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Thème

Nonlinear lmplicit Genealized Hilfer- Type Fractional Differential Equations with Non-Instantaneous Impulses in Banch Spaces.

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ملخص

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Résumé

Dans ce travail, nous donnons des conditions suffisante qui assurent l'éxistence de solutions de probléme initial pour des équations différentielles fractionnaires implicites non linéaires avec des impulsions non instantaées et dérrivée fractionnaire de Hilfer généralisée dans les espace de Banach.

Les résultats sont basés sur les théorèmes du point fixe de Darbo et de Monch associés a la tichnique de la mesure de la non compacité. Nous donons un exemple pour montrer l'applicabilité de notre résultat.

Abstract

In the present work, we give some sufficient conditions wich guarantee the existence of solutions for a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces.

The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. An example is included to show the applicability of our results.

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List of symbols and Notations

List of symbols:

We use the following notations throughout this thesis Acronyms

•FC: Fractional calculus.

•FD : Fractional derivative.

•FDE : Fractional differential equation.

•FI : Fractional integral.

Notaions:

•$\mathbb{N}$: Set of natural numbers.

•$\mathbb{R}$: Set of real numbers.

•C: Set of complex numbers.

•Re(α) : the real part of numbre $\alpha \in$ C

•D^n (or $\frac{d^\pi}{d'}$) : order derivative n.

.

Introduction

Fractional calculus can be seen as a generalization of classical calculus. It should be noted that the fractional calculus is now more attractive and many monographs and conferences are devoted to this subject, although it is an old subject and known since the 17th century. The advantage of fractional derivatives is that they are nonlocal operators describing the memory and hereditary properties of many materials and processes. Recently, fractional calculus is introduced in mathematical psychology to describe human behavior since the manner he reacts to external influences depends on the experiences he had in the past [10].

Many authors have shown that derivatives of fractional order are better suited to the description of various real materials and that the introduction of the fractional calculation in the modeling reduces the number of parameters required. While fractional integral can be used for example in order to better describe the accumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of the regression model [23].

Due to these facts, differential equations involving fractional derivatives are more adequate to describe many phenomena in different fields of applied sciences and engineering such as in control, signal processing, electrochemistry, viscoelasticity, rheology, chaotic dynamics, statistical physics, biosciences, [10, 17].

We must mention that there is no general applicable method to discuss the classical questions related to an arbitrary given fractional differential equation and that to study the existence, uniqueness and properties of solutions, different methods are used. This includes the upper and lower solutions method, the Mawhin theory, the decomposition method, the variational iteration method, the homotopy method... [6, 9, 12, 13, 14, 15] Another important question regarding solutions for fractional differential equations is their stability. Note that the analysis of the stability of fractional differential equations is more complex than ordinary differential equations, due to the fact that fractional derivatives are nonlocal and have a singular kernel. The literature on the stability of fractional differential equations is limited and concentrated on a fractional order. We can cite some articles dealing with the stability of solutions for systems of fractional differential equations or for fractional differential equations [7, 8, 11, 21]. Most of them used Lyapunov direct or indirect method without finding the explicit form of the solution.

This memory is devoted to the study of sufficient condition for the existence of solutions nonlinear fractional differential equations using fixed point theory. An example is included to show the applicability of our results.

presentation

This work contains three chapters .

In chapter:1

We introduce some functions that are of fundamental importance in the theory of fractional differential equations, Gamma function , Beta function and Mittag-Leffler.We give a characterization of a compact set in the space of continuous functions and, We give some properties and lemmas and fixed point theorems

In chapter:2

We prove the existence result of solutions for a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces.

In chapter:3

We give aur main result, We have to change the continuous condition in g studied in chapter two by lipschitz condition

$(A'x4)$ The functions $g_k \in C\left(\tilde{I}_k, E\right), k = 1, \ldots, m$, and there exists $l^* > 0$ such that

$$\|g_k(t,u) - g_k(t,v)\| \leq l^* \|u - v\| \text{ for each } u, v \in E, k = 1, \ldots, m$$

Chapter 1

Preliminaries

1.1 Introduction

In this parte we introduce some important functions which are used in fractional calculus. The gamma, beta and Mittag-Leffler functions that will be used ,these functions play a very important role in the fractioal calculation theory [7, 11, 16, 24]

1.2 Basic functions

Definition 1.2.1 *E'eulre gamma function is a function which naturally extends the factorial to numbres real and even to numbres complex*
The Gamma function $\Gamma(.)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

Which converges in the right half of the complex plane, that is, $Re(z) > 0$.

The Gamma function satisfies

$$\Gamma(z+1) = z\Gamma(z), \; Re(z) > 0,$$

and for any integer $n \geq 0$, we have

$$\Gamma(n+1) = n!.$$

A limit definition of the Gamma function is given by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^2}{z(z+1)...(z+n)}, \quad Re(z) > 0,$$

Some particular values of the gamma function:

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
2. $\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$
3. $\Gamma(-1) = (-1)! = +\infty$
4. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
5. $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$
6. $\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\Gamma\left(\frac{1}{2}\right) = \frac{(2n)!}{2^{2\pi n}}\sqrt{\pi}$
7. $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$
8. $\Gamma(1) = 0! = 1$

Definition 1.2.2 *For every z, w such that $Re(z) > 0$, $Re(w) > 0$, the Beta function is defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt.$$

The beta function is symmetric :

$$B(z,w) = B(w,z) \qquad \mathrm{Re}(z) > 0 \qquad \mathrm{Re}(w) > 0,$$

an interesting formula relating the Gamma and Beta functions is

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \qquad Re(z) > 0, \qquad Re(w) > 0.$$

Definition 1.2.3 *A two-parameter Mittag-Leffler function, $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$, is defined by*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}.$$

For $\beta = 1$, we have the one-parameter Mittag-Leffler function by means of the following series:

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}$$

Definition 1.2.4 *Let X a Banach space, we say that $f : X \to X$ is Liphizien if and only if*

$$\|f(x) - f(y)\| \leq L\|x - y\| \quad x, y \in X\ , L > 0$$

1.3 fractional integrals and derivatives

In this section, we focus on the Riemann-Liouville integrals and derivatives and the Caputo derivative since they are the most used ones in applications. We will formulate the conditions of their equivalence and derive the most important properties. There is several types of fractional derivatives Hadamard fractional derivative.

Definition 1.3.1 *[23] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (a, +\infty) \to \mathbb{R}$ is given by:*

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right side is pointwise defined on $(a, +\infty)$.

Definition 1.3.2 *[23] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (a, +\infty) \to \mathbb{R}$ is given by:*

$$D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_a^x \frac{f(s)}{(t-s)^{\alpha-n+1}} ds = \left(\frac{d}{ds}\right)^n I_{a^+}^{n-\alpha} f(s),$$

provided that the right side is pointwise defined on $(a, +\infty)$, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 1.3.1 *[16] Let $\alpha \geq \beta > 0$, then for $f \in L^P[a,b]\,(1 \leq p \leq \infty)$ the relation*

$$\left(D_{a^+}^{\beta} I_{a^+}^{\alpha} f\right)(t) = I_{a^+}^{\alpha-\beta} f(t).$$

holds almost everywhere on $[a,b]$. In particular if $\alpha = \beta$ we get

$$\left(D_{a^+}^{\alpha} I_{a^+}^{\alpha} f\right)(t) = f(t).$$

Lemma 1.3.2 *[16] The fractional integral operator $I_{a^+}^{\alpha}$ is bounded from $L^p(a,b)$ $(1 \leq p \leq \infty)$ into itself*

$$\|I_{a^+}^{\alpha} f\|_{L^P} \leq k\, \|f\|_{L^p}, \qquad k = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}$$

Definition 1.3.3 *[3] Let $\alpha > 0$ and $n = [\alpha] + 1$, for a function $f \in AC^n([a,b], \mathbb{R})$ the Caputo fractional derivative of order α of f is defined by:*

$$\begin{aligned} \left({}^C D_{a^+}^{\alpha} f\right)(t) &= I^{n-\alpha} D^{(n)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds. \end{aligned}$$

Where $D = \frac{d}{dt}$ *denotes the classical derivative and* $AC^n[a,b] = \{f \in C^{n-1}[a,b], f^{(n-1)}$ *absolutely continuous function*$\}$.

