الجمهورية الجزائرية الديمقراطية الشعبية
Democratic and Popular Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research

N° d'enregistrement
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جامعة غـرداية
Ghardaïa University

كلية العلوم والتكنولوجيا
Faculty of Science and Technology

قسم الرياضيات والإعلام الآلي
Department of Mathematics and Computer Science

Thesis Submitted in Partial Fulfilment of the Requirements for the

Master's Degree

Mathematics and Computer Science Field, Mathematics Stream
Functional Analysis Specialty
Theme

Integration of Differential Forms on Manifolds With Locally-Finite Variations

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Publicly supported the 23/06/2025

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Year 2024/2025

الملخص بالعربية

نظرًا لأهمية التكامل في حساب المسافات، فإن هناك حاجة دائمة إلى تطوير هذا المفهوم، خاصةً بسبب تنوّع وتعقيد الأشكال في حياتنا. اعتمد علماء الهندسة على توسيع مفهوم التكامل ليشمل الأشكال المعقّدة. وقد ظهرت أول مشكلة عندما لم يعد تكامل ريمان وليبزغ صالحًا للتطبيق على الأسطح غير المستوية.

تدخّل علماء الهندسة التفاضلية، ومن أبرزهم السير جورج ستوكس، الذي قدّم صيغة تُعرف الآن بمبرهنة ستوكس، والتي مثّلت حلاً جزئيًا للمشكلة عبر البرهنة على علاقة بين التكامل على الفضاء وحدوده. لكن هذا لم يكن كافيًا لحل جميع الإشكالات.

ولهذا، اتّخذ العالم أليكسي بوتيبون مسارًا مختلفًا، حيث قام بتغيير قواعد التكامل على المستعمَلات، وذلك لملاءمة الحالات ذات التغيّر المحلي المحدود. كان هذا التحدي كبيرًا، لكن نتائجه سمحت بفتح آفاق جديدة لحساب التكاملات على أنواع أعقد من الكائنات الهندسية.

هيا نكتشف معًا ما الذي حدث!

الكلمات المفتاحية : الأشكال التفاضلية، متعددات الشعب ذات التغير المحدود محليًا، التكامل على متعددات الشعب، التباين الهندسي، نظرية الدرجات الطوبولوجية، التغير المحدود محليًا.

Abstract in English

Given the fundamental role of integration in measuring distances, there has always been a need to develop this concept, especially due to the diversity and complexity of shapes in the real world. Geometers have historically sought to extend integration to more complex geometric objects.

The first major obstacle arose when Riemann and Lebesgue integrals failed to apply on non-flat surfaces. This prompted the intervention of differential geometers, most notably George Stokes, who formulated the famous Stokes's theorema partial resolution connecting integration over a manifold with its boundary.

However, this did not fully solve the issue. Therefore, Alexey V. Potepun approached the problem differently by modifying the foundations of integration on manifolds, adapting it to the framework of locally-finite variation. This was a major challenge, but it allowed for powerful new generalizations of the integral.

Let us now explore what exactly happened.

Keywords : Differential Forms, Manifolds with Locally-Finite Variations, Integration on Manifolds, Geometric Measure Theory, Topological Degree Theory, Locally-Finite Variation.

Résumé en Français

Étant donné l'importance du concept d'intégration dans le calcul des distances, il est toujours nécessaire de développer cette notion, surtout à cause de la diversité et de la complexité des formes dans la vie réelle. Les géomètres ont donc travaillé à généraliser l'intégrale aux objets géométriques plus complexes.

Le premier problème est apparu lorsque les intégrales de Riemann et de Lebesgue ont montré leurs limites sur des surfaces non planes. Cela a conduit à l'intervention des géomètres différentiels, notamment George Stokes, qui a formulé le célèbre théorème de Stokes, apportant une solution partielle au problème.

Cependant, ce n'était pas une solution complète. Ainsi, Alexey V. Potepun a emprunté une autre voie en modifiant les règles d'intégration sur les variétés, en les adaptant au cadre de la variation localement finie. Ce fut un défi majeur, mais ses travaux ont ouvert la voie à de nouvelles extensions puissantes de l'intégrale.

Découvrons ensemble ce qui s'est passé !

Mots-clés : Formes différentielles, Variétés à variation localement finie, Intégration sur les variétés, Théorie de la mesure géométrique, Degré topologique, Variation localement finie.

Dedication

To our respected professors and doctors in Department of Mathematics and Computer Science at the University of Ghardaia, I dedicate this work with deep gratitude. Your dedication, patience, and passion for teaching have shaped not only our knowledge but also our character. You taught us how to dive into the depth of mathematics a vast and complex world and inspired us to keep going, even when understanding seemed far away.

To my dear friend and study companion, **Noussiba Mellakh**, thank you for being more than a friend a true sister and supporter through every challenge. Your presence made this journey brighter.

To **Dr. Latreche Ismail**, who was not just a teacher, but a father figure and role model throughout my academic path. Thank you for your unwavering support, guidance, and kindness that helped me grow both academically and personally.

To **Dr. Chikh Salah Abdelouahab**, for giving me the opportunity to experience a research journey similar to preparing a doctoral thesis. I am deeply honored by your mentorship and grateful for all the academic skills and research methodologies you shared with me.

To my beloved mother, **Khiar Chahrazed**, for your endless love, trust, and encouragement that gave me strength in every step.

To my father, **Rahmani Larbi**, for instilling in me the value of education and for supporting every decision I made with confidence and wisdom.

And finally, I dedicate this work to **myself**, for never giving up, for every effort made, and for every sacrifice endured to reach this moment.

Contents

Introduction

The classical theory of integration of differential forms is well-established for smooth manifolds, where integration is defined using local parametrizations and partitions of unity. This theory can be extended to piecewise-smooth manifolds and even to rectifiable curves in one dimension. However, when moving beyond one dimension or relaxing smoothness conditions, traditional approaches such as the use of currents or Lipschitz parametrizations may fail to provide a robust integration framework.

