = \{\varrho\in BC_\varsigma : \|\varrho\|_{BC_\varsigma} \le R_*\}$$

into itself. From Lemma 6.2.2, the solution of problem (6.1) is equivalent to find the solution of the operator equation $K(\rho)=\rho$. We check that the operator $K: BC_\varsigma \to BC_\varsigma$ satisfies all assumptions of Theorem 1.4.1. The proof is divided into multiple steps :

Step 1. K is continuous.

Let $\{\rho_n\}_{n\in N}$ be a sequence such that $\rho_n\to\rho$ in B_{R_*}.

Then, for each $t \in \mathbb{R}_+$, we have

$$
\begin{aligned}
&\Big|((\psi(t)-\psi(0))^{1-\varsigma}\left(K\rho_n\right)(t)-((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)\Big| \\
&\leq \Bigg|\varrho(t,\rho_n(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho_n(s))ds\Bigg\} \\
&-\varrho(t,\rho(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\Bigg\}\Bigg| \\
&\leq \Bigg|\varrho(t,\rho_n(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho_n(s))ds\Bigg\} \\
&-\varrho(t,\rho(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho_n(s))ds\Bigg\} \\
&+\varrho(t,\rho(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho_n(s))ds\Bigg\} \\
&-\varrho(t,\rho(t))\Bigg\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\Bigg\}\Bigg| \\
&\leq \Bigg|\varrho(t,\rho_n(t))-\varrho(t,\rho(t))\Bigg|\Bigg|\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1} \\
&\times w(s,\rho_n(s))ds\Bigg|+|\varrho(t,\rho(t))|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)} \\
&\times\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho_n(s))-w(s,\rho(s))|ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|((\psi(t)-\psi(0))^{1-\varsigma}\left(K\rho_n\right)(t)&-((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)| \\
&\leq \Bigg|\varrho(t,\rho_n(t))-\varrho(t,\rho(t))\Bigg|\Bigg\{\Bigg|\tfrac{\rho_0}{\varrho(0,\rho(0))}\Bigg| \\
+\tfrac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}&\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho_n(s))|ds\Bigg\}
\end{aligned}
$$

$$
+L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho_n(s))-w(s,\rho(s))|ds. \tag{6.3}
$$

Case 1. If $t \in [0,d]$, then taking into account of the facts that $\rho_n \to \rho$ as $n \to \infty$ v and w are continuous, by the Lebesgue dominated convergence Theorem, from the equation (6.3), we have

$$
\|K\left(\rho_n\right)-K\left(\rho\right)\|_{BC_\varsigma} \to 0 \quad \text{as} \quad n \to \infty.
$$

Case 2. If $t \in (d,\infty)$, then from the hypotheses and (6.3), we have

$$
\begin{aligned}
&\Big|((\psi(t)-\psi(0))^{1-\varsigma}\left(K\rho_n\right)(t)-((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)\Big| \\
&\leq \Bigg|\varrho(t,\rho_n(t))-\varrho(t,\rho(t))\Bigg|\Bigg\{\Bigg|\frac{\rho_0}{\varrho(0,\rho(0))}\Bigg| \\
&+\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\Bigg\} \\
&+2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds.
\end{aligned}
$$

Then

$$\left|((\psi(t)-\psi(0))^{1-\varsigma}\left(K\rho_n\right)(t)-((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)\right|$$
$$\leq\left|\varrho(t,\rho_n(t))-\varrho(t,\rho(t))\right|\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right|+((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\right\}$$
$$+2L((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t). \tag{6.4}$$

Since $\rho_n\to\rho$ as $n\to\infty$,ϱ is continuous and $((\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\to 0$ as $t\to\infty$, it follows from (6.4) that

$$\|K\left(\rho_n\right)-K(\rho)\|_{BC_\varsigma}\to 0 \quad \text{as} \quad n\to\infty.$$

Step 2. $K\left(B_{R_*}\right)$ is uniformly bounded, and equicontinuous on all compact subset $[0,d]$ of $\mathbb{R}_+$, $d>0$.

We have $K\left(B_{R_*}\right)\subset B_{R_*}$ and B_{R_*} is bounded, so $K\left(B_{R_*}\right)$ is uniformly bounded.

Next, for each $t_1,t_2\in[0,d], t_1<t_2$, and $\rho\in B_{R_*}$, we have

$$\left|((\psi(t_2)-\psi(0))^{1-\varsigma}\left(K\rho\right)(t_2)-((\psi(t_1)-\psi(0))^{1-\varsigma}(K\rho)(t_1)\right|$$
$$\leq\Bigg|\varrho(t_2,\rho(t_2))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}$$
$$-v(t_1,\rho(t_1))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}\Bigg|$$
$$\leq\Bigg|\varrho(t_2,\rho(t_2))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}$$
$$-\varrho(t_1,\rho(t_1))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}$$
$$+\varrho(t_1,\rho(t_1))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}$$
$$-\varrho(t_1,\rho(t_1))\left\{\frac{\rho_0}{\varrho(0,\rho(0))}+\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right\}\Bigg|.$$

Thus

$$\left|((\psi(t_2)-\psi(0))^{1-\varsigma}\left(K\rho\right)(t_2)-((\psi(t_1)-\psi(0))^{1-\varsigma}(K\rho)(t_1)\right|$$
$$\leq\left|\varrho(t_2,\rho(t_2))-\varrho(t_1,\rho(t_1))\right|\Bigg|\frac{\rho_0}{\varrho(0,\rho(0))}$$
$$+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\Bigg|$$
$$+\left|\varrho(t_1,\rho(t_1))\right|\Bigg|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds$$
$$+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}w(s,\rho(s))ds$$
$$-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\Bigg|.$$