Property: Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$I_{a^+}^{\alpha}(x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}.$$

$$D_{a^+}^{\alpha}(x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.$$

$${}^{C}D_{a^+}^{\alpha}(x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \qquad \beta > n.$$

On the other hand, for $k = 1, 2, ..., n$, we have

$$D_{a^+}^{\alpha}(x-a)^{\alpha-k}(t) = 0,$$

and for $k = 0, 1, ..., n-1$

$${}^{C}D_{a^+}^{\alpha}(x-a)^{k}(t) = 0,$$

in particular,

$${}^{C}D_{a^+}^{\alpha}(1) = 0.$$

The Riemann-Liouville fractional derivative of a constant is in general not equal to zero, in fact

$$D_{a^+}^{\alpha}(1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1.$$

Lemma 1.3.3 *[23] Let* $\alpha > 0$, $n = [\alpha] + 1$ *and* $f : [a,b] \to \mathbb{R}$ *be a given fonction. Assume that* $D_{a^+}^{\alpha}f$ *and* ${}^{C}D_{a^+}^{\alpha}f$ *exist. Then*

$${}^{C}D_{a^+}^{\alpha}f(t) = D_{a^+}^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.$$

Lemma 1.3.4 *[23] Let* $\alpha > 0$, *then the fractional differential equation*

$$D_{0^+}^{\alpha}f(t) = 0.$$

has $f(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + ... + c_nt^{\alpha-n}$, $c_i \in \mathbb{R}, i = 1, 2, ..., n$ *as solution.*

Lemma 1.3.5 *[23] Let* $\alpha > 0$, $n = [\alpha]+1$. *If* $f \in L^1[a,b]$ *and* $f_{n-\alpha} \in AC^n[a,b]$, *then the equality*

$$\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha}f\right)(t) = f(t) - \sum_{j=1}^{n}\frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j}.$$

holds almost everywhere on $[a,b]$. *In particular, if* $0<\alpha<1$, *then*

$$\left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f\right)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1},$$

where $f_{n-\alpha} = I_{a^+}^{n-\alpha} f$ *and* $f_{1-\alpha} = I_{a^+}^{1-\alpha} f$.

Theorem 1.3.1 *[24] Let* $\beta>\alpha>0$, *then we have*

$$\left(I_{a^+}^{\alpha C} D_{a^+}^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

$$\left(D_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) = I_{a^+}^{\beta-\alpha} f(t).$$

$$D^m D_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha+m} f(t), m \in N.$$

Definition 1.3.4 *[1] The Hadamard fractional integral of order* $\alpha>0$ *of a function* f *is defined by:*

$$\boldsymbol{I}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \qquad a<t<b.$$

A more general fractional integral referred as Hadamard fractional integral of order α *is given by*

$$\boldsymbol{I}_{a^+}^{\alpha,\mu} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{s}{t}\right)^{\mu} \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \qquad a<t<b, \qquad \mu\in\mathbb{R}.$$

Definition 1.3.5 *[2] The Hadamard fractional derivative of order* $\alpha>0$ *of a function* f *is defined by:*

$$\boldsymbol{D}_{a^+}^{\alpha} f(t) = \left(t\frac{d}{dt}\right)^n \boldsymbol{I}_{a^+}^{n-\alpha} f(t), \qquad a<t<b, \qquad n=[\alpha]+1.$$

A more general fractional derivative referred as Hadamard fractional derivative of order α *is given by:*

$$\boldsymbol{D}_{a^+}^{\alpha,\mu} f(t) = t^{-\mu}\left(t\frac{d}{dt}\right)^n t^{\mu} \boldsymbol{I}_{a^+}^{n-\alpha,\mu} f(t), \qquad a<t<b, \qquad n=[\alpha]+1.$$

Definition 1.3.6 *[18](Generalized fractional integral)Let* $\alpha\in\mathbb{R}_+$ *and* $g\in L^1(J)$. *The generalized fractional integral of order* α *is defined by:*

$$\left({}^{\rho}\mathcal{J}_{a^+}^{\alpha} g\right)(t) = \int_a^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \qquad t>a, \qquad \rho>0.$$

Where $\Gamma(\cdot)$ *is the Euler gamma function defined by:* $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt, \qquad \alpha>0.$

Definition 1.3.7 *[18](Generalized fractional derivative) Let $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}$ and $\rho > 0$. The generalized fractional derivative ${}^{\rho}\mathcal{D}_{a^+}^{\alpha}$ of order α is defined by;*

$$\begin{aligned} \left({}^{\rho}\mathcal{D}_{a}^{\alpha} g\right)(t) &= \delta_{\rho}^{n} \left({}^{\rho}\mathcal{J}_{a^+}^{n-\alpha} g\right)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^{n} \int_{a}^{t} s^{\rho-1} \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds, \qquad t > a, \end{aligned}$$

where $n = [\alpha] + 1$ and $\delta_{\rho}^{n} = \left(t^{1-\rho}\frac{d}{dt}\right)^{n}$.

Definition 1.3.8 *[22] Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$, and $k = 0, \ldots, m$. The generalized Hilfer-type fractional derivative, with $\rho > 0$ of a function $g \in C_{\gamma,\rho}(I_k)$, is defined by*

$$\begin{aligned} \left({}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta} g\right)(t) &= \left({}^{\rho}\mathcal{J}_{s_k^+}^{\beta(n-\alpha)} \left(t^{\rho-1}\frac{d}{dt}\right)^{n} {}^{\rho}\mathcal{J}_{s_k^+}^{(1-\beta)(n-\alpha)} g\right)(t) \\ &= \left({}^{\rho}\mathcal{J}_{s_k^+}^{\beta(n-\alpha)} \delta_{\rho}^{n\rho} \mathcal{J}_{s_k^+}^{(1-\beta)(n-\alpha)} g\right)(t). \end{aligned}$$

In this work we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property [18] The fractional derivative ${}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$), Hilfer-Hadamard ($\rho \to 0^+$), generalized ($\beta = 0$), Caputo-type ($\beta = 1$), Riemann-Liouville ($\beta = 0, \rho \to 1$), Hadamard ($\beta = 0, \rho \to 0^+$), Caputo ($\beta = 1, \rho \to 1$), Caputo-Hadamard ($\beta = 1, \rho \to 0^+$) Liouville ($\beta = 0, \rho \to 1, a = 0$) and Weyl ($\beta = 0, \rho \to 1, a = -\infty$). Consider the following parameters α, β, γ satisfying

$$\gamma = \alpha + \beta - \alpha\beta, \qquad 0 < \alpha, \beta, \gamma < 1$$

Definition 1.3.9 *[4] Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \longrightarrow [0, \infty)$ defined by:*

$$\mu(M) = \inf\left\{\epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, \operatorname{diam}(M_j) \leq \epsilon\right\},$$

where $M \in \Omega_X$. The map μ satisfies the following properties:

- $\mu(M) = 0 \Leftrightarrow \bar{M}$ is compact (M is relatively compact).

- $\mu(M) = \mu(\bar{M})$.

- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

- $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

-$\mu(cM) = |c|\mu(M), \qquad c \in \mathbb{R}$.

- $\mu(\text{convM}) = \mu(M)$.

Lemma 1.3.6 *[4] Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then (i) the function $t \to \mu(D(t))$ is continuous on J, and*

$$\mu_{PC_{\gamma,\rho}} = \max\left\{\max_{k=0,\dots,m}\left\{\sup_{t\in I_k}\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t)\right)\right\}, \max_{k=1,\dots,m}\left\{\sup_{t\in I_k}\mu(u(t))\right\}\right\},$$

ii) $\mu\left(\int_a^b u(s)ds : u \in D\right) \leq \int_a^b \mu(D(s))ds$, where

$$D(t) = \{u(t) : t \in D\}, t \in J$$

1.4 Generalized fractional integral and derivative

Katugampola in [3] introduced a new type of fractional derivative generalizing Riemann-Liouville and Hadamard fractional derivatives. Later, Almeida and all in [5], introduced a generalization of the derivative as the left inverse of Katugampola's fractional integral and which retains some of the fundamental properties of the fractional derivatives of Caputo and Caputo-Hadamard, the new derivative is called Caputo-Katugampola fractional derivative [3, 5, 19, 20]