This work introduces a novel class of geometric objects: n-dimensional manifolds embedded in $\mathbb{R}^m$ with locally-finite variations . These manifolds are not necessarily smooth or Lipschitz-regular, but they support a vector-valued measure that encodes both orientation and geometry. This measure allows for the definition of an integral of differential forms in a way that retains key properties of the Lebesgue integral including countable additivity, measurability of forms, and convergence theorems.

A crucial feature of this approach is that it goes beyond the class of rectifiable manifolds. While every one-dimensional manifold with locally-finite variation corresponds to a rectifiable curve (and hence admits a natural parametrization), in higher dimensions there exist manifolds with locally-finite variation that do not admit Lipschitz parametrizations . Thus, these objects lie outside the scope of classical geometric measure theory and the theory of rectifiable currents.

The main result of the first part of the thesis establishes that every continuous differential form is integrable over any compact subset of an orientable manifold with locally-finite variations , and that the integral is finite. This confirms that such manifolds represent a broad and natural class for which a well-behaved integration theory exists.

The author conjectures that manifolds with locally-finite variation form the most general class of manifolds for which integration of differential forms can be consistently defined while preserving essential analytical properties such as limit theorems and the integrability of continuous forms.

1

Preliminaries

The references for this chapter are [**25**, **45**, **14**, **42**, **33**, **50**, **36**, **41.**]

1.0.1 σ-algebra

Definition 1.0.1 (Algebra)**.** *Let Ω be a non-empty set. An **algebra** (or field) of subsets of Ω is a collection $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ such that:*

1. $\Omega \in \mathcal{F}$,
2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closed under complementation),*
3. *If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$ (closed under finite unions).*

Definition 1.0.2 (σ-Algebra)**.** *Let Ω be a non-empty set. A σ**-algebra** (or σ-field) of subsets of Ω is a collection $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ such that:*

1. $\Omega \in \mathcal{F}$,
2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closed under complementation),*
3. *If $A_1, A_2, A_3, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (closed under countable unions).*

Remark 1.0.1. *Every σ-algebra is an algebra, but not every algebra is a σ-algebra. The main distinction lies in the closure property:*
*- An algebra is closed only under **finite unions**.*
*- A σ-algebra is closed under **countable unions**, making it suitable for defining measures and probabilities on infinite sample spaces.*

Theorem 1.0.1. *Fubini's theorem*
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, and let

$$f : X \times Y \to \mathbb{R}$$

be a measurable function such that

$$\int_{X\times Y} |f(x, y)|\, d(\mu \times \nu)(x, y) < \infty.$$

Then, the iterated integrals exist and satisfy

$$\int_{X\times Y} f(x, y)\, d(\mu \times \nu)(x, y) = \int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y)\, d\mu(x) \right) d\nu(y).$$

Intuition: *This theorem allows you to compute the integral over the product space $X \times Y$ by iteratively integrating with respect to each variable. The main requirement is that the function f is absolutely integrable with respect to the product measure $\mu \times \nu$.*

Definition 1.0.3. *Equivalence Class: Let $\sim$ be an equivalence relation on a set A. For any element $a \in A$, the* ***equivalence class*** *of a is defined as:*

$$[a] = \{x \in A \mid x \sim a\}$$

That is, the set of all elements in A that are related to a under $\sim$.

A relation R on a set A is **reflexive** if:

$$\forall a \in A, \quad a\ R\ a$$

That is, every element is related to itself.

A relation R on a set A is **symmetric** if:

$$\forall a, b \in A, \quad a\ R\ b \implies b\ R\ a$$

This means if a is related to b, then b is also related to a.

A relation R on a set A is **transitive** if:

$$\forall a, b, c \in A, \quad (a\ R\ b \text{ and } b\ R\ c) \implies a\ R\ c$$

This means if a is related to b, and b is related to c, then a must be related to c.

Partition of unity:

In order to generalize the notion of integration to n-forms on an arbitrary manifold M, we will need the concept of a *partition of unity*.

Definition 1.0.4. *Let G be open subset of $\mathbb{R}^n$. The pre image of a point $y \in \mathbb{R}^n$ under the function $f : G \to \mathbb{R}^n$ is the set of all points $x \in G$ such that $f(x) = y$*

$$f^{-1}(y) = \{x \in G \quad | \quad f(x) = y\}$$

Definition 1.0.5. *Let the subset $E \subset G$. We take the pre-image of y under f and then intesection it with the subset E we get :*

$$f^{-1}(y) \cap E = \{x \in E \quad | \quad f(x) = y\}$$

Definition 1.0.6. *The diameter of a subset $A \subset \mathbb{R}^n$ is defined as*

$$diam(A) = \sup_{x,y \in A} \rho(x, y) \quad \text{where } \rho \text{ the metric distance in } \mathbb{R}^n$$

Definition 1.0.7. ***Summable function***

A measurable function $g : \Omega \to \mathbb{R}$ where (Ω, A, μ) is a measure space is said to be summable if the Lebesgue integral of the absolute value of g exists and is finite:

$$\int_\Omega |g|\, d\mu < +\infty.$$

An alternative way of expressing this condition is to assert that $g \in L^1(\Omega)$.

Remark 1.0.2. *Note that some authors distinguish between integrable and summable: an integrable function is one for which the above integral exists; a summable function is one for which the integral exists and is finite. see([**42**])*

1.0.2 Continuity theorem (Paul Lévy)

Characteristic functions and weak convergence

Definition 1.0.8. *The **characteristic function** of a random variable X is defined by:*

$$\varphi(t) = \mathbb{E}(e^{itX})$$

Understanding the expression $\mathbb{E}(e^{itX_n})$ this let's break down what each part means:

- $\mathbb{E}$ *: is the **expected value** (or average) of a random quantity. It tells us what value we would expect to see ``on average" if we observed the random variable many times.*
- e *: The base of the natural logarithm , used here in an exponential function.*
- i *: The imaginary unit, defined by $i^2 = -1$. It appears in complex numbers.*
- t*: A real number parameter (can be any real value). It's the input to the characteristic function.*
- X_n *: A random variable, typically one element in a sequence $X_1, X_2, \ldots$.It might represent the result of an experiment that changes each time.*
- *This is a complex exponential expression. Using Euler's formula:*

$$e^{itX_n} = \cos(tX_n) + i\sin(tX_n)$$

So, e^{itX_n} is a complex number that lies on the unit circle and depends on the value of tX_n.