Hence
$$\begin{aligned}
&\Big|((\psi(t_2)-\psi(0))^{1-\varsigma}(K\rho)(t_2)-((\psi(t_1)-\psi(0))^{1-\varsigma}(K\rho)(t_1)\Big|\\
&\leq \Big|\varrho(t_2,\rho(t_2))-\varrho(t_1,\rho(t_1))\Big|\bigg(\Big|\frac{\rho_0}{\varrho(0,\rho(0))}\Big|\\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\bigg)\\
&+L\bigg(\int_0^{t_1}\Big|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\Big|\\
&|w(s,\rho(s))|ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\bigg)\\
&\leq \Big|\varrho(t_2,\rho(t_2))-\varrho(t_1,\rho(t_1))\Big|\bigg(\Big|\frac{\rho_0}{\varrho(0,\rho(0))}\Big|\\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\bigg)\\
&+L\bigg(\int_0^{t_1}\Big|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\Big|\\
&T(s)ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}T(s)ds\bigg).
\end{aligned}$$

From the continuity of the functions T and v, by setting $T_*=\sup_{t\in[0,d]}T(t)$, we get
$$\begin{aligned}
&\Big|((\psi(t_2)-\psi(0))^{1-\varsigma}(K\rho)(t_2)-((\psi(t_1)-\psi(0))^{1-\varsigma}(K\rho)(t_1)\Big|\\
&\leq \Big|\varrho(t_2,\rho(t_2))-\varrho(t_1,\rho(t_1))\Big|\bigg(\Big|\frac{\rho_0}{\varrho(0,\rho(0))}\Big|\\
&+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\bigg)\\
&+LT_*\bigg(\int_0^{t_1}\Big|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\Big|\\
&ds+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}ds\bigg)\\
&\leq \Big|\varrho(t_2,\rho(t_2))-\varrho(t_1,\rho(t_1))\Big|\bigg(\Big|\frac{\rho_0}{\varrho(0,\rho(0))}\Big|\\
&+\frac{T_*(\psi(t_2)-\psi(0))^{1-\varsigma+\lambda}}{\Gamma(\lambda+1)}\bigg)+LT_*\bigg(\int_0^{t_1}\Big|\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_2)-\psi(s))^{\lambda-1}\\
&-\frac{(\psi(t_1)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\psi'(s)(\psi(t_1)-\psi(s))^{\lambda-1}\Big|ds\\
&+\frac{(\psi(t_2)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda+1)}(\psi(t_2)-\psi(t_1))^{\lambda}\bigg).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the inequality tends to zero.

Step 3. $K\left(B_{R_*}\right)$ is equiconvergent.
Let $\rho \in B_{R*}$. Then, for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\left|\left((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t)\right|\right. \leq |\varrho(t,\rho(t))| \left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right|\right.\\
&\left.+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}w(s,\rho(s))ds\right|\right\}\\
&\leq |\varrho(t,\rho(t))| \left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right|\right.\\
&\left.+\left|\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right|\right\}\\
&\leq L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + (\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)\right\}.
\end{aligned}$$

Since

$$(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t) \to 0 \; as \;\; t \to +\infty,$$

we find

$$|(K\rho)(t)| \leq L\left\{\left|\frac{\rho_0}{(\psi(t)-\psi(0))^{1-\varsigma}\varrho(0,\rho(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}\left(I_{0^+}^{\lambda;\psi}T\right)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}}\right\}.$$

Then,

$$|(K\rho)(t)-(K\rho)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty,$$

in sight of Lemma 1.3.8 as a consequence of Steps $1-4$, we conclude that $K : B_{R_*} \to B_{R_*}$ is compact and continuous. The Theorem 1.4.1, ensures that K has a fixed point ρ, which is a solution of problem (6.1) on $\mathbb{R}_+$.

Step 4. The uniform local attractivity of solutions.
We assume that ρ_* is a solution of problem (6.1).

Set $\rho \in B(\rho_*, 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2T^*\right\})$, we have

$$\begin{aligned}
&\left|((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t) - ((\psi(t)-\psi(0))^{1-\varsigma}(\rho_*)(t)\right| \\
&\leq \left|((\psi(t)-\psi(0))^{1-\varsigma}(K\rho)(t) - ((\psi(t)-\psi(0))^{1-\varsigma}(K\rho_*)(t)\right| \\
&\leq \left|\varrho(t,\rho(t)) - \varrho(t,\rho_*((t))\right|\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| \right.\\
&\left.+ \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\right\} \\
&+ L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s)) - w(s,\rho_*((s))|ds \\
&\leq 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + \frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds\right\} \\
&+ 2L\frac{(\psi(t)-\psi(0))^{1-\varsigma}}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}T(s)ds \\
&\leq 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2T^*\right\}.
\end{aligned}$$

Thus, we get

$$\|K(\rho) - \rho_*\|_{BC_\varsigma} \leq 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2T^*\right\}.$$

Concluding that K is a continuous function so that

$$K\left(B\left(\rho_*, 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2T^*\right\}\right)\right) \subset B\left(\rho_*, 2L\left\{\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2T^*\right\}\right).$$

Additionally, if ρ is a solution of problem (6.1), then

$$\begin{aligned}
|\rho(t) - \rho_*(t)| &= |(K\rho)(t) - (K\rho_*)(t)| \\
&\leq \left|\varrho(t,\rho(t)) - \varrho(t,\rho_*((t))\right|\left\{(\psi(t)-\psi(0))^{\varsigma-1}\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| \right.\\
&\left.+ \frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\right\} \\
&+ \frac{L}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s)) - w(s,\rho_*((s))|ds \\
&\leq 2L\left\{(\psi(t)-\psi(0))^{\varsigma-1}\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| \right.\\
&\left.+ \frac{1}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s))|ds\right\} \\
&+ \frac{L}{\Gamma(\lambda)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\lambda-1}|w(s,\rho(s)) - w(s,\rho_*((s))|ds \\
&\leq 2L\left\{(\psi(t)-\psi(0))^{\varsigma-1}\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right| + 2(I_{0^+}^{\lambda;\psi}T)(t)\right\}.
\end{aligned}$$