Definition 1.4.1 *[18] (Katugampola fractional integrals) Let a, b be two real and $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional integrals of order $\alpha > 0$, parameter $\rho > 0$, of f is defined as*

$$I_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s)\, ds.$$

Definition 1.4.2 *[18] (Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real $f : [a, b] \to \mathbb{R}$ be an integrable function. The Katugampola fractional derivative of order $\alpha > 0$, and parameter $\rho > 0$, is defined as*

$$\begin{aligned} D_{a^+}^{\alpha,\rho} f(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\rho} f(t) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{n-\alpha-1} f(s)\, ds. \end{aligned}$$

proposition 1.4.1 *We have the following properties for Katugompola fractional integral and derivative.*

$$\begin{aligned}
D_{a^+}^{\alpha,\rho}\left(I_{a^+}^{\alpha,\rho}\right)f(t) &= f(t),\\
I_{a^+}^{\alpha,\rho}\left(I_{a^+}^{\beta,\rho}\right)f(t) &= I_{a^+}^{\alpha+\beta,\rho}f(t),\\
\lim_{\rho\to 1} I_{a^+}^{\alpha,\rho}f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds,\\
\lim_{\rho\to 0^+} I_{a^+}^{\alpha,\rho}f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},\\
\lim_{\rho\to 0^+} D_{a^+}^{\alpha,\rho}f(t) &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}f(s)\frac{ds}{s},\\
\lim_{\rho\to 1} D_{a^+}^{\alpha,\rho}f(t) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}f(s)\,ds.
\end{aligned}$$

Definition 1.4.3 *[3](Caputo-Katugampola fractional derivative) Let $0 < a < b < \infty$ be two real, $\rho > 0$ be a positive real number and $f \in AC^n([a,b],\mathbb{R})$. The Caputo-Katugampola fractional derivative of order $\alpha > 0$ of the function f is defined by:*

$$\begin{aligned}
{}^C D_{a^+}^{\alpha,\rho}f(t) &= I_{a^+}^{n-\alpha,\rho}\left(t^{1-\rho}\frac{d}{dt}\right)^n f(t)\\
&= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{n-\alpha-1}\left(t^{1-\rho}\frac{d}{dt}\right)^n f(s)\,ds\\
&= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t \frac{s^{(\rho-1)(1-n)}f^{(n)}(s)}{\left(t^\rho - s^\rho\right)^{\alpha-n+1}}ds,
\end{aligned}$$

where n is the smallest integer greater than α.

Property:

1- When $\rho = 1$, the Caputo-Katugampola derivative coincides with Caputo derivative.

2- In the case $0 < \alpha < 1$ and $\rho > 0$, then

$${}^C D_{a^+}^{\alpha,\rho}f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}t^{1-\rho}\frac{d}{dt}\int_a^t \frac{s^{(\rho-1)}\left(f(s)-f(a)\right)}{\left(t^\rho - s^\rho\right)^\alpha}ds,$$

3- If $f \in C[a,b]$ then

$${}^C D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho}f(t) = f(t),$$

and if $f \in C^1[a,b]$ then

$$I_{a^+}^{\alpha,\rho C} D_{a^+}^{\alpha,\rho}f(t) = f(t) - f(a).$$

4- If $f(a)=0$, then the Caputo Katugampola and the Katugampola fractional derivatives coincide. Moreover if both types of derivatives exist then

$$^{C}D_{a^+}^{\alpha,\rho} f(t) = D_{a^+}^{\alpha,\rho} f(t) - \frac{f(a)\,\rho^{\alpha}\,(t^{\rho}-s^{\rho})^{-\alpha}}{\Gamma(1-\alpha)}.$$

We give some property and lemmas:

Lemma 1.4.1 *[3] Let $t > s_k, k = 0, \ldots, m$. Then, for $\alpha \geq 0$ and $\beta > 0$, we have*

$$\left[\rho \mathcal{J}_{s_k^+}^{\alpha} \left(\tfrac{s^{\rho}-s_k^{\rho}}{\rho}\right)^{\beta-1}\right](t) = \tfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\tfrac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha+\beta-1},$$
$$\left[\rho_{\mathcal{D}_k^+}^{\alpha} \left(\tfrac{s^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-1}\right](t) = 0, \quad 0<\alpha<1.$$

Lemma 1.4.2 *[18, 22] Let $\alpha > 0, 0 \leq \gamma < 1$ and $k = 0, \ldots, m$. Then, ${}^{\rho}\mathcal{J}_{s_k^+}^{\alpha}$ is bounded from $C_{\gamma,\rho}(I_k)$ into $C_{\gamma,\rho}(I_k)$*

Proof: Let $\alpha > 0, 0 \leq \gamma < 1$, $k = 0, \ldots, m$ and $u \in C_{\gamma,\rho}(I_k)$, we have

$$\begin{aligned}
\left|\left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{1-\gamma} \left({}^{\rho}\mathcal{J}_{s_k^+}^{\alpha} u\right)(t)\right| &\leq \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{1-\gamma} \int_{s_k^+}^{t} s^{\rho-1} \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} \frac{|u(s)|}{\Gamma(\alpha)} ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{s_k^+}^{t} \left| s^{\rho-1} \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} u(s) \left(\frac{s^{\rho}-s_k^{\rho}}{\rho}\right)^{1-\gamma} \left(\frac{s^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1}\right| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_{s_k^+}^{t} s^{\rho-1} \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} \left(\frac{s^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1} ds \\
&\leq \frac{M}{\Gamma(\alpha)} . {}^{\rho}\mathcal{J}_{s_k^+}^{\alpha} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1} \\
&\leq M . \frac{\Gamma(1-(1-\alpha))}{\Gamma(\alpha-(1-\gamma)+1)} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-\gamma+1} \\
&\leq M . \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-\gamma+1} \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{1-\alpha} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-\gamma+1} \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-2\gamma+2} \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{t_{k+1}^{\rho}-s_k^{\rho}}{\rho}\right)^{\alpha-2\gamma+2} \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-2\gamma+2}
\end{aligned}$$

$$< \infty.$$

Then ${}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}u \in C_{\gamma,\rho}(I_k)$.

Lemma 1.4.3 *[22] Let $0 < a < b < \infty, \alpha > 0, 0 \leq \gamma < 1$, $u \in C_{\gamma,\rho}(I_k)$ and $k = 0, \ldots, m$. If $\alpha > 1-\gamma$, then ${}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}u \in C([s_k, t_{k+1}], E)$ and*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}u\right)(s_k) = \lim_{t \to s_k^+}\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}u\right)(t) = 0$$

Proof: Since $u \in C_{\gamma,\rho}[s_k, s_{k+1}]$, then $\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}u(t)$ is continuous on $[s_k, s_{k+1}]$ and

$$\left|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}u(t)\right| \leq M, \quad x \in [s_k, s_{k+1}]$$

for some positive constant M. Consequently,

$$\left|\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}u\right)(t)\right| \leq \frac{1}{\Gamma(\alpha)}\int_{s_k^+}^{t}\left|s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}u(s)\right|ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{s_k^+}^{t}\left|s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}u(s)\left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}\right|ds$$

$$\leq \frac{M}{\Gamma(\alpha)}\int_{s_k^+}^{t}s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}ds$$

$$\leq \frac{M}{\Gamma(\alpha)}.{}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1}$$

and by lemma 1.4.1, we can write

$$\leq M.\frac{\Gamma(1-(1-\alpha))}{\Gamma(\alpha-(1-\gamma)+1)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha-\gamma+1} \tag{1.1}$$

$$\leq M.\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha-\gamma+1}$$

as the right-hand side of 1.1 tends to zero when $t \to s_k^+$

Lemma 1.4.4 *[18] Let $\alpha > 0, 0 \leq \gamma < 1, k = 0, \ldots, m$, and $g \in C_{\gamma,\rho}(I_k)$. Then, $\left({}^{\rho}\mathcal{D}^{\alpha}_{s_k^+}{}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\right)(t) = g(t), \quad$ for all $\quad t \in I_k, k = 0, \ldots$ sm*

Lemma 1.4.5 *[18] Let $0 < \alpha < 1, 0 \leq \gamma < 1, k = 0, \ldots, m$. If $g \in C_{\gamma,\rho}(I_k)$ and ${}^{\rho}\mathcal{J}^{1-\alpha}_{s_k^+}g \in C^1_{\gamma,\rho}(I_k)$, then for all $t \in I_k, k = 0, \ldots, m$,*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}{}^{\rho}\mathcal{D}^{\alpha}_{s_k^+}g\right)(t) = g(t) - \frac{\left({}^{\rho}\mathcal{J}^{1-\alpha}_{s_k^+}g\right)(s_k)}{\Gamma(\alpha)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha-1}$$

Lemma 1.4.6 *[18] Let $0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta - \alpha\beta$ and $k = 0, \ldots, m$. If $u \in C^{\gamma}_{\gamma,\rho}(I_k)$, then*

$$^{\rho}\mathcal{J}^{\gamma}_{s_k^+}\, {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u = {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\, {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u$$

and

$$^{\rho}\mathcal{D}^{\gamma}_{s_k^+}\, {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} u = {}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{s_k^+} u$$

1.5 Fixed point theorems

Fixed point theory is an important topic with a large number of applications in various fields of mathematics. The fixed point theorems concern a function f satisfying some conditions and admits a fixed point, that is $f(x) = x$. Knowledge of the existence of fixed points has pertinent applications in many branches of analysis and topology. Following if the conditions are imposed on the function or on the set, different fixed point theorems are given, we cite the following[16, 21, 22, 24]

Theorem 1.5.1 *(Banach contraction principle)*
Let T be a contraction on a Banach space X. Then T has a unique fixed point.