It uniquely determines the distribution of X. That is, two random variables with the same characteristic function have the same distribution.

Proposition 1.0.1. *Let $X = (X_1, X_2, \ldots, X_d)$ be a random vector in $\mathbb{R}^d$. Then its characteristic function is:*

$$\varphi(t) = \mathbb{E}(e^{it\cdot X}) = \mathbb{E}\left(e^{i\sum_{k=1}^{d} t_k X_k}\right) \quad \text{for } t = (t_1, \ldots, t_d) \in \mathbb{R}^d$$

This function also uniquely determines the joint distribution of X.

Proposition 1.0.2 (Cramér-Wold device)**.** *To determine the distribution of a random vector X, it suffices to know the distributions of all linear combinations:*

$$a_1X_1 + \cdots + a_dX_d \quad \text{for all } a \in \mathbb{R}^d.$$

Hence, the characteristic function $\varphi(t)$ for all $t \in \mathbb{R}^d$ determines the distribution of X.

Theorem 1.0.2 (Continuity theorem (Paul Lévy))**.** *Let (X_n) be a sequence of real-valued random variables. Suppose:*

$$\mathbb{E}(e^{itX_n}) \longrightarrow \varphi(t) \quad \text{for all } t \in \mathbb{R}$$

Then the following are equivalent:

1. *(X_n) is tight, i.e.,*

$$\lim_{x\to\infty} \sup_n P(|X_n| > x) = 0.$$

*The symbol P stands for the **probability measure**. It is shorthand for:*

The probability that the random variable $|X_n|$ exceeds the value x.

Mathematically, this is written as:

$$P(|X_n| > x)$$

2. *Weak convergence:* $X_n \xrightarrow{d} X$ *for some random variable* $X \in \mathbb{R}^n$.

3. φ *is a characteristic function of some* $X \in \mathbb{R}^n$*, i.e.,*

$$\varphi(t) = \mathbb{E}(e^{itX}).$$

4. φ *is a continuous function of* t.

5. φ *is continuous at* $t = 0$.

If all the conditions (1)-(5) hold, then $X_n \xrightarrow{d} X$ *for* X *as in (3).*

For more details, see ([**14**, **41**])

Whitneys approximation theorem

For any continuous function f defined on smooth manifold M , there exists , for any $\varepsilon > 0$ such that

$$\sup_{x\in M} |f(x) - g(x)| < \varepsilon \quad \text{for all } p \in M.$$

This theorem guarantees that continous function can be approximated arbitrarily closely by differentiable function

Topological reminder

As it is well known, the continuity of a map can be characterised by the concept of open sets. A map φ is continuous if and only if the preimage $\varphi^{-1}(G)$ is open for each open subset G. A map φ on a manifold maps to $\mathbb{R}^n$. There, the usual notion of distance defines what the open subsets are. In this section we will choose the open subsets of a manifold so that the maps are continuous.

Definition 1.0.9. *A* topological space $[M, \mathcal{G}]$ *is a set* M *together with a collection* $\mathcal{G}$ *of open subsets of* M *such that:*

(G1) $M, \emptyset \in \mathcal{G}$,

(G2) For $\{G_i\} \subset \mathcal{G}$*, we have* $\bigcup_i G_i \in \mathcal{G}$,

(G3) For $G_1, \ldots, G_m \in \mathcal{G}$*, we have* $G_1 \cap \cdots \cap G_m \in \mathcal{G}$.

The collection $\mathcal{G}$ *is called a* topology*, and its members are called* open sets.

The concept of topology enables one to define the notion of convergence:

- A sequence of elements $P_1, P_2, \ldots$ in a topological space M converges to an element $P \in M$ if for each $G \in \mathcal{G}$ such that $P \in G$, there exists an index m_0 such that for every natural number $m > m_0$, $P_m \in G$.

The uniqueness of the limit is enforced by a so-called separation axiom:

Definition 1.0.10. *A topological space $[M, \mathcal{G}]$ is called a* Hausdorff space *if for every two distinct $P, Q \in M$, there exist open sets $G, H \in \mathcal{G}$ such that $P \in G$ and $Q \in H$ with $G \cap H = \emptyset$.*

- For two different limiting values P and Q of the same sequence in a Hausdorff space, we could choose disjoint open sets G and H with $P \in G$ and $Q \in H$, and then for large m, all P_m would have to lie in both G and H, a contradiction, since $G \cap H = \emptyset$.
- An atlas creates a topology on the base set M with the goal of making the maps homeomorphisms, that is, they must be continuous in both directions.

Definition 1.0.11. *A subset G of a set M equipped with an atlas is called* open *if for each chart (U, φ), the subset*

$$\varphi(U \cap G) = \{\varphi(P) : P \in U \wedge P \in G\}$$

of $\mathbb{R}^n$ is open.

Definition 1.0.12. *Let M, N be two topological spaces. A continuous map $f : M \to N$ is a **homeomorphism** if, in addition, it is invertible and its inverse $f^{-1} : N \to M$ is also continuous.*

Definition 1.0.13. *A map f from an open set $U \subset \mathbb{R}^n$ into an open set $V \subset \mathbb{R}^n$ is a **diffeomorphism** if:*

- *f is bijective,*
- *f is differentiable on U,*
- *f^{-1} is differentiable on V.*

Remark 1.0.3. *A map f is called a **diffeomorphism of class** C^k, $k > 0$, if f is differentiable of class C^k, and there exists $g : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$, differentiable of class C^k, such that $g \circ f = \mathrm{Id}_{\mathbb{R}^n}$ and $f \circ g = \mathrm{Id}_{\mathbb{R}^n}$. We write $g = f^{-1}$. In the case $k = 0$, we say that f is a homeomorphism.*