Therefore,

$$|\rho(t)-\rho_*(t)|\leq 2L\left\{(\psi(t)-\psi(0))^{\varsigma-1}\left|\frac{\rho_0}{\varrho(0,\rho(0))}\right|+2\frac{(\psi(t)-\psi(0))^{1-\varsigma}(I_{0^+}^{\lambda;\psi}T)(t)}{(\psi(t)-\psi(0))^{1-\varsigma}}\right\}. \quad (6.5)$$

By using (6.5) and the fact that

$$\lim_{t\to\infty}(\psi(t)-\psi(0))^{1-\varsigma}(I_{0^+}^{\lambda;\psi}T)(t)=0,$$

we conclude

$$\lim_{t\to\infty}|\rho(t)-\rho_*(t)|=0.$$

Hence, all solutions of problem (6.1) are uniformly locally attractive.

6.3 An Example

We consider the following problem for a ψ-Hilfer fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{2};\psi}\frac{\rho(t)}{\varrho(t,\rho(t))}=w\left(t,\rho(t)\right), a.e. \quad t\in\mathbb{R}_+, \\ (\psi(t)-\psi(0))^{\frac{1}{4}}\rho(t)\mid_{t=0}=1, \end{cases} \quad (6.6)$$

where $\psi:[0,1]\to\mathbb{R}$ with $\psi(t)=\sqrt{t+3}$,

$$\varrho(t,\rho)=\frac{1}{(1+t)(1+|\rho|)},$$

$$\begin{cases} w(t,\rho)=\frac{\beta\left((\psi(t)-\psi(0)\right)^{\frac{-1}{4}}\sin t}{64(1+\sqrt{t})(1+|\rho|)}, \quad t\in(0,\infty), \quad \rho\in\mathbb{R}, \\ w(0,\rho)=0, \quad \rho\in\mathbb{R}, \end{cases}$$

and

$$\beta=\frac{9\sqrt{\pi}}{16}.$$

Clearly, the function w is continuous. Hypothesis (H_2) is satisfied with

$$\begin{cases} T(t)=\frac{\beta\left((\psi(t)-\psi(0)\right)^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})}, \quad t\in(0,\infty), \\ T(0)=0. \end{cases}$$

In addition, we have

$$\begin{aligned}\left((\psi(t)-\psi(0)\right)^{\frac{1}{4}}\left(I_{0^+}^{\frac{1}{2};\psi}T\right)(t) &= \frac{\left((\psi(t)-\psi(0)\right)^{\frac{1}{4}}}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\frac{-1}{2}}T(\tau)d\tau \\ &\leq \frac{1}{4}\left((\psi(t)-\psi(0)\right)^{\frac{-1}{4}} \to 0 \quad \text{as} \quad t\to\infty.\end{aligned}$$

All conditions of Theorem 6.2.1 are satisfied. So, problem (6.6) has at least one solution defined on $\mathbb{R}_+$, and all the solutions of this problem are uniformly locally attractive.

Chapter 7

Successive Approximations for Caputo-Fabrizio Fractional Differential Equations

7.1 Introduction

In this chapter, we discuss uniqueness result of solutions for a class of fractional differential equations involving Caputo-Fabrizio derivative. We provide a result on the global convergence of successive approximations.

The convergence of successive approximations for nonlinear functional equations,and on global convergence of successive approximations of problem for functional differential equations have attracted many researchers. In 1968, Browder [41] gave a brief and transparent proof of a generalization of the classical Picard-Banach contraction principle by using the convergence of successive approximations. In 1981, Chen [44] used the successive approximations method to prove the existence of solutions for the functional integral equations

$$\begin{cases} x(t) = f(t);\ t \in [\sigma - r, \sigma] \\ x(t) = f(t) + \int_\sigma^t g(t,s,x_s)ds;\ t \in [\sigma, b]. \end{cases}$$

In [46], Czlapiiǹski studied the global convergence of successive approximations of the Darboux problem for partial functional differential equations with infinite delay, and in [57], Faina studied the generic property of global convergence of successive approximations for functional differential equations with infinite delay.

In this chapter, we begin the study of the global convergence of successive approximations for Caputo-Fabrizio fractional differential equation(CFFDE)

$$\begin{cases} ({}^{CF}D_0^s\rho)(t) = \varphi(t,\rho(t));\quad t \in \Upsilon := [0,\lambda] \\ \rho(0) = \rho_0. \end{cases} \tag{7.1}$$

Here ${}^{\mathrm{CF}}D_t^s$ is for the CFFDE, $0 < s < 1$, $\varphi : [0,\lambda] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\rho_0 \in \mathbb{R}$.

7.2 Successive Approximations for Caputo-Fabrizio Fractional Differential Equations

Lemma 7.2.1 *[17, 94] Let $\varphi \in L^1(\Upsilon)$. Then the problem*

$$\begin{cases} ({}^{CF}D_0^s\rho)(t) = \varphi(t); \quad t \in \Upsilon := [0, \lambda] \\ \rho(0) = \rho_0, \end{cases} \tag{7.2}$$

admits a unique solution which is given by

$$\rho(t) = \rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t) - \varphi(0)) + \frac{2s}{(2-s)M(s)} \int_0^t \varphi(\tau)d\tau. \tag{7.3}$$

The family which contains all real valued and continuous functions on the interval Υ which is a Banach space supplied with the norm

$$\|\rho\| = \sup_{t \in \Upsilon} |\rho(t)|.$$

This section is devoted to the main result of the global convergence of successive approximations.