Theorem 1.5.2 *(Schauder fixed point theorem)*
Let Ω be a nonempty closed bounded and convex subset of a normed space. Let N be a continuous mapping from Ω into a compact subset of Ω, then N has a fixed point in Ω.

Theorem 1.5.3 *(Krasnoselskii fixed point theorem)*
. Let Ω be a closed bounded and convex nonempty subset of a Banach space X. Suppose that A and B map Ω into X such that
(i) A is continuous and compact.
(ii) B is a contraction mapping.
(iii) $x, y \in \Omega$, implies $Ax + By \in \Omega$.
Then there exists $x \in \Omega$ with $x = Ax + Bx$.

The criteria for compactness for sets in the space of continuous functions $C([a, b])$ is the following.

Theorem 1.5.4 *(Arzela-Ascoli theorem) A set $\Omega \subset C([a, b])$ is relatively compact in $C([a, b])$ if the functions in Ω are uniformly bounded and equicontinuous on $[a, b]$.*

We recall that a family Ω of continuous functions is uniformly bounded if there exists $M > 0$ such that

$$\|f\| = \max_{x\in[a,b]} |f(x)| \leq M, \; f \in \Omega.$$

The family Ω is equicontinuous on $[a, b]$, if $\forall\, \varepsilon > 0$, $\exists \delta > 0$ such that $\forall t_1, t_2 \in [a, b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

Theorem 1.5.5 *(Mönch fixed point Theorem)*
Let D be a closed, bounded and convex subset of a Banach space X such that $0 \in D$, and let T be a continuous mapping of D into itself. If the implication

$$V = \overline{\mathrm{con}\, v} T(V), \qquad or \qquad V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0$$

holds for every subset V of D, then T has a fixed point.

Theorem 1.5.6 *(Darbo's fixed point Theorem)*
Let D be a non-empty, closed, bounded and convex subset of a Banach space X, and let T be a continuous mapping of D into itself such that for any non-empty subset C of D,

$$\mu(T(C)) \leq k\mu(C)$$

where $0 \leq k < 1$, and μ is the Kuratowski measure of noncompactness on X. Then T has a fixed point in D. Now, we consider the Ulam stability for problem.

Chapter 2

Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with the continuous g condition

2.1 Introduction

This work was studied by A.salim and all, we establish existence results for the initial value problem of a nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses,

$$\left({}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}u\right)(t)=f\left(t,u(t),\left({}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}u\right)(t)\right);t\in I_k,k=0,\ldots,m \tag{2.1}$$

$$u(t)=g_k(t,u(t));t\in\tilde{I}_k,k=1,\ldots,m \tag{2.2}$$

$$\left({}^{\rho}\mathcal{J}_{a+}^{1-\gamma}u\right)\left(a^+\right)=\phi_0 \tag{2.3}$$

Where ${}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}$ and ${}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}$ are, respectively, the generalized Hilfer-type fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$ and generalized fractional integral of order $1-\gamma,(\gamma=\alpha+\beta-\alpha\beta),\rho>0,\phi_0\in E$ $I_k:=(s_k,t_{k+1}];k=0,\ldots,m,\tilde{I}_k:=(t_k,s_k];k=1,\ldots,m,a=s_0<t_1\leq s_1<t_2\leq s_2<\ldots\leq s_{m-1}<t_m\leq s_m<t_{m+1}=b<\infty,u\left(t_k^+\right)=\lim_{\epsilon\to0^+}u\left(t_k+\epsilon\right)$ and $u\left(t_k^-\right)=\lim_{\epsilon\to0^-}u\left(t_k+\epsilon\right)$ represent the right and left hand limits of $u(t)$ at $t=t_k$, $f:I_k\times E\times E\to E$ is a given function and

$g_k : \tilde{I}_k \times E \to E; k = 1, \ldots, m$, are given continuous functions such that $\left(\rho\mathcal{J}_{s_k^+}^{1-\gamma} g_k\right)(t, u(t))\Big|_{t=s_k} = \phi_k \in E$, where $(E, \|\cdot\|)$ is a real Banach space.

Consider the weighted Banach space

$$C_{\gamma,\rho}\left(I_k\right) = \left\{u : I_k \to E : t \to \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t) \in C\left(\left[s_k, t_{k+1}\right], E\right)\right\},$$

Where $0 \leq \gamma < 1, k = 0, \ldots, m$, and

$$C_{\gamma,\rho}^n\left(I_k\right) = \left\{u \in C^{n-1}\left(I_k\right) : u^{(n)} \in C_{\gamma,\rho}\left(I_k\right)\right\}, n \in \mathbb{N},$$
$$C_{\gamma,\rho}^0\left(I_k\right) = C_{\gamma,\rho}\left(I_k\right).$$

Also consider the Banach space $PC_{\gamma,\rho}(J) = \Big\{u : (a, b] \to E : u \in C_{\gamma,\rho}\left(\cup_{k=0}^m I_k\right) \cap C\left(\cup_{k=1}^m \tilde{I}_k, E\right)$ and there exist

$$u\left(t_k^-\right), u\left(t_k^+\right), u\left(s_k^-\right), \text{and } u\left(s_k^+\right) \text{ with } u\left(t_k^-\right) = u\left(t_k\right)\Big\}, 0 \leq \gamma < 1,$$

and

$$PC_{\gamma,\rho}^n(J) = \left\{u \in PC^{n-1}(J) : u^{(n)} \in PC_{\gamma,\rho}(J)\right\}, n \in \mathbb{N}$$
$$PC_{\gamma,\rho}^0(J) = PC_{\gamma,\rho}(J)$$

With the norm

$$\|u\|_{PC_{\gamma,\rho}} = \max\left\{\max_{k=0,\ldots,m}\left\{\sup_{t\in I_k}\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t)\right\|\right\}, \max_{k=1,\ldots,m}\left\{\sup_{t\in\tilde{I}_k}\|u(t)\|\right\}\right.$$

By $L^1(J)$, we denote the space of Bochner-integrable functions $f : J \longrightarrow E$ wit et

$$\|f\|_1 = \int_a^b \|f(t)\|dt.$$

2.2 Existence of solutions

Definition 2.2.1 *[24] Let $f : I_k \times E \to E$ be a function such that $f\left(\cdot, u(\cdot), {}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} u(\cdot)\right) \in C\left(I_k, E\right), k = 0, \ldots, m$, for any $u \in C_{\gamma,\rho}\left(I_k\right)$. The function $u \in C_{\gamma,\rho}^\gamma\left(I_k\right)$ is a solution of the differential equation, for $0 < \alpha < 1, 0 \leq \beta \leq 1$,*

$$\left(\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} u\right)(t) = f\left(t, u(t), {}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} u(t)\right), \text{ for each }, t \in I_k, k = 0, \ldots, m \tag{2.4}$$

if and only if u satisfies the following Volterra integral equation,

$$u(t)=\frac{\left(\rho\mathcal{J}_{s_k^+}^{1-\gamma}u\right)\left(s_k^+\right)}{\Gamma(\gamma)}\left(\frac{t^\rho-s_k^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f\left(s,u(s),{}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta}u(s)\right)ds \tag{2.5}$$

where $\gamma=\alpha+\beta-\alpha\beta$.