Definition 1.0.14. *An **immersion of class** C^k from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is a map $f : U \to \mathbb{R}^n$, such that for all $p \in U$, its differential at p, Df_p, is injective.*

*A **submersion of class** C^k from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is a map $f : U \to \mathbb{R}^n$ of class C^k, such that for all $p \in U$, its differential at p, Df_p, is surjective.*

Theorem 1.0.3 (Local inversion theorem)**.** *Let f be a map from an open set U of $\mathbb{R}^n$ into an open set V of $\mathbb{R}^n$ of class C^k, $k \geq 1$, and let $a \in U$ such that df_a is invertible. Then there exists a neighborhood U_a of a in U and a neighborhood $V_{f(a)}$ of $f(a)$ in V such that the restriction of f to U_a is a C^k-diffeomorphism from U_a to $V_{f(a)}$.*

Definition 1.0.15 (Local chart)**.** *A topological space M that is Hausdorff and separable is a **topological manifold** of dimension n if for every $p \in M$, there exists an open neighborhood U of p, an open set $V \subset \mathbb{R}^n$, and a map $\varphi : U \to V$ that is a homeomorphism. The pair (U, φ) is called a **local chart** of M at the point p. For every $p \in U$, the coordinates of $\varphi(p)$ in $\mathbb{R}^n$ are the **coordinates of** p **in the chart** (U, φ).*

Definition 1.0.16 (Atlas)**.** *Let M be a topological manifold and $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a family of local charts of M. We say that $\mathcal{A}$ is an **atlas of** M if $M = \bigcup_{\alpha \in I} U_\alpha$.*

Definition 1.0.17 (Change of chart maps)**.** *Let now $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be an atlas of M, and $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ two charts such that $U_\alpha \cap U_\beta \neq \emptyset$. The maps*

$$\Phi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

*are called **transition maps** or **change of chart maps**.*

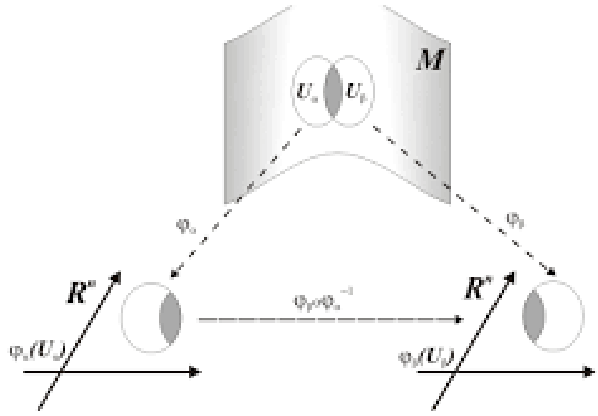


Figure 1.1: The chart transition homeomorphism

Definition 1.0.18. *Let M be a topological manifold and $A = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ an atlas of M. We say that A is of class C^k, $1 \leq k \leq \infty$, if for all α and β in I, the change of chart maps*

$$\Phi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$$

are diffeomorphisms of class C^k from $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

Definition 1.0.19. *Let M be a topological manifold and let A_1, A_2 be two atlases of class C^k on M. We say that A_1, A_2 are C^k-compatible if $A_1 \cup A_2$ is still an atlas of class C^k on M.*

Remark 1.0.4. *The relation of C^k-compatibility is an equivalence relation on the set of class C^k atlases on M. The union of all atlases in the same equivalence class is called a **saturated (or complete) C^k-atlas**. Every class C^k atlas on M is therefore contained in a unique saturated C^k-atlas.*

Definition 1.0.20 (Differentiable maps between manifolds)**.** *Let M and N be two manifolds of dimensions m and n respectively, and let $f : M \to N$ be a map. We say that f is of class C^k on M, with $k > 1$, if for every point $p \in M$, every chart (U, φ) of M at the point p, and every chart (V, ψ) of N at the point $f(p)$ such that $f(U) \subset V$, the map*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is of class C^k.

The map f is a submersion if $\psi \circ f \circ \varphi^{-1}$ is a submersion from $\varphi(M)$ into $\psi(N)$. Since the chart transition maps are of class C^k, the map f is of class C^k on M if and only if, for every point $p \in M$, there exists a chart (U, φ) of M at p, and a chart (V, ψ) of N at $f(p)$ such that $f(U) \subset V$, and

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is of class C^k.

In particular, if $N = \mathbb{R}$, endowed with its natural structure as a C^k-manifold, then $f : M \to \mathbb{R}$ is of class C^k.

2

Multilinear forms

The references for this chapter are [**25**, **31**, **12**, **18**, **5.**]

2.1 Symmetric group S_n

Definition 2.1.1 (Symmetric group S_n)**.** *Let $n \in \mathbb{N}^*$, we denote by S_n the set of bijections from the set $A_n = \{1, \ldots, n\}$ to itself. S_n is a finite set with cardinality card$(S_n) = n!$. The elements of S_n are called permutations. An element $\sigma \in S_n$ is represented by the matrix*

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

S_n equipped with the composition of functions $\circ$ is a non-commutative group.
If $\sigma, \mu \in S_n$, we denote $\sigma\mu := \sigma \circ \mu$.

Definition 2.1.2 (Support of a permutation)**.** *Let $\sigma \in S_n$, we call the support of σ the set, denoted supp(σ), of elements of A_n that are not fixed by σ, i.e.*

$$\text{supp}(\sigma) = \{i \in A_n, \sigma(i) \neq i\}$$

Two permutations from A_n are not equal if their supports are disjoint.

Definition 2.1.3 (p-cycle)**.** *A cycle of length p is a permutation $\sigma \in S_n$ defined by a subset $\{i_1, \ldots, i_p\} \subset A_n$ such that*

$$\begin{cases} \sigma(i_1) = i_2, \ldots, \sigma(i_{p-1}) = i_p, & \sigma(i_p) = \sigma(i_1) \\ \quad \sigma(j) = j, & \forall j \notin \{i_1, \ldots, i_p\}. \end{cases}$$

The cycle σ is represented by the matrix row $(i_1, \ldots, i_p)$. Two cycles $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_q)$ are said to be disjoint if $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$.