Definition 7.2.1 *The solution of the problem* (7.1) *is a function $\rho \in C(\Upsilon)$ which satisfies the equation $({}^{CF}D_0^s\rho)(t) = \varphi(t, \rho(t))$ on Υ and initial condition $\rho(0) = \rho_0$.*

Set $\Upsilon_\varrho := [0, \varrho\lambda]$; for any $\varrho \in [0, 1]$. We cite some hypotheses

(H_1) The functions $\varphi : \Upsilon \times \mathbb{R} \to \mathbb{R}$ are continuous.

(H_2) There exist a constant $\varsigma > 0$ and a continuous function $h : \Upsilon \times [0, \varsigma] \to \mathbb{R}_+$ such that $h(t, \cdot)$ is nondecreasing for all $t \in \Upsilon$, and the inequality

$$|\varphi(t, \rho) - \varphi(t, \overline{\rho})| \leq h(t, |\rho - \overline{\rho}|), \tag{7.4}$$

holds for all $t \in \Upsilon$ and $\rho, \overline{\rho} \in \mathbb{R}$ such that $|\rho - \overline{\rho}| \leq \varsigma$,

(H_3) $R \equiv 0$ is the function in $C(\Upsilon_\delta, [0, \varsigma])$ which satisfies the inequality

$$R(t) \leq \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho) \in \Upsilon_\delta \times [0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h(\tau, R(\tau))d\tau \tag{7.5}$$

with $\varrho \leq \delta \leq 1$.

We define the successive approximations :

$$\rho_0(t) = \rho_0; \ t \in \Upsilon,$$

$$\rho_{n+1}(t) = \rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t, \rho_n(t)) - \varphi(0, \rho_0)) + \frac{2s}{(2-s)M(s)} \int_0^t \varphi(\tau, \rho_n(\tau))d\tau; \ t \in \Upsilon.$$

Theorem 7.2.1 *Assume $(H_1) - (H_3)$ hold. Then the successive approximations $\rho_n; n \in \mathbb{N}$ are well defined and converge to the unique solution of problem* (7.1) *uniformly on Υ.*

Proof. There exist $\eta > 0$ such that

$$\|\rho_n\|_\infty \leq \eta.$$

Let

$$\varpi = \sup_{(t,\rho)\in\Upsilon\times[0,\eta]} |\varphi(t,\rho)|.$$

For each $t_1, t_2 \in \Upsilon$ with $t_1 < t_2$, and for all $t \in \Upsilon$

$$\begin{aligned}
|\rho_n(t_2) - \rho_n(t_1)| &\leq \Bigg|\rho_0 + \frac{2(1-s)}{(2-s)M(s)}\left(\varphi(t_2, \rho_{n-1}(t_2)) - \varphi(0,\rho_0)\right) \\
&+ \frac{2s}{(2-s)M(s)} \int_0^{t_2} \varphi(\tau, \rho_{n-1}(\tau))d\tau - \Bigg[\rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t_1, \rho_{n-1}(t_1)) - \varphi(0,\rho_0)) \\
&+ \frac{2s}{(2-s)M(s)} \int_0^{t_1} \varphi(\tau, \rho_{n-1}(\tau))d\tau\Bigg]\Bigg| \\
&\leq \Bigg|\frac{2(1-s)}{(2-s)M(s)}(\varphi(t_2, \rho_{n-1}(t_2)) - \varphi(t_1, \rho_{n-1}(t_1))) \\
&+ \frac{2s}{(2-s)M(s)} \left(\int_0^{t_2} \varphi(\tau, \rho_{n-1}(\tau))d\tau - \int_0^{t_1} \varphi(\tau, \rho_{n-1}(\tau))d\tau\right)\Bigg| \\
&\leq \Bigg|\frac{2(1-s)}{(2-s)M(s)}(\varphi(t_2, \rho_{n-1}(t_2)) - \varphi(t_1, \rho_{n-1}(t_1))) \\
&+ \frac{2s}{(2-s)M(s)} \int_{t_1}^{t_2} \varphi(\tau, \rho_{n-1}(\tau))d\tau\Bigg| \\
&\leq \frac{2(1-s)}{(2-s)M(s)}|\varphi(t_2, \rho_{n-1}(t_2)) - \varphi(t_1, \rho_{n-1}(t_1))| \\
&+ \frac{2s}{(2-s)M(s)}\varpi \int_{t_1}^{t_2} d\tau.
\end{aligned}$$

From the continuity of the function φ, we have

$$\begin{aligned}
|\rho_n(t_2) - \rho_n(t_1)| \leq &\frac{2(1-s)}{(2-s)M(s)}|\varphi(t_2, \rho_{n-1}(t_2)) - \varphi(t_1, \rho_{n-1}(t_1))| \\
&+ \frac{2s}{(2-s)M(s)}\varpi(t_2 - t_1) \\
&\longrightarrow 0, \ \ as \ t_1 \to t_2.
\end{aligned}$$

Hence

$$\|\rho_n(t_2) - \rho_n(t_1)\| \longrightarrow 0, \quad as \quad t_1 \to t_2.$$

And we get the equi-continuous on Υ of the sequence $\{\rho_n(t);\ n \in \mathbb{N}\}$.

Let

$$\nu := \sup\{\varrho \in [0,1] : \{\rho_n(t)\} \text{ converges uniformly on } \Upsilon_\varrho\}.$$

If $\nu = 1$, and then we will obtain the global convergence of successive approximations. We suppose that $\nu < 1$, and the sequence $\{\rho_n(t)\}$ is equicontinuous on Υ_ν, converging uniformly towards a function $\tilde{\rho}(t)$. If we show that there exists $\delta \in (\nu, 1]$ such that $\{\rho_n(t)\}$ converges uniformly on Υ_δ, then it leads to a contradiction.