We consider the following linear fractional differential equation studied by A.Salim and al

$$\left({}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta}u\right)(t)=\psi(t),\quad t\in I_k,k=0,\dots,m \tag{2.6}$$

Where $0<\alpha<1,0\leq\beta\leq1,\rho>0$, with the conditions

$$u(t)=g_k(t,u(t)),t\in\tilde{I}_k,k=1,\dots,m \tag{2.7}$$

$$\left({}^\rho\mathcal{J}_{a+}^{1-\gamma}u\right)\left(a^+\right)=\phi_0 \tag{2.8}$$

Where $\gamma=\alpha+\beta-\alpha\beta$ and $\phi_0\in E$, and let $\phi^*=\max\left\{\|\phi_k\|:k=0,\dots,m\right\}$. The following theorem shows that the problem (2.6),(2.8) has a unique solution given by

$$u(t)=\begin{cases}\frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho-s_k^\rho}{\rho}\right)^{\gamma-1}+\left(\rho\mathcal{J}_{s_k^+}^{\alpha}\psi\right)(t), & t\in I_k,k=0,\dots,m\\ g_k(t,u(t)), & t\in\tilde{I}_k,k=1,\dots,m\end{cases} \tag{2.9}$$

Lemma 2.2.1 *[22] Let* $\alpha>0,\beta>0,1\leq p\leq\infty,0<a<b<\infty$. *Then, for* $g\in L^1\left([s_k,t_{k+1}]\right),k=0,\dots,m$, *we have*

$$\left({}^\rho\mathcal{J}_{s_k^+}^{\alpha}\,{}^\rho\mathcal{J}_{s_k^+}^{\beta}\right)(t)=\left({}^\rho\mathcal{J}_{s_k^+}^{\alpha+\beta}g\right)(t)$$

Theorem 2.2.1 *[24] Let* $\gamma=\alpha+\beta-\alpha\beta$, *where* $0<\alpha<1$ *and* $0\leq\beta\leq1$. *If* $\psi:I_k\to E$ $k=0,\dots,m$, *is a function such that* $\psi(\cdot)\in C\left(I_k,E\right)$, *then* $u\in PC_{\gamma,\rho}^{\gamma}(J)$ *satisfies the problem* $(2.6)-(2.8)$ *if and only if it satisfies (2.9).*

Proof: *Assume u satisfies* $(2.6)-(2.8)$. *If* $t\in I_0$, *then*

$$\left({}^\rho\mathcal{D}_{a^+}^{\alpha,\beta}u\right)(t)=\psi(t)$$

definition2.2.1 implies we have the solution can be written as

$$u(t)=\frac{\left(\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a)}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds$$

If $t \in I_1$, then we have $u(t) = g_1(t, u(t))$.

If $t \in I_1$, then definition2.2.1 implies

$$\begin{aligned} u(t) &= \frac{\left(\rho \mathcal{J}_{s_1^+}^{1-\gamma} u\right)(s_1)}{\Gamma(\gamma)} \left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_1}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \psi(s) ds \\ &= \frac{\phi_1}{\Gamma(\gamma)} \left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho \mathcal{J}_{s_1^+}^{\alpha} \psi\right)(t) \end{aligned}$$

If $t \in \tilde{I}_2$, then we have $u(t) = g_2(t, u(t))$.

If $t \in I_2$, then definition2.2.1implies

$$\begin{aligned} u(t) &= \frac{\left({}^\rho \mathcal{J}_{s_2^+}^{1-\gamma} u\right)(s_2)}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_2}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \psi(s) ds \\ &= \frac{\phi_2}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho \mathcal{J}_{s_2^+}^{\alpha} \psi\right)(t) \end{aligned}$$

Repeating the process in this way, the solution $u(t)$ for $t \in (a, b]$ can be written as

$$u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho \mathcal{J}_{s_k^+}^{\alpha} \psi\right)(t), & t \in I_k, k = 0, \ldots, m \\ g_k(t, u(t)), & t \in \tilde{I}_k, k = 1, \ldots, m \end{cases}$$

Conversely, for $t \in I_0$, applying ${}^\rho \mathcal{J}_{a+}^{1-\gamma}$ on both sides of (2.9) and using Lemma1.4.1 and lemma2.2.1 We get

$$\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)(t) = \phi_0 + \left({}^\rho \mathcal{J}_{a^+}^{1-\gamma+\alpha} \psi\right)(t) \tag{2.10}$$

Next, taking the limit as $t \to a^+$ of (2.10) and using Lemma1.4.3 with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)\left(a^+\right) = \phi_0 \tag{2.11}$$

which shows that the initial condition $\left({}^\rho \mathcal{J}_{a^+}^{1-\gamma} u\right)(a^+) = \phi_0$, is satisfied. Next, for $t \in I_k, k = 0, \ldots, m$, apply operator ${}^\rho \mathcal{D}_{s_k^+}^{\gamma}$ on both sides of (2.9) Then, from Lemma1.4.1 and Lemma1.4.6 we obtain

$$\left({}^\rho \mathcal{D}_{s_k^+}^{\gamma} u\right)(t) = \left({}^\rho \mathcal{D}_{s_k^+}^{\beta(1-\alpha)} \psi\right)(t) \tag{2.12}$$

Since $u \in C_{\gamma,\rho}^{\gamma}(I_k)$ and by definition of $C_{\gamma,\rho}^{\gamma}(I_k)$, we have ${}^\rho \mathcal{D}_{s_k^+}^{\gamma} u \in C_{\gamma,\rho}(I_k)$, and then (2.12) implies that

$$\left({}^\rho \mathcal{D}_{s_k^+}^{\gamma} u\right)(t) = \left(\delta_\rho {}^\rho \mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)} \psi\right)(t) = \left({}^\rho \mathcal{D}_{s_k^+}^{\beta(1-\alpha)} \psi\right)(t) \in C_{\gamma,\rho}(I_k) \tag{2.13}$$

As $\psi(\cdot) \in C\left(I_k, E\right)$ and from Lemma1.4.2, it follows that

$$\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}\left(I_k\right), k=0,\ldots,m \tag{2.14}$$

From (2.13. 2.14)a and by the definition of the space $C_{\gamma,\rho}^n\left(I_k\right)$, we obtain

$$\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}^1\left(I_k\right), k=0,\ldots,m$$

Applying operator ${}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)}$ on both sides of (2.12) and using Lemma1.4.5 . Lemma1.4.3 and the next property

property[24] The operator ${}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta}$ can be written as

$${}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta} = {}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)}\delta_\rho{}^\rho\mathcal{J}_{s_k^+}^{1-\gamma} = {}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)\rho}\mathcal{D}_{s_k^+}^{\gamma}, \quad \gamma=\alpha+\beta-\alpha\beta, k=0,\ldots,m$$

We have

$$\begin{aligned}\left({}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta}u\right)(t) &= {}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)}\left({}^\rho\mathcal{D}_{s_k^+}^{\gamma}u\right)(t)\\ = \psi(t) - \frac{\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right)(s_k)}{\Gamma(\beta(1-\alpha))}&\left(\frac{t^\rho-s_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1}\\ &= \psi(t),\end{aligned}$$

That is, 2.6 holds. Also, we have easily for $u \in C\left(\tilde{I}_k, E\right)$,

$$u(t) = g_k\left(t, u\left(t_k^-\right)\right), t \in \tilde{I}_k, k=1,\ldots,m$$

This completes the proof.

As a consequence of Theorem 2.2.1, we have the following result:

Lemma 2.2.2 *[24] Let* $\gamma=\alpha+\beta-\alpha\beta$ *where* $0<\alpha<1, 0\leq\beta\leq1$, *and* $k=0,\ldots,m$, *let* $f: I_k\times E\times E\to E$, *be a function such that* $f(\cdot,u(\cdot),w(\cdot)) \in C\left(I_k,E\right)$, *for any* $u,w\in PC_{\gamma,\rho}(J)$. *If* $u\in PC_{\gamma,\rho}(J)$, *then* u *satisfies the problem* (2.1) − (2.3) *if and only if* u *is the fixed point of the operator* $\Psi: PC_{\gamma,\rho}(J)\to PC_{\gamma,\rho}(J)$ *defined by*

$$\Psi u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho-s_k^\rho}{\rho}\right)^{\gamma-1} + \left(\rho\mathcal{J}_{s_k^+}^{\alpha}h\right)(t), \quad t\in I_k, k=0,\ldots,m\\ g_k(t,u(t)), \quad t\in\tilde{I}_k, k=1,\ldots,m\end{cases} \tag{2.15}$$

Where $h\in C\left(I_k,E\right), k=0,\ldots,m$ *is a function satisfying the functional equation*

$$h(t) = f(t,u(t),h(t))$$

Also, by Lemma 1.4.2, $\Psi u \in PC_{\gamma,\rho}(J)$.