Definition 2.1.4 (Transposition)**.** *A transposition is a cycle τ of length 2 defined by (i, j), i.e. $\tau(i) = j, \tau(j) = i$, and $\tau(k) = k, \forall k \notin \{i, j\}$.*

Remark 2.1.1. *If σ is a cycle of length p and τ is a transposition, then:*

1. $\sigma^p = \sigma \circ \cdots \circ \sigma$ *(p times) = Id.*
2. $\sigma^{-1} = \sigma^{p-1}$.

3. $\tau^{-1} = \tau$.

Lemma 2.1.1. *Every cycle $\sigma = (i_1, \ldots, i_p)$ can be decomposed into transpositions.*

Definition 2.1.5 (Signature of a permutation)**.** *The signature of a permutation $\sigma \in S_n$ is defined by the formula*

$$sign(\sigma) = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}$$

We denote $\varepsilon(\sigma) = sign(\sigma)$.

Remark 2.1.2. $sign(\sigma) = \pm 1$.

Theorem 2.1.1. *The signature of a p-cycle from $\{1, \ldots, n\}$ is $(-1)^{p-1}$.*

Lemma 2.1.2. *Let σ and μ be two permutations, then*

$$sign(\sigma\mu) = sign(\sigma \circ \mu) = sign(\sigma)sign(\mu).$$

Remark 2.1.3.
1. *If τ is a transposition, then $sign(\tau) = -1$.*
2. *If σ is a permutation, then $sign(\sigma) = (-1)^k$, where k is the number of transpositions that decompose σ (i.e., $\sigma = \tau_1 \ldots \tau_k$).*
3. $\varepsilon(Id) = 1$.
4. $\varepsilon(\sigma^{-1}) = \varepsilon(\sigma)$.

Example 2.1.1. *Let $\sigma \in S_7$ defined by*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 5 & 1 & 7 & 6 \end{pmatrix}$$

$$supp(\sigma) = \{1, 2, 4, 5, 6, 7\}$$

To calculate $\varepsilon(\sigma)$, we have the decomposition into a product of transpositions:

$$\sigma = (1, 2, 4, 5)(6, 7) = (1, 2)(2, 4)(4, 5)(6, 7)$$

so

$$\varepsilon(\sigma) = (-1)^{4-1}(-1)^{2-1} = (-1)^4 = 1$$

Now calculate σ^{-1}

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 2 & 4 & 7 & 6 \end{pmatrix}$$

2.2 Linear k-forms

Definition 2.2.1. *Let E be a vector space over $\mathbb{K}$ with dimension n. A map $\omega : E \to F$ is said to be linear if*

$$\forall x, y \in E, \forall \lambda, \mu \in K, \quad \omega(\lambda x + \mu y) = \lambda\omega(x) + \mu\omega(y).$$

We denote the set of linear maps from E to F by $\mathcal{L}(E, F)$.

Remark 2.2.1. *If $F = \mathbb{K}$, we say that ω is a linear form, denoted by*

$$E^* = \mathcal{L}(E, \mathbb{K}).$$

Definition 2.2.2. *(k-Linear forms) Let E be a vector space over $\mathbb{K}$ and $k \geq 1$ a natural number. A map ω is called a k-linear form or simply a k-form on E if it is a map*

$$\omega : \overbrace{E \times \ldots \times E} = E^k \to \mathbb{K}$$
$$\text{defined by} \quad (x_1, x_2, \ldots, x_k) \mapsto \omega\,(x_1, x_2, \ldots, x_k)$$

for all $(x_1, x_2, \ldots, x_k) \in E^k$. It satisfies the following condition:

$$\forall (x_1, x_2, \ldots, x_k) \in E^k, \quad \forall \lambda, \mu \in \mathbb{K},$$

$$\omega(x_1, \ldots, x_{i-1}, \lambda x + \mu y, x_{i+1}, \ldots, x_k) = \lambda\, \omega(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k) + \mu\, \omega(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k),$$

i.e., ω is linear in each variable.

Example 2.2.1.

1. *The dot product of two vectors in $\mathbb{R}^3$:*

$$\omega(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

2. *$\omega(x, y, z) = f(x) f(y) f(z)$ where $f \in \mathcal{L}(E, \mathbb{K})$.*

Proposition 2.2.1. *The k-linear forms on E form a vector subspace of the vector space $\mathcal{F}(E^k, \mathbb{K})$ of maps from E^k to $\mathbb{K}$.*
We denote $\mathcal{L}_k(E, K)$ or $\mathcal{L}_k(E)$ as the $\mathbb{K}$-vector space of k-linear forms on E.

Definition 2.2.3. *(Symmetric k-linear form)*
A map $\omega \in \mathcal{L}_k(E, \mathbb{K})$ is called symmetric if $\omega(x_1, \ldots, x_k)$ is invariant under the exchange of two vectors, i.e.,

$$\forall (x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_k) \in E^k,$$
$$\omega(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = \omega(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).$$

The set of symmetric k-linear forms is denoted by $S_k(E, \mathbb{K})$.

Definition 2.2.4. *(Antisymmetric k-linear form) A map $\omega \in \mathcal{L}_k(E)$ is called antisymmetric if $\omega(x_1, \ldots, x_k)$ changes sign under the exchange of two vectors, i.e.,*

$$\forall (x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_k) \in E^k,$$

$$\omega(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = -\omega(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).$$

Proposition 2.2.2. *(Alternating k-Linear Form)*
We say that ω is an alternating k-linear form if ω is zero on any set of vectors where at least two vectors are equal:

$$\exists i \neq j \in \{1, \ldots, n\}, \quad x_i = x_j \quad \Rightarrow \quad \omega(x_1, \ldots, x_i, x_j, \ldots, x_k) = 0.$$

Definition 2.2.5. *(Alternating k-Linear Form with Permutation)*
A k-linear form $\omega \in \mathcal{L}_k(E)$ is said to be alternating if for any $x_1, \ldots, x_k \in E$ and any permutation $\sigma \in S_k$,

$$\omega(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \varepsilon(\sigma) \omega(x_1, \ldots, x_k),$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ.
The set of alternating k-linear forms is a vector subspace of $\mathcal{L}_k(E, \mathbb{K})$, denoted by $\mathcal{A}_k(E, \mathbb{K})$.