Put $\rho(t) = \tilde{\rho}(t)$; for all $t \in \Upsilon_\nu$. From (H_2), there exist a constant $\zeta > 0$ and a continuous function $h : \Upsilon \times [0, \zeta] \to \mathbb{R}_+$ checking inequality (7.4). So therefore, there exist $\delta \in [\nu, 1]$ and $n_0 \in \mathbb{N}$, such that for all $t \in \Upsilon_\delta$ and $n, m > n_0$, we get

$$|\rho_n(t) - \rho_m(t)| \le \zeta.$$

For all $t \in \Upsilon_\delta$, put

$$R^{(n,m)}(t) = |\rho_n(t) - \rho_m(t)|,$$

$$R_k(t) = \sup_{n,m \ge k} R^{(n,m)}(t).$$

$R_k(t)$ is the non-increasing sequence, thus it converges to a function $R(t)$ and that for all $t \in \Upsilon_\delta$. From the equicontinuity of $\{R_k(t)\}$ we have $\lim_{k\to\infty} R_k(t) = R(t)$ uniformly on Υ_δ. And further, for each $t \in \Upsilon_\delta$ and $n, m \ge k$, we get

$$\begin{aligned}
&R^{(n,m)}(t) = |\rho_n(t) - \rho_m(t)| \\
&\le \Bigg| \rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t, \rho_{n-1}(t)) - \varphi(0, \rho_0)) + \frac{2s}{(2-s)M(s)} \int_0^t \varphi(\tau, \rho_{n-1}(\tau))d\tau \\
&- \rho_0 - \frac{2(1-s)}{(2-s)M(s)}(\varphi(t, \rho_{m-1}(t)) - \varphi(0, \rho_0)) - \frac{2s}{(2-s)M(s)} \int_0^t \varphi(\tau, \rho_{m-1}(\tau))d\tau \Bigg| \\
&\le \frac{2(1-s)}{(2-s)M(s)} |\varphi(t, \rho_{n-1}(t)) - \varphi(t, \rho_{m-1}(t))| \\
&+ \frac{2s}{(2-s)M(s)} \int_0^t |\varphi(\tau, \rho_{n-1}(\tau)) - \varphi(\tau, \rho_{m-1}(\tau))| d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^t |\varphi(\tau, \rho_{n-1}(\tau)) - \varphi(\tau, \rho_{m-1}(\tau))| d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} |\varphi(\tau, \rho_{n-1}(\tau)) - \varphi(\tau, \rho_{m-1}(\tau))| d\tau,
\end{aligned}$$

therefore, by (7.4) we get

$$\begin{aligned}
R^{(n,m)}(t) &\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| \\
&+ \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h(\tau, |\rho_{n-1}(\tau) - \rho_{m-1}(\tau)|) d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h(\tau, R^{(n-1,m-1)}(\tau)) d\tau,
\end{aligned}$$

then

$$R_k(t) \le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h(\tau, R_{k-1}(\tau)) d\tau.$$

By the Lebesgue dominated convergence theorem we have

$$R(t) \le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h(\tau, R(\tau)) d\tau.$$

By (H_1) and (H_3) we have $R \equiv 0$ on Υ_δ, which gives that $\lim_{k\to\infty} R_k(t) = 0$ uniformly on Υ_δ. Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on Υ_δ. So $\{\rho_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on Υ_δ which gives the contradiction.

Thusly $\{\rho_k(t)\}_{k=1}^{\infty}$ converges uniformly on Υ to a continuous function $\rho_*(t)$. By the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t,\rho_k(t)) - \varphi(0,\rho_0)) + \frac{2s}{(2-s)M(s)}\int_0^t \varphi(\tau,\rho_k(\tau))d\tau$$

$$= \rho_0 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t,\rho_*(t)) - \varphi(0,\rho_0)) + \frac{2s}{(2-s)M(s)}\int_0^t \varphi(\tau,\rho_*(\tau))d\tau,$$

for all $t \in \Upsilon$. This leads to ρ_* being a solution to the problem (7.1).

Now, we demonstrate the uniqueness of the solutions of the previous problem (7.1). Let ρ_1 and ρ_2 be two solutions of (7.1). Then put

$$\overline{\nu} := \sup\{\varrho \in [0,1] : \rho_1(t) = \rho_2(t) \ for \ t \in \Upsilon_\varrho\},$$

and suppose that $\overline{\nu} < 1$. There exist a constant $\zeta > 0$ and a function $h : \Upsilon_{\overline{\nu}\times[0,\zeta]\to\mathbb{R}_+}$ verifying inequality (7.4). We choose $\delta \in (\varrho, 1)$ such as

$$|\rho_1(t) - \rho_2(t)| \le \zeta \ ;$$

for $t \in \Upsilon_\delta$. Then for all $t \in \Upsilon_\delta$, we get

$$\begin{aligned}
|\rho_1(t) - \rho_2(t)| &\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&+ \frac{2s}{(2-s)M(s)}\int_0^{\delta t} |\varphi(\tau,\rho_0(\tau)) - \varphi(\tau,\rho_1(\tau))|d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&+ \frac{2s}{(2-s)M(s)}\int_0^{\delta t} h(\tau, |\rho_0(\tau) - \rho_1(\tau)|)d\tau.
\end{aligned}$$

Again, by (H_1) and (H_3) we get $\rho_1 - \rho_2 \equiv 0$ on Υ_δ. This gives $\rho_1 = \rho_2$ on Υ_δ, which leads to a contradiction. So, $\overline{\nu} = 1$ the problem (7.1) admits a unique solution on Υ.