The following hypotheses will be used in the sequel:

$(Ax1)$ *The function* $t \mapsto f(t,u,w)$ *is measurable on* $I_k, k = 0,\dots,m$, *for each* $u,w \in E$, *and the functions* $u \mapsto f(t,u,w)$ *and* $w \mapsto f(t,u,w)$ *are continuous on* E *for a.e.* $t \in I_k, k = 0,\dots,m$, *and*

$$f(\cdot,u(\cdot),w(\cdot)) \in C_{\gamma,\rho}^{\beta(1-\alpha)}(I_k) \text{ for any } u,w \in PC_{\gamma,\rho}(J)$$

$(Ax2)$ *There exists a continuous function* $p : [a,b] \longrightarrow [0,\infty)$ *such that* $\|f(t,u,w)\| \leq p(t)$, *for a.e.* $t \in I_k, k = 0,\dots,m$, *and for each* $u,w \in E$

$(Ax3)$ *For each bounded set* $B \subset E$ *and for each* $t \in I_k, k = 0,\dots,m$, *we have*

$$\mu\left(f\left(t,B,\left({}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}B\right)\right)\right) \leq p(t)\mu(B)$$
$$\text{where } {}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}B = \left\{{}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta}w : w \in B\right\}$$

$(Ax4)$ *The functions* $g_k \in C\left(\tilde{I}_k, E\right), k = 1,\dots,m$, *and there exists* $l^* > 0$ *such that*

$$\|g_k(t,u)\| \leq l^*\|u\| \text{ for each } u \in E, k = 1,\dots,m$$

$(Ax5)$ *For each bounded set* $B \subset E$ *and for each* $t \in \tilde{I}_k, k = 1,\dots,m$, *we have*

$$\mu\left(g_k(t,B)\right) \leq l^*\mu(B), k = 1,\dots,m$$
$$\text{Set } p^* = \sup_{t\in[a,b]} p(t)$$

We are now in a position to state and prove our existence result for the problem (2.1]-(2.3) based on Mönch's fixed point theorem.

Theorem 2.2.2 *[24] Assume (Ax1) (Ax5) hold. If*

$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - \alpha^\rho}{\rho}\right)^\alpha\right\} < 1 \tag{2.16}$$

then the problem $2.1 - 2.3$ *has at least one solution in* $PC_{\gamma,\rho}(J)$

proof: Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in 2.15 and the ball $B_R := B(0,R) = \left\{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\right\}$, such that

$$R \geq \frac{\phi^*}{(1-l^*)\Gamma(\gamma)} + \frac{p^*}{(1-l^*)\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}$$

For any $u \in B_R$, and each $t \in I_k, k = 0, \ldots, m$, we have

$$\|\Psi u(t)\| \leq \frac{\|\phi_k\|}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}_{s_k^+}^\alpha \|h(s)\|\right)(t)$$
$$\leq \frac{\phi^*}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|h(s)\|ds$$

By Lemma 1.4.1 we have

$$\|\Psi u(t)\| \leq \frac{\|\phi_k\|}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha}$$

Ther fore

$$\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \Psi u(t)\right\| \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}$$

And for $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\|(\Psi u)(t)\| \leq l^*\|u(t)\| \leq l^* R$$

Hence

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq l^* R + \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq R.$$

Ther fore

$$\frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq (1 - l^*)R.$$
$$\frac{\phi^*}{(1-l^*)\Gamma(\gamma)} + \frac{p^*}{(1-l^*)\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq R.$$

This proves that Ψ transforms the ball B_R into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theorem 1.5.5The rest of the proof will be given in several steps.

Step 1: $\Psi : B_R \to B_R$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k, k = 0, \ldots, m$, we have.

$$\left\|\left((\Psi u_n)(t) - (\Psi u)(t)\right) \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\right\| \leq \left(\frac{t^p - s_k^S}{\rho}\right)^{1-\gamma} \left(\rho\mathcal{J}_{s_k^+}^\alpha \|h_n(s) - h(s)\|\right)(t),$$

Where $h_n, h \in C(I_k, E); k = 0, \ldots, m$, such that

$$h_n(t) = f(t, u_n(t), h_n(t))$$

$$h(t) = f(t, u(t), h(t))$$

For each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have.

$$\|((\Psi u_n)(t) - (\Psi u)(t))\| \leq \|(g_k(t, u_n(t)) - g_k(t, u(t)))\|$$

Since $u_n \to u$ then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in (a, b]$, and since f and g_k are continuous. then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty$$

Step 2 : $\Psi(B_R)$ is bounded and equicontinuous.

Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.

Next. let $\epsilon_1, \epsilon_2 \in I_k, k = 0, \ldots, m, \epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\begin{aligned}
&\left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\| \\
\leq &\left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\rho\mathcal{J}_{s_k^+}^\alpha h(\tau)\right)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\rho\mathcal{J}_{s_k^+}^\alpha h(\tau)\right)(\epsilon_2)\right\| \\
\leq &\left(\frac{\epsilon_2^\rho - s_k^e}{\rho}\right)^{1-\gamma}\left(\rho\mathcal{J}_{\epsilon_1^+}^\alpha \|h(\tau)\|\right)(\epsilon_2) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{\epsilon_1}\left\|\tau^{\rho-1}H(\tau)h(\tau)\right\| d\tau
\end{aligned}$$

Where

$$H(\tau) = \left[\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_1^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\right]$$

Then by Lemma 1.4.1, we have

$$\begin{aligned}
&\left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\| \\
\leq &\frac{p^*}{\Gamma(1+\alpha)}\left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho}\right)^{\alpha} + p^*\int_{s_k}^{\epsilon_1}\left\|H(\tau)\frac{\tau^{\rho-1}}{\Gamma(\alpha)}\right\|\left(\frac{\tau^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} d\tau,
\end{aligned}$$

and for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\|(\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2)\| \leq \|(g_k(\epsilon_1, u(\epsilon_1))) - (g_k(\epsilon_2, u(\epsilon_2)))\|$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication of Theorem 1.5.5 holds.

Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ are continuous on J. By $(Ax3)$,$Ax5$) and the properties of the measure μ.

for each $t \in I_k, k=0,\ldots,m.$ we have

$$\begin{aligned}
\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \cup \{0\}\right) \\
&\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t)\right) \\
&\leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{s_k^+}^\alpha p(s)\mu(D(s))\right)(t) \\
&\leq p^*\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\rho^\rho\mathcal{J}_{s_k^+}^\alpha d(s)\right)(t) \\
&\leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right] \|d\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

And for each $t \in \tilde{I}_k, k=1,\ldots,m$, we have

$$d(t) \leq \mu\left(g_k(t, D(t))\right) \leq l^* d(t)$$

Thus for each $t \in (a,b]$, we have

$$\|d\|pC_{\gamma,\rho} \leq L\|d\|p_{C_{\gamma,p}}^{\cdot}$$

From (2.16) we get $\|d\|pC_{\gamma,\rho} = 0$, that is, $d(t) = \mu(D(t)) = 0$, for each $t \in (a,b]$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R Applying now Theorem 1.5.5 we conclude that Ψ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$. which is solution of the problem (2.1)-(2.3).

Step 4: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC_{\gamma,\rho}^\gamma(J)$. Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$. then for each $t \in J$, we have

$$\Psi u^*(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left(\rho\mathcal{J}_{s_k^+}^\alpha h\right)(t), & t \in I_k, k=0,\ldots,m, \\ g_k\left(t, u^*(t)\right), \quad t \in \tilde{I}_k, k=1,\ldots,m. \end{cases}$$

Where $h \in C\left(I_k, E\right); k=0,\ldots,m$, such that

$$h(t) = f\left(t, u^*(t), h(t)\right)$$

For $t \in I_k; k=0,\ldots,m$, applying $\rho\mathcal{D}_{s_k^+}^\gamma$ to both sides and by Lemma 1.4.1 and Lemma 1.4.6, we have

$$\begin{aligned}
\rho\mathcal{D}_{s_k^+}^\gamma u^*(t) &= \left(\rho\mathcal{D}_{s_k^+}^\gamma \rho\mathcal{J}_{s_k^+}^\alpha f\left(s, u^*(s), h(s)\right)\right)(t) \\
&= \left(\rho\mathcal{D}_{s_k^+}^{\beta(1-\alpha)} f\left(s, u^*(s), h(s)\right)\right)(t)
\end{aligned}$$

Since $\gamma \geq \alpha$, by $(Ax1)$. the right hand side is in $C_{\gamma,\rho}(I_k)$ and thus $\rho\mathcal{D}^{\gamma}_{s_k^+}u^* \in C_{\gamma,\rho}(I_k)$ which implies that $u^* \in C^{\gamma}_{\gamma,\rho}(I_k)$. And since $g_k \in C\left(\tilde{I}_k, E\right); k = 1, \ldots, m$, then $u^* \in PC^{\gamma}_{\gamma,\rho}(J)$. As a consequence of Steps 1 to 4 together with Theorem (3.2.1), we can conclude that the problem (2.1) (2.3) has at least one solution in $PC_{\gamma,\rho}(J)$

Our second existence result for the problem (2.1)-(2.3) is based on Darbo's fixed point theorem.

Theorem 2.2.3 *[24] Assume (Ax1) (Ax5) hold. If*

$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - \alpha^\rho}{\rho}\right)^{\alpha}\right\} < 1. \tag{2.17}$$

Ther the problem (2.1)-(2.3) at least one solution in $PC_{\gamma,\rho}(J)$.