Proposition 2.2.3. *Let $\omega \in \mathcal{A}_k(E, \mathbb{K})$. If there is a vector x_i such that x_i is a linear combination of other vectors $x_1, \ldots, x_k$, then*

$$\omega(x_1, \ldots, x_k) = 0.$$

2.3 Pulback application

Definition 2.3.1. *Let E and F be two vector spaces over $\mathbb{R}$. Let $u : E \to F$ be a smooth map C^∞ and ω a k-linear form on F.*

We define the reciprocal image of ω, denoted $u^*(\omega)$, by:

$$u^*(\omega) : E^k \to \mathbb{R},$$
$$(x_1, \dots, x_k) \mapsto u^*(\omega)(x_1, \dots, x_k) = \omega(u(x_1), \dots, u(x_k))$$

Remark 2.3.1.

- $$u^* : \mathcal{L}_k(F) \to \mathcal{L}_k(E),$$
  $$\omega \mapsto u^*(\omega)$$

  where $\omega \mapsto u^*(\omega)$ is a linear map, i.e.:

  $$u^*(\alpha\omega + \beta\omega') = \alpha u^*(\omega) + \beta u^*(\omega')$$

  where $\alpha, \beta \in \mathbb{R}$ and $\omega, \omega' \in \mathcal{L}_k(F)$.

- *Let E, F, G be vector spaces over $\mathbb{R}$. If $u : E \to F$ and $v : F \to G$ are two smooth maps, then*

  $$(v \circ u)^* = u^* \circ v^*$$

  and

  $$(v \circ u)^* : \mathcal{L}_k(G) \to \mathcal{L}_k(E),$$
  $$\omega \mapsto u^*(v^*(\omega))$$

Definition 2.3.2. *(Antisymmetrization) Let ω be a k-linear form. The map $A(\omega)$ defined by*

$$A(\omega)(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma)\omega(x_{\sigma(\mu(1))}, \dots, x_{\sigma(\mu(k))})$$

The map

$$A : \mathcal{L}_k(E) \to \mathcal{A}_k(E)$$
$$\omega \mapsto A(\omega)$$

is called the antisymmetrization of ω.

2.4 Tensor product

Definition 2.4.1. *Let E be a vector space over $\mathbb{R}$, ω a k-linear form on E, and ω' a p-linear form on E. We define the tensor product $\omega \otimes \omega'$ by:*

$$\omega \otimes \omega' : E^k \times E^p \to \mathbb{R}, \quad (x_1, \dots, x_k, x_{k+1}, \dots, x_{k+p}) \mapsto \omega(x_1, \dots, x_k)\omega'(x_{k+1}, \dots, x_{k+p})$$

Proposition 2.4.1.

- *If $\omega \in \mathcal{L}_k(E)$, $\omega' \in \mathcal{L}_p(E)$, then $\omega \otimes \omega' \in \mathcal{L}_{k+p}(E)$.*
- *The tensor product is not commutative, i.e. $\omega \otimes \omega' \neq \omega' \otimes \omega$.*
- *In general, if $\omega_1 \in \mathcal{L}_{k_1}(E), \dots, \omega_m \in \mathcal{L}_{k_m}(E)$, we define the tensor product as*

  $$\omega_1 \otimes \dots \otimes \omega_m : (x_1^1, \dots, x_1^{k_1}, \dots, x_m^1, \dots, x_m^{k_m}) \mapsto \omega_1(x_1^1, \dots, x_1^{k_1}) \dots \omega_m(x_m^1, \dots, x_m^{k_m})$$

  with $x_j^i \in E$.

Definition 2.4.2. *(Dual space)*
Let E be a vector space over $\mathbb{R}$. The set of 1-linear forms is called the dual space of E, denoted E^* or E', i.e.:

$$E^* = \mathcal{L}_1(E) = \mathcal{L}(E;\mathbb{R})$$

Proposition 2.4.2. *If E is of dimension n, then E^* is a vector space of dimension n.*

Definition 2.4.3. *(Dual basis) Let E be a finite-dimensional vector space of dimension n. If $(e_1,\dots,e_n)$ is a basis of E, we define a dual basis $(e_1^*,\dots,e_n^*)$ on E^* by the formula:*

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proposition 2.4.3. *Let E be a finite-dimensional vector space of dimension n.*
If $(e_1,\dots,e_n)$ is a basis of E, then $\{e_i^* \otimes e_j^*\}_{1\le i,j\le n}$ is a basis of the vector space of bilinear forms $\mathcal{L}_2(E)$.
From this proposition, we deduce that

$$\dim(\mathcal{L}_2(E)) = (\dim E)^2 = n^2$$

Proposition 2.4.4. *Let E be a finite-dimensional vector space of dimension n. If $(e_1,\dots,e_n)$ is a basis of E, then*

$$\{e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*\}_{1\le i_1,i_2,\dots,i_k\le n}$$

is a basis of the vector space of k-linear forms. From proposition 2.4.3, we deduce:

- $\dim(\mathcal{L}_k(E)) = (\dim E)^k = n^k$
- *If $\omega \in \mathcal{L}_k(E)$, then*

$$\omega = \sum_{i_1,\dots,i_k=1}^{n} \omega_{i_1\dots i_k} e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* = \sum_{i_1,\dots,i_k=1}^{n} \omega(e_{i_1},\dots,e_{i_k}) e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$$

2.5 Exterior product

Definition 2.5.1. *Let $\omega \in \mathcal{A}_p(E)$ and $\omega' \in \mathcal{A}_q(E)$. The exterior product of ω and ω', denoted $\omega \wedge \omega'$, is a (p + q)-linear alternating form defined by*

$$\omega \wedge \omega' = \frac{(p+q)!}{p!q!} A(\omega \otimes \omega')$$

$$\omega \wedge \omega'(x_1,\dots,x_p,x_{p+1},\dots,x_{p+q}) = \frac{1}{p!q!} \sum_{\sigma\in S_{p+q}} \varepsilon(\sigma)\omega((x_{\sigma(1)},\dots,x_{\sigma(p)}))\omega'((x_{\sigma(p+1)},\dots,x_{\sigma(p+q)}))$$