7.3 An Example

We consider the following Caputo-Fabrizio fractional differential Cauchy problem :

$$\begin{cases} ({}^{CF}D_0^s\rho)(t) = \varphi(t,\rho(t)); \quad t \in \Upsilon := [0,1], \ s \in (0,1), \\ \rho(0) = 1, \end{cases} \tag{7.6}$$

where

$$\varphi(t,\rho(t)) = \left(e^{t-1} + |\rho(t)|\right)\frac{t}{(1+t^2)(1+|\rho(t)|)}.$$

For each $\rho, \ \overline{\rho} \in \mathbb{R}$ and $t \in \Upsilon$, we have

$$|\varphi(t,\rho) - \varphi(t,\overline{\rho})| \le t(1+e^{t-1})|\rho - \overline{\rho}|.$$

This leads to the condition (7.4) that holds for each $t \in \Upsilon$, $\zeta > 0$ and the function $h : [0,1] \times [0,\zeta] \to [0,\infty)$ such as

$$h(t,\rho) = t(1+e^{t-1})|\rho|.$$

Then, Theorem 7.2.1 leads us to the successive approximations $\rho_n;\ n \in \mathbb{N}$, defined by

$$\rho_0(t) = 1;\ t \in \Upsilon,$$

$$\rho_{n+1}(t) = 1 + \frac{2(1-s)}{(2-s)M(s)}(\varphi(t,\rho_n(t)) - \varphi(0,1)) + \frac{2s}{(2-s)M(s)}\int_0^t \varphi(\tau,\rho_n(\tau))d\tau;\ t \in \Upsilon,$$

converges uniformly on Υ to the unique solution of the problem (7.6).

Chapter 8

Hilfer-Hadamard Fractional Differential Equations ; Existence and Attractivity

8.1 Introduction

This chapter studies a class of Hilfer-Hadamard differential equations. We prove a result of existence and attractivity of solutions.

In [11], Abbas *et al.* study some existence and Ulam stability results of the following problem

$$\begin{cases} ({}^H D_{1^+}^{\tau,\theta} i)(t) = \chi(t, i(t)); & t \in [1, T], \\ ({}^H I_{1^+}^{1-\varrho} i)(1) = d, & \varrho = \tau + \theta(1-\tau). \end{cases}$$

We devote this work to the existence and attractivity of solutions of the following problem

$$\begin{cases} ({}^H D_{c^+}^{\tau,\theta} i)(t) = \chi(t, i(t)); & t \in [c, +\infty),\ c > 0, \\ ({}^H I_{c^+}^{1-\varrho} i)(c) = d, & \varrho = \tau + \theta(1-\tau), \end{cases} \tag{8.1}$$

where $d \in \mathbb{R}$, $\chi : [c, +\infty) \times \mathbb{R} \to \mathbb{R}$, ${}^H I_{c^+}^{1-\varrho}$ is the left-sided Hadamard fractional integral of order $\tau > 0$ and ${}^H D_{c^+}^{\tau,\theta}$ is the Hilfer-Hadamard derivative operator of order τ $(0 < \tau < 1)$ and type θ $(0 \leq \theta \leq 1)$.

8.2 Existence and Attractivity Results

We introduce some spaces. We denote by $C_{\varrho,\log}[c,e]$, $(0 < c < e < \infty)$, the space $C_{\varrho,\log}[c,e] = \{\iota : (c,e] \to \mathbb{R} : (\log \frac{t}{c})^{1-\varrho}\ \iota(t) \in C[c,e]\}$, with the norm

$$\|\iota\|_{C_{\varrho,\log}} = \sup_{t\in[c,e]} \left| \left(\log \frac{t}{c}\right)^{1-\varrho} \iota(t) \right|.$$

$BC^* := BC([c, +\infty))$ denotes the space continuous and bounded functions $\iota : [c, +\infty) \to \mathbb{R}$.
$BC_\varrho = \{\iota : (c, +\infty) \to \mathbb{R} : (\log \frac{t}{c})^{1-\varrho}\iota(t) \in BC^*\}$, with the norm

$$\|\iota\|_{BC_\varrho} := \sup_{t\in[c,+\infty)} \left| \left(\log \frac{t}{c}\right)^{1-\varrho} \iota(t) \right|.$$

Denote $\|\iota\|_{BC_\varrho}$ by $\|\iota\|_{BC^*}$.

Corollaire 8.2.1 *[72] Let* $\sigma \in C_{\varrho,\log}(I)$. *The problem*

$$\begin{cases} ({}^H D_{c^+}^{\tau,\theta} i)(t) = \sigma(t), \quad t \in I := [c,e] \\ ({}^H I_{c^+}^{1-\varrho} i)(c) = d, \end{cases}$$

admits the following unique solution

$$i(t) = \frac{d}{\Gamma(\varrho)} \left(\log \frac{t}{c}\right)^{\varrho-1} + \left({}^H I_{c^+}^{\tau} \sigma\right)(t). \tag{8.2}$$

Lemma 8.2.1 *[72] Let* $\chi : (c,e] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $\chi(\cdot, i(\cdot)) \in BC_\varrho$ *for any* $i \in BC_\varrho$. *The problem (8.1) is equivalent to the integral equation*

$$i(t) = \frac{d}{\Gamma(\varrho)} \left(\log \frac{t}{c}\right)^{\varrho-1} + \left({}^H I_{c^+}^{\tau} \chi(\cdot, i(\cdot))\right)(t). \tag{8.3}$$

Definition 8.2.1 *A function* $i \in BC_\varrho$ *is a solution of (8.1) if it verifies* $({}^H I_{c^+}^{1-\varrho} i)(c) = d$, *and the equation* $({}^H D_{c^+}^{\tau,\theta} i)(t) = \chi(t, i(t))$ *on* $[c, +\infty)$.

We assume the following hypotheses :

(H_1) The function $t \mapsto \chi(t,i)$ is measurable on $[c,+\infty)$ for each $i \in BC_\varrho$, and $i \mapsto \chi(t,i)$ is continuous.