Proof: Consider the operator Ψ defined in (2.15). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuons and that $\Psi(B_R)$ is equicontinuous. We need to prove that the operator Ψ is an $L-$ contraction. Let $D \subset B_R$ and $t \in I_k, k = 0, \ldots, m$. Then we have

$$\begin{aligned}
&\mu\left(\left(\frac{t^\rho - s_k^p}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) = \mu\left(\left(\frac{t^p - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t) : u \in D\right) \\
&\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left\{\left(\rho\mathcal{J}^{\alpha}_{s_k^+}p^*\mu(u(s))\right)(t), u \in D\right\}
\end{aligned}$$

By Lemma 1.4.1 we have for $t \in I_k, k = 0, \ldots, m$.

$$\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) \leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right]\mu_{PC_{\gamma,\rho}}(D)$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\mu((\Psi D)(t)) \leq \mu\left(g_k(t, D(t))\right) \leq l^*\mu(D(t))$$

Hence. for each $t \in (a, b]$, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq L\mu_{PC_{\gamma,\rho}}(D)$$

So. by (2.16) the operator Ψ is an L contraction. As consequence of Theorem (1.5.6) and using Step 4 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (2.1)-(2.3).

Chapter 3

Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations whit the lipschitz condition on g

3.1 Introduction

We have to change the continuous condition in g by lipschitz condition $(A'x4)$

3.2 Existence of solutions

We consider the following linear fractional differential equation studied in chapter two given by :

$$\left({}^{\rho}\mathcal{D}_{s_k^+}^{\alpha,\beta} u\right)(t)=\psi(t), \quad t \in I_k, k=0, \ldots, m \tag{3.1}$$

Where $0<\alpha<1,0 \leq \beta \leq 1, \rho>0$, with the conditions

$$u(t)=g_k(t, u(t)), t \in \tilde{I}_k, k=1, \ldots, m \tag{3.2}$$

$$\left({}^{\rho}\mathcal{J}_{a+}^{1-\gamma} u\right)\left(a^{+}\right)=\phi_0 \tag{3.3}$$

Where $\gamma=\alpha+\beta-\alpha\beta$ and $\phi_0 \in E$, and let $\phi^*=\max\left\{\left\|\phi_k\right\|: k=0, \ldots, m\right\}$. but we consider The following hypotheses will be used in the sequel:

$(Ax3)$ For each bounded set $B \subset E$ and for each $t \in I_k, k = 0, \ldots, m$, we have

$$\mu\left(f\left(t, B, \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}B\right)\right)\right) \leq p(t)\mu(B)$$
$$\text{where } {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}B = \left\{{}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}w : w \in B\right\}$$

$(A'x4)$ The functions $g_k \in C\left(\tilde{I}_k, E\right), k = 1, \ldots, m$, and there exists $l^* > 0$ such that

$$\|g_k(t,u) - g_k(t,v)\| \leq l^*\|u - v\| \text{ for each } u, v \in E, k = 1, \ldots, m$$

$(Ax5)$ For each bounded set $B \subset E$ and for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\mu\left(g_k(t,B)\right) \leq l^*\mu(B), k = 1, \ldots, m$$
$$\text{Set } p^* = \sup_{t\in[a,b]} p(t)$$

We are now in a position to give our maine result for the problem (3.1]-(3.3) based on Mönch's fixed point theorem.

Theorem 3.2.1 *[24] Assume (Ax1)-(Ax3),$(A'x4)$and(Ax5) hold. If*

$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - \alpha^\rho}{\rho}\right)^\alpha\right\} < 1 \tag{3.4}$$

then the problem $3.1 - 3.3$ *has at least one solution in* $PC_{\gamma,\rho}(J)$.

proof: Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in 2.15 and the ball $B_R := B(0,R) = \left\{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\right\}$, such that

$$R \geq \frac{\phi^*}{(1-l^*)\,\Gamma(\gamma)} + \frac{p^*}{(1-l^*)\,\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{M}{1-l^*}$$
$$M = \|g_k(t,0)\|$$

For any $u \in B_R$, and each $t \in I_k, k = 0, \ldots, m$, we have

$$\|\Psi u(t)\| \leq \frac{\|\phi_k\|}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\|h(s)\|\right)(t) \leq \frac{\phi^*}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + p^*\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}(1)\right)(t)$$

By Lemma 1.4.1 we have

$$\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\Psi u(t)\right\| \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}$$

And for $t \in \tilde{I}_k, k=1,\ldots,m$, we have

$$\|(\Psi u)(t)\| \leq \|(\Psi u)(t) - \|(\Psi u)(0)\| + \|(\Psi u)(0)\| \leq l^* \|u\| + M \leq l^* R + M$$

$$M = \|g_k(t,0)\|$$

Hence

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq l^* R + \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} + M \leq R.$$
$$M = \|g_k(t,0)\|$$

This proves that Ψ transforms the ball B_R into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theorem 1.5.5The rest of the proof will be given in several steps.

Step 1: $\Psi : B_R \to B_R$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k, k=0,\ldots,m$, we have.

$$\left\|\left((\Psi u_n)(t) - (\Psi u)(t)\right)\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\right\| \leq \left(\frac{t^p - s_k^S}{\rho}\right)^{1-\gamma}\left(\rho \mathcal{J}_{s_k^+}^\alpha \|h_n(s) - h(s)\|\right)(t),$$

Where $h_n, h \in C(I_k, E); k=0,\ldots,m$, such that

$$h_n(t) = f(t, u_n(t), h_n(t))$$

$$h(t) = f(t, u(t), h(t))$$

For each $t \in \tilde{I}_k, k=1,\ldots,m$, we have.

$$\|((\Psi u_n)(t) - (\Psi u)(t))\| \leq \|(g_k(t, u_n(t)) - g_k(t, u(t)))\| \leq l^* \|u_n - u\|$$

Since $u_n \to u$ then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in (a,b]$, and since f and g_k are continuous. then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,p}} \to 0 \text{ as } n \to \infty$$

Step 2 : $\Psi(B_R)$ is bounded and equicontinuous.

Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.

Next. let $\epsilon_1, \epsilon_2 \in I_k, k=0,\ldots,m, \epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\left\|\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\|$$

$$\leq \left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\epsilon_1} s^\rho \left(\frac{\epsilon_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} h(s)ds, - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\epsilon_2} s^\rho \left(\frac{\epsilon_2^\rho - s^\rho}{\rho}\right)^{1-\gamma} h(s)ds, \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\epsilon_1} s^\rho \left[\left(\frac{\epsilon_1^\rho - s^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right] \|h(s)ds\|$$

$$+ \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_{\epsilon_1}^{\epsilon_2} s^\rho \left(\frac{\epsilon_2 - s^\rho}{\rho}\right)^{\alpha-1} \|h(s)ds\|$$

$$\leq \left(\frac{\epsilon_2^\rho - s_k^e}{\rho}\right)^{1-\gamma} \left(\rho \mathcal{J}_{\epsilon_1^+}^\alpha \|h(\tau)\|\right)(\epsilon_2) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\epsilon_1} \left\|\tau^{\rho-1} H(\tau) h(\tau)\right\| d\tau$$

Where

$$H(\tau) = \left[\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_1^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_2^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \right]$$

Then by Lemma 1.4.1, we have

$$\left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_2) \right\|$$

$$\leq \frac{p^*}{\Gamma(1+\alpha)} \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho}\right)^{\alpha} + p^* \int_{s_k}^{\epsilon_1} \left\| H(\tau) \frac{\tau^{\rho-1}}{\Gamma(\alpha)} \right\| \left(\frac{\tau^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} d\tau,$$

and for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\|(\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2)\| \leq \|(g_k(\epsilon_1, u(\epsilon_1))) - (g_k(\epsilon_2, u(\epsilon_2)))\|$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication of Theoren 1.5.5 holds.

Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ are continuous on J. By $(Ax3)$,$Ax5$) and the properties of the measure μ. for each $t \in I_k, k = 0, \ldots, m.$ we have

$$\begin{aligned}
\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \cup \{0\}\right) \\
&\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t)\right) \\
&\leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}_{s_k^+}^\alpha p(s)\mu(D(s))\right)(t) \\
&\leq p^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left(\rho^\rho \mathcal{J}_{s_k^+}^\alpha d(s)\right)(t) \\
&\leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right] \|d\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$d(t) \leq \mu\left(g_k(t, D(t))\right) \; \leq l^* d(t)$$

Thus for each $t \in (a, b]$, we have

$$\|d\|pC_{\gamma,\rho} \leq L\|d\|p_{C_{\gamma,p}}$$

From (3.7) we get $\|d\|pC_{\gamma,\rho} = 0$, that is, $d(t) = \mu(D(t)) = 0$, for each $t \in (a, b]$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R Applying now Theorem 1.5.5 we conclude that Ψ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$. which is solution of the problem (3.1)-(3.3).