Proposition 2.5.1. *Let $\omega_1,\omega_2 \in \mathcal{A}_p(E)$, $\omega_1',\omega_2' \in \mathcal{A}_q(E)$, and $\lambda \in \mathbb{R}$, then we have:*

1. $(\omega_1 + \omega_2) \wedge \omega_1' = \omega_1 \wedge \omega_1' + \omega_2 \wedge \omega_1'$
2. $(\lambda\omega_1) \wedge \omega_1' = \omega_1 \wedge (\lambda\omega_1') = \lambda\omega_1 \wedge \omega_1'$

Proposition 2.5.2. *If $\omega \in \mathcal{A}_1(E)$ and $\omega' \in \mathcal{A}_1(E)$ are two linear forms, then $\omega \wedge \omega' \in \mathcal{A}_2(E)$ and we have:*

$$\omega \wedge \omega' = -\omega' \wedge \omega$$

Proof 2.5.1. *We have $S_2 = \{id, (1, 2)\}$, thus:*

$$\omega \wedge \omega'((x_1, x_2)) = \omega(x_1)\omega'(x_2) - \omega(x_2)\omega'(x_1) = -\omega' \wedge \omega((x_1, x_2))$$

Remark 2.5.1. *We have:*

$$\omega \wedge \omega'(x_1, x_2) = det \begin{pmatrix} \omega(x_1) & \omega'(x_1) \\ \omega(x_2) & \omega'(x_2) \end{pmatrix}$$

Proposition 2.5.3. *Let $\omega, \omega', \omega'' \in \mathcal{A}_1(E)$, then:*

$$(\omega \wedge \omega') \wedge \omega'' = \omega \wedge (\omega' \wedge \omega'')$$

We denote:

$$\omega \wedge \omega' \wedge \omega'' = \omega \wedge (\omega' \wedge \omega'')$$

Definition 2.5.2. *Let $\omega_1, \dots, \omega_p \in \mathcal{A}_1(E)$. We define the exterior product $\omega_1 \wedge \cdots \wedge \omega_p$ by:*

$$(\omega_1 \wedge \cdots \wedge \omega_p)(x_1, \dots, x_p) = \sum_{\sigma \in S_p} \varepsilon(\sigma)\omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_p(x_{\sigma(p)})$$

$$= det \begin{vmatrix} \omega_1(x_1) & \dots & \omega_1(x_p) \\ \vdots & \ddots & \vdots \\ \omega_p(x_1) & \dots & \omega_p(x_p) \end{vmatrix}$$

Remark 2.5.2. *Let $(e_1, \dots, e_n)$ be a basis of the vector space E, and $(e_1^*, \dots, e_n^*)$ the dual basis of E^*.*
Let $\omega_1, \dots, \omega_p \in \mathcal{A}_p(E)$. We have:

1. *By the properties of the determinant, if $i, j \in \{1, \dots, n\}$ such that $i \neq j$ and $\omega_i = \omega_j$, then:*

$$\omega_1 \wedge \cdots \wedge \omega_p = 0$$

2. *If $p > n$, then the system $\{e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*\}$ is linearly dependent, hence:*

$$e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = 0$$

Thus, $A_p(E) = 0$.

Proposition 2.5.4. *Let E be a finite-dimensional vector space and $(e_1, \dots, e_n)$ a basis of E. If $(e_1^*, \dots, e_n^*)$ denotes the dual basis of E^*, then:*

$$B = \{e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*\}_{1 \leq i_1 < \cdots < i_p \leq n}$$

is a basis of the vector space $A^p(E)$.

Proposition 2.5.5. *Let $\omega_1, \dots, \omega_p, \omega_1', \dots, \omega_q' \in \mathcal{L}_1(E)$, then:*

$$(\omega_1 \wedge \cdots \wedge \omega_p) \wedge (\omega_1' \wedge \cdots \wedge \omega_q') = (-1)^{pq}(\omega_1' \wedge \cdots \wedge \omega_q') \wedge (\omega_1 \wedge \cdots \wedge \omega_p)$$

Proposition 2.5.6. *If ω is a p-linear form on E, and ω' is a q-linear form on E, then $\omega \wedge \omega' \in A^{p+q}(E)$ and we have:*

$$\omega \wedge \omega' = (-1)^{pq}\omega' \wedge \omega$$

2.6 Interior product

Definition 2.6.1. *Let $x \in E$. The inner product i_x is a map defined by:*

$$i_x : \mathcal{L}_p(E) \to \mathcal{L}_{p-1}(E)$$

$$\omega \mapsto i_x(\omega)$$

such that $i_x(\omega)$ is a $(p-1)$-linear alternating form given by:

$$i_x(\omega) : \mathcal{A}_p \to \mathcal{A}_{p-1},$$
$$\omega \mapsto i_x(\omega)(x_1, \dots, x_{p-1}) := \omega(x, x_1, \dots, x_{p-1})$$

Proposition 2.6.1. *Let $x \in E$, $\lambda \in \mathbb{R}$, and $\omega, \omega' \in \mathcal{A}_p(E)$, then:*

1. $i_x(\omega + \omega') = i_x(\omega) + i_x(\omega')$
2. $i_x(\lambda\omega) = \lambda i_x(\omega)$

Proposition 2.6.2. *If $x \in E$, $\omega, \omega' \in \mathcal{A}_p(E)$, then:*

$$(\omega \wedge \omega')i_x = i_x(\omega) \wedge \omega' + (-1)^p \omega \wedge i_x(\omega')$$

3

Differential forms

*The references for this chapter are [**55**, **25**, **2**, **5**, **31**, **48**.]*