(H_2) There exists a continuous function $l : [c,+\infty) \to [0,+\infty)$ such that

$$|\chi(t,i)| \le \frac{l(t)}{1+|i|} \quad \text{for a.e.} \quad t \in [c,+\infty) \quad \text{and} \quad \text{each} \quad i \in \mathbb{R},$$

and

$$\lim_{t\to\infty} \left(\log \frac{t}{c}\right)^{1-\varrho} \left({}^H I_{c^+}^{\tau} l\right)(t) = 0.$$

Set

$$l^* = \sup_{t\in[c,+\infty)} \left(\log \frac{t}{c}\right)^{1-\varrho} \left({}^H I_{c^+}^{\tau} l\right)(t).$$

Theorem 8.2.1 *Assume* (H_1) *and* (H_2) *hold. Then (8.1) has at least one solution which is uniformly locally attractive.*

Proof. Define the operator L by

$$(Li)(t) = \frac{d}{\Gamma(\varrho)} \left(\log \frac{t}{c}\right)^{\varrho-1} + \frac{1}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s}\right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s}.$$

We prove that the operator L maps BC_ϱ into BC_ϱ. Indeed; the map $L(i)$ is continuous on $[c,+\infty)$, and for any $i \in BC_\varrho$ and, for each $t \in [c,+\infty)$, we get

$$\begin{aligned}
\left|\left(\log \frac{t}{c}\right)^{1-\varrho} (Li)(t)\right| &\le \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s}\right)^{\tau-1} |\chi(s,i(s))| \frac{ds}{s} \\
&\le \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s}\right)^{\tau-1} l(s) \frac{ds}{s} \\
&\le \frac{|d|}{\Gamma(\varrho)} + l^* \\
&:= R^*,
\end{aligned}$$

so

$$\|L(i)\|_{BC_\varrho} \leq R^*. \tag{8.4}$$

Therefore, $L(i) \in BC_\varrho$, which proves that the operator $L\left(BC_\varrho\right) \subset BC_\varrho$. Equation (8.4) implies that L maps

$$B_{R^*} := B(0, R^*) = \left\{ v \in BC_\varrho : \|v\|_{BC_\varrho} \leq R^* \right\}$$

into itself.

Step 1. L is continuous.

Let $\{i_n\}_{n\in N}$ be a sequence converging to i in B_{R^*}. Then,

$$\left| \left(\log \frac{t}{c}\right)^{1-\varrho} (Li_n)(t) - \left(\log \frac{t}{c}\right)^{1-\varrho} (Li)(t) \right|$$

$$\leq \frac{1}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s}\right)^{\tau-1} \left| \left(\log \frac{t}{c}\right)^{1-\varrho} \chi\left(s, i_n(s)\right) - \left(\log \frac{t}{c}\right)^{1-\varrho} \chi(s, i(s)) \right| \frac{ds}{s}. \tag{8.5}$$

Case 1. If $t \in [c, T], T > 0$, then $i_n \to i$ as $n \to \infty$ and from the continuity of χ, we get

$$\|L\left(i_n\right) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$

Case 2. If $t \in (T, \infty), T > 0$, then (8.5) implies that

$$\left| \left(\log \frac{t}{c}\right)^{1-\varrho} (Li_n)(t) - \left(\log \frac{t}{c}\right)^{1-\varrho} (Li)(t) \right| \leq 2 \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \times \int_c^t \left(\log \frac{t}{s}\right)^{\tau-1} l(s) \frac{ds}{s}, \tag{8.6}$$

since $i_n \to i$ as $n \to \infty$ and $\left(\log \frac{t}{c}\right)^{1-\varrho} \left({}^H I_{c^+}^\tau l\right)(t) \to 0$ as $t \to \infty$, it follows from (8.6) that

$$\|L\left(i_n\right) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. $L\left(B_{R^*}\right)$ is uniformly bounded and equicontinuous.

Since $L\left(B_{R^*}\right) \subset B_{R^*}$ and B_{R^*} is bounded, then $L\left(B_{R^*}\right)$ is uniformly bounded. Next let $t_1, t_2 \in [c, T], t_1 < t_2$, and let $i \in B_{R^*}$. This yields

$$\begin{aligned}
&\left| \left(\log \frac{t_2}{c}\right)^{1-\gamma} (Li)(t_2) - \left(\log \frac{t_1}{c}\right)^{1-\varrho} (Li)(t_1) \right| \\
&\leq \left| \left(\log \frac{t_2}{c}\right)^{1-\varrho} \left[\frac{d}{\Gamma(\varrho)} \left(\log \frac{t_2}{c}\right)^{\varrho-1} + \frac{1}{\Gamma(\tau)} \int_c^{t_2} \left(\log \frac{t_2}{s}\right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right] \right. \\
&\left. - \left(\log \frac{t_1}{c}\right)^{1-\varrho} \left[\frac{d}{\Gamma(\varrho)} \left(\log \frac{t_1}{c}\right)^{\varrho-1} + \frac{1}{\Gamma(\tau)} \int_c^{t_1} \left(\log \frac{t_1}{s}\right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right] \right| \\
&\leq \left| \frac{\left(\log \frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^{t_2} \left(\log \frac{t_2}{s}\right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right. \\
&\left. - \frac{\left(\log \frac{t_1}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^{t_1} \left(\log \frac{t_1}{s}\right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right|.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}(Li)(t_2)-\left(\log\frac{t_1}{c}\right)^{1-\varrho}(Li)(t_1)\right| \\
&\leq \frac{\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}|\chi(s,i(s))|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1}-\left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right||\chi(s,i(s))|\frac{ds}{s} \\
&\leq \frac{\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}l(s)\frac{ds}{s} \\
&+\frac{1}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1}-\left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|l(s)\frac{ds}{s}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}(Li)(t_2)-\left(\log\frac{t_1}{c}\right)^{1-\varrho}(Li)(t_1)\right| \\
&\leq \frac{l_*\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}\frac{ds}{s} \\
&+\frac{l_*}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1}-\left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|\frac{ds}{s} \\
&\leq \frac{l_*\left(\log\frac{T}{c}\right)^{1-\varrho}}{\Gamma(\tau+1)}\left(\log\frac{t_2}{t_1}\right)^{\tau} \\
&+\frac{l_*}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1}-\left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|\frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the inequality tends to zero.