Step 4: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC^{\gamma}_{\gamma,\rho}(J)$. Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$. then for each $t \in J$, we have

$$\Psi u^*(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left(\rho\mathcal{J}^{\alpha}_{s_k^+} h\right)(t), & t \in I_k, k = 0, \ldots, m, \\ g_k\left(t, u^*(t)\right), & t \in \tilde{I}_k, k = 1, \ldots, m. \end{cases}$$

Where $h \in C\left(I_k, E\right); k = 0, \ldots, m$, such that

$$h(t) = f\left(t, u^*(t), h(t)\right)$$

For $t \in I_k; k = 0, \ldots, m$, applying $\rho\mathcal{D}^{\gamma}_{s_k^+}$ to both sides and by Lemma 1.4.1 and Lemma 1.4.6, we have

$$\begin{aligned} \rho\mathcal{D}^{\gamma}_{s_k^+} u^*(t) &= \left(\rho\mathcal{D}^{\gamma}_{s_k^+} \rho\mathcal{J}^{\alpha}_{s_k^+} f\left(s, u^*(s), h(s)\right)\right)(t) \\ &= \left(\rho\mathcal{D}^{\beta(1-\alpha)}_{s_k^+} f\left(s, u^*(s), h(s)\right)\right)(t) \end{aligned}$$

Since $\gamma \geq \alpha$, by $(Ax1)$. the right hand side is in $C_{\gamma,\rho}\left(I_k\right)$ and thus $\rho\mathcal{D}^{\gamma}_{s_k^+} u^* \in C_{\gamma,\rho}\left(I_k\right)$ which implies that $u^* \in C^{\gamma}_{\gamma,\rho}\left(I_k\right)$. And since $g_k \in C\left(\tilde{I}_k, E\right); k = 1, \ldots, m$, then $u^* \in PC^{\gamma}_{\gamma,\rho}(J)$. As a consequence of Steps 1 to 4 together with Theorem (2.2.1), we can conclude that the problem (3.1) (3.3) has at least one solution in $PC_{\gamma,\rho}(J)$ Our second existence result for the problem (3.1)-(3.3) is based on Darbo's fixed point theorem.

Theorem 3.2.2 *[24] Assume (Ax1)-(Ax3),$(A'x4)$and (Ax5) hold. If*

$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right\} < 1$$

ther the problem (3.1)-(3.3) at least one solution in $PC_{\gamma,\rho}(J)$.

Proof: Consider the operator Ψ defined in (2.15). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuons and that $\Psi(B_R)$ is equicontinuous. We need to prove that the operator Ψ is an $L-$ contraction. Let $D \subset B_R$ and $t \in I_k, k = 0, \ldots, m$. Then we have

$$\mu\left(\left(\frac{t^\rho - s_k^p}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) = \mu\left(\left(\frac{t^p - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t) : u \in D\right)$$
$$\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left\{\left(\rho\mathcal{J}_{s_k^+}^\alpha p^*\mu(u(s))\right)(t), u \in D\right\}$$

By Lemma 1.4.1 we have for $t \in I_k, k = 0, \ldots, m$.

$$\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right) \leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right]\mu_{PC_{\gamma,\rho}}(D)$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\mu((\Psi D)(t)) \leq \mu\left(g_k(t, D(t))\right) \leq l^*\mu(D(t))$$

Hence. for each $t \in (a, b]$, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq L\mu_{PC_{\gamma,\rho}}(D)$$

So. by (3.7) the operator Ψ is an L contraction. As consequence of Theorem (1.5.6) and using Step 4 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (3.1)-(3.3)

3.3 Example

Let

$$E = l^1 = \left\{v = (v_1, v_2, \ldots, v_n, \ldots), \sum_{n=1}^\infty |v_n| < \infty\right\}$$

Be the Banach space with the norm

$$\|v\| = \sum_{n=1}^\infty |v_n|.$$

Consider the following initial value problem with non-instantaneous impulses

$$\left({}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}u\right)(t) = f\left(t, u(t), \left({}^1\mathcal{D}_{s_k^+}^{\frac{1}{2},0}u\right)(t)\right), t \in (1,2] \cup (e,3], k \in \{0,1\} \tag{3.5}$$

$$u(t) = g(t, u(t)), t \in (2, e] \tag{3.6}$$

$$\left({}^{1}\mathcal{J}_{1^{+}}^{\frac{1}{2}}u\right)(1^{+})=0 \tag{3.7}$$

Where

$$a=t_0=s_0=1<t_1=2<s_1=e<t_2=3=b,$$

$$u=(u_1,u_2,\ldots,u_n,\ldots)$$

$$f=(f_1,f_2,\ldots,f_n,\ldots)$$

$${}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u=\left({}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u_1,\ldots,{}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u_2,\ldots,{}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u_n,\ldots\right),$$

$$g=(g_1,g_2,\ldots,g_n,\ldots),$$

$$f_n\left(t,u_n(t),\left({}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u_n\right)(t)\right)=\frac{\left(2t^3+5e^{-2}\right)|u_n(t)|}{183e^{-t+3}\left(1+\|u(t)\|+\left\|\left({}^{1}\mathcal{D}_{s_k^{+}}^{\frac{1}{2},0}u\right)(t)\right\|\right)}$$

with $t\in(1,2]\cup(e,3], k\in\{0,1\}, n\in\mathbb{N},$ and

$$g_n(t,u_n(t))=\frac{|\sin u_n(t)|}{105e^{-t+5}+1}, t\in(2,e], n\in\mathbb{N}$$

We have

$$C_{\gamma,\rho}^{\beta(1-\alpha)}((1,2])=C_{\frac{1}{2},1}^{0}((1,2])=\{h:(1,2]\to E:(\sqrt{t-1})h\in C([1,2],E)\},$$

and

$$C_{\gamma,\rho}^{\beta(1-\alpha)}((e,3])=C_{\frac{1}{2},1}^{0}((e,3])=\{h:(e,3]\to E:(\sqrt{t-e})h\in C([e,3],E)\}$$

With $\gamma=\alpha=\frac{1}{2},\rho=1,\beta=0$ and $k\in\{0,1\}$. Clearly, the continuous function $f\in C_{\frac{1}{2},1}^{0}((1,2])\cup C_{\frac{1}{2},1}^{0}((e,3])$. Hence the condition $(Ax1)$ is satisfied. For each $u,w\in E$ and $t\in(1,2]\cup(e,3]$,

$$\|f(t,u,w)\|\leq\frac{2t^3+5e^{-2}}{183e^{-t+3}}.$$

Hence condition $(Ax2)$ is satisfied with

$$p(t)=\frac{2t^3+5e^{-2}}{183e^{-t+3}},$$

and

$$p^{*}=\frac{54+5e^{-2}}{183}.$$

And for each $u\in E$ and $t\in(2,e]$ we have

$$\|g(t,u)-g(t,v)\|\leq\frac{\|u-v\|}{105e^{5-e}+1}.$$

And so the condition $(A'x4)$ is satisfied with $l^{*}=\frac{1}{105e^{5-e}+1}$ The condition(3.4) of Theorem (3.2.1) is satisfied, for

$$L:=\max\left\{l^{*},\frac{p^{*}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right\}\approx 0.7489295248<1.$$

Let Ω be a bounded set in E where ${}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}\Omega = \left\{ {}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+} v : v \in \Omega \right\}, k \in \{0,1\}$. Then by the properties of the Kuratowski measure of noncompactness, for each $u \in \Omega$ and $t \in (1,2] \cup (e,3]$, we have

$$\mu\left(f\left(t, \Omega, {}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}\Omega\right)\right) \leq p(t)\mu(\Omega)$$

and for each $t \in (2,e]$,

$$\mu(g(t,\Omega)) \leq l^*\mu(\Omega)$$

Hence conditions $(Ax3)$ and $(Ax5)$ are satisfied. Then the problem $(3.5)-(3.7)$ has at least one solution in $PC_{\frac{1}{2},1}([1,3])$

CONCLUSION

We have studied the existence of solutions for a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces. We have change the continuous condition in g by lipschitz condition. our main results are based on Darbo and Mönch fixed point theorems associated with the technique of measure of noncompactness.

In perspective, we will project these results in other fractional differential problems.

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