Step 3. $L(B_{R^*})$ is equiconvergent.
Let $t \in [c,+\infty)$ and let $i \in B_{R^*}$. We have

$$\begin{aligned}
\left|\left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)\right| &\leq \frac{|d|}{\Gamma(\varrho)}+\frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}|\chi(s,i(s))|\frac{ds}{s} \\
&\leq \frac{|d|}{\Gamma(\varrho)}+\frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}l(s)\frac{ds}{s} \\
&\leq \frac{|d|}{\Gamma(\varrho)}+\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_c^{\tau} l\right)(t).
\end{aligned}$$

Since

$$\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_{c^+}^{\tau} l\right)(t)\to 0\; as\;\; t\to+\infty,$$

we find

$$|(Li)(t)| \leq \frac{|d|}{\left(\log\frac{t}{c}\right)^{1-\varrho}\Gamma(\varrho)}+\frac{\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_{c^+}^{\tau} l\right)(t)}{\left(\log\frac{t}{c}\right)^{1-\varrho}}\to 0 \quad \text{as} \quad t\to+\infty.$$

Hence

$$|(Li)(t)-(Li)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps $1-3$, $L : B_{R^*} \to B_{R^*}$ is compact and continuous. Using Schauder's fixed point Theorem, we get that L has a fixed point i, which is a solution of problem (8.1) on $[c,+\infty)$.

Step 4. Assume that i_0 is solution of (8.1). Set $i \in B\left(i_0, 2l^*\right)$, we have

$$\begin{aligned}
&\left|\left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)-\left(\log\frac{t}{c}\right)^{1-\varrho} i_0(t)\right| \\
&=\left|\left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)-\left(\log\frac{t}{c}\right)^{1-\varrho}(Li_0)(t)\right| \\
&\leq \frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}\left|\chi(s,i(s))-\chi(s,i_0(s))\right|\frac{ds}{s} \\
&\leq \frac{2\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1} l(s)\frac{ds}{s} \\
&\leq 2l^*.
\end{aligned}$$

We get

$$\|L(i)-i_0\|_{BC_\varrho} \leq 2l^*.$$

So L is a continuous function such that

$$L\left(B\left(i_0, 2l^*\right)\right) \subset B\left(i_0, 2l^*\right).$$

Moreover, if i is a solution of problem (8.1), then

$$\begin{aligned}
|i(t)-i_0(t)| &= \left|(Li)(t)-\left(Li_0\right)(t)\right| \\
&\leq \frac{1}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}\left|\chi(s,i(s))-\chi\left(s,i_0(s)\right)\right|\frac{ds}{s} \\
&\leq 2\left({}^H I_{c^+}^{\tau} l\right)(t).
\end{aligned}$$

Therefore,

$$|i(t)-i_0(t)| \leq \frac{2\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_{c^+}^{\tau} l\right)(t)}{\left(\log\frac{t}{c}\right)^{1-\varrho}}. \tag{8.7}$$

By (8.7) and

$$\lim_{t\to\infty}\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_{c^+}^{\tau} l\right)(t)=0,$$

we get

$$\lim_{t\to\infty}|i(t)-i_0(t)|=0.$$

Hence, solutions of (8.1) are uniformly locally attractive.

8.3 An Example

Consider the problem

$$\begin{cases} ({}^H D_{1^+}^{\frac{1}{2},\frac{1}{2}} i)(t) = \chi(t, i(t)); \qquad t \in [1, +\infty), \\ ({}^H I_{1^+}^{\frac{1}{4}} i)(1) = 1, \end{cases} \tag{8.8}$$

where

$$\begin{cases} \chi(t,i) = \frac{\Lambda(\log t)^{-1/4}\cos t}{64(1+\sqrt{t})(1+|i|)}, \quad t \in (1,\infty), \quad i \in \mathbb{R}, \\ \chi(1,i) = 0, \;\; i \in \mathbb{R}, \end{cases} \tag{8.9}$$

and

$$\Lambda = \frac{9\sqrt{\pi}}{16}.$$

Clearly, the function χ is continuous. (H_2) is satisfied with

$$\begin{cases} l(t) = \frac{\Lambda(\log t)^{-1/4}|\cos t|}{64(1+\sqrt{t})}; \quad t \in (1,\infty), \\ l(1) = 0, \end{cases} \tag{8.10}$$

and

$$\begin{aligned} (\log t)^{\frac{1}{4}} \, {}^H I_1^{1/2} l(t) &= \frac{(\log t)^{1/4}}{\Gamma\left(\frac{1}{2}\right)} \int_1^t \left(\log \frac{t}{s}\right)^{-1/2} \frac{l(s)}{s} ds \\ &\le \frac{1}{4} (\log t)^{-1/4} \to 0 \quad \text{as} \quad t \to \infty. \end{aligned}$$

Hence, the problem (8.8) has at least one solution which is uniformly locally attractive.

Conclusion and perspective

In this thesis, we have proved some results on the existence and attractivity of the solution for two classes of nonlinear ψ-Hilfer hybrid and Hilfer-Hadamard fractional differential equations. We used the Schauder fixed point theorem and we proved that all solutions are uniformly locally attractive. Other results on the global convergence of successive approximations and the uniqueness of the solution for initial value problems involving implicit Caputo q-difference equations, Caputo-Fabrizio, random coupled Hilfer and ψ-Hilfer hybrid Caputo fractional differential equations are also considered.

For further research, we can study the global convergence of successive approximations for more general problems, such as studying problems involving Hilfer-Katugampola. We can also extend the considered problems to the Fréchet space setting.

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