3.1 Differential forms on $\mathbb{R}^n$

Definition 3.1.1. *Let U be an open subset of $\mathbb{R}^n$. A differential form of degree p on U (or simply a p-differential form) is any map*

$$\omega : U \to A_p(\mathbb{R}^n)$$
$$x \mapsto \omega_x$$

- *If $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$ and ω is a p-differential form on $U \subset \mathbb{R}^n$, then we have*

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \dots i_p} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$$

- *where $\omega_{i_1 \dots i_p} : U \to \mathbb{R}$ are differentiable functions on U, and $\{e^*_1, \dots, e^*_n\}$ denotes the canonical dual basis of $\mathbb{R}^n$.*

- *For all $x \in U$,after Proprety 2.5.4 we have:*

$$\omega_x := \omega(x) = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \dots i_p}(x) e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$$

Definition 3.1.2. *Let $\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \dots i_p} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$ be a p-differential form on $\mathbb{R}^n$.*

1. *ω is said to be continuous if the functions $\omega_{i_1 \dots i_p} : U \to \mathbb{R}$ are continuous for $1 \le i_1 < \cdots < i_p \le n$.*

2. *ω is said to be differentiable of class C^k if the functions $\omega_{i_1 \dots i_p} : U \to \mathbb{R}$ are differentiable of class C^k for $1 \le i_1 < \cdots < i_p \le n$.*

Proposition 3.1.1. *Let $U \subseteq \mathbb{R}^n$ be an open set, $\omega_1 = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega^1_{i_1 \dots i_p} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$,*
$\omega_2 = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega^2_{i_1 \dots i_p} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$ two p-differential forms of class C^k on U, and
$\omega = \sum_{1 \le j_1 < \cdots < j_q \le n} \omega_{j_1 \dots j_q} e^*_{j_1} \wedge \cdots \wedge e^*_{j_q}$ a q-differential form on U, then:

1. $\omega_1 + \omega_2 = \sum_{1 \leq i_1 < \cdots < i_p \leq n} (\omega^1_{i_1 \ldots i_p} + \omega^2_{i_1 \ldots i_p}) e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$ *is a p-differential form of class* C^k *on* U.

2. $\omega_1 \wedge \omega$ *is a* $(p+q)$*-differential form of class* C^k *on* U*, where*

$$\omega_1 \wedge \omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \sum_{1 \leq j_1 < \cdots < j_q \leq n} \omega^1_{i_1 \ldots i_p} \omega_{j_1 \ldots j_q} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p} \wedge e^*_{j_1} \wedge \cdots \wedge e^*_{j_q}$$

Definition 3.1.3. *Let* U *be an open set of* $\mathbb{R}^n$*. If we denote by* $\Omega^k_p(U)$ *the set of p-differential forms of class* C^k *on* U*,*

$C^k(U)$ *denotes the ring of real functions of class* C^k *on* U*.*

$C^0(U)$ *denotes the ring of continuous real functions on* U*.*

3.2 Characterization of differential forms

Let $i \in \{1, \ldots, n\}$ *and* P_i *the* i*-th projection defined by*

$$P_i : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_1, \ldots, x_n) \mapsto x_i$$

Then dP_i *is a 1-differential form of class* C^∞ *denoted* dx_i*,*

$$dx_i : \mathbb{R}^n \to A_1(\mathbb{R}^n), \quad x \mapsto dP_i|_x = e^*_i$$

If ω *is a 1-differential form on an open set* $U \subseteq \mathbb{R}^n$*, then* ω *is written as*

$$\omega = \sum_{i=1}^{n} \omega_i dx_i$$

For $x \in U$*, we have*

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x) dx_i$$

Example 3.2.1.

1. *If* $n = 2$*, then every 1-differential form* ω *on an open set* $U \subseteq \mathbb{R}^2$ *can be written in the form*

$$\omega = \omega_1 dx + \omega_2 dy$$

where ω_1, ω_2 *are functions on* U*.*

2. $\omega = y dx + x dy$ *is a 1-differential form on* $\mathbb{R}^2$*.*

In general, if ω *is a p-differential form, then* ω *is written in the form:*

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \omega_{i_1 \ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

Remark 3.2.1. *For the following, we write* $dx_{i_1} \ldots dx_{i_p}$ *instead of* $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$*. Thus, the formula is written*

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \omega_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p}$$

Theorem 3.2.1. *If $j \in \{i_1, \dots, i_p\}$, then*

$$dx_{i_1} \dots dx_j \dots dx_{i_p} = 0$$

Example 3.2.2. *If U is an open set of $\mathbb{R}^2$, then:*

1. $\Omega_0^k(U) = C^k(U)$
2. $\Omega_1^k(U) = \{fdx + gdy; f, g \in C^k(U)\}$
3. $\Omega_2^k(U) = \{fdxdy; f \in C^k(U)\}$
4. $\Omega_p^k(U) = \{0\}$ *if* $p \geq 3$

Definition 3.2.1. *Let U be an open set of $\mathbb{R}^n$. The n-differential form*

$$\omega = fdx_1 \wedge \dots \wedge dx_n$$

where f is a non-vanishing differentiable real function, is called the volume form on U.

3.3 Exterior derivative

Definition 3.3.1. *Let U be an open set of $\mathbb{R}^n$ and ω a p-differential form of class C^k on U $(1 \leq k)$. The exterior derivative of ω is the $(p+1)$-differential form.*

- ***$d\omega$ defined by:***

$$d : \Omega_p^k(U) \to \Omega_{p+1}^k(U); \quad \omega \mapsto d\omega$$

$$d\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} d\omega_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_p}$$

Remark 3.3.1.

$$d\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \sum_{j=1}^{n} \frac{\partial \omega_{i_1 \dots i_p}}{\partial x_j} dx_j dx_{i_1} \dots dx_{i_p}$$

$$d(dx_{i_1} \dots dx_{i_p}) = 0$$

Example 3.3.1. *Let U be an open set in $\mathbb{R}^3$, we have:*

1. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$
2. $d(fdx + gdy + hdz) = \frac{\partial g}{\partial x} dxdy + \frac{\partial h}{\partial x} dxdz + \frac{\partial f}{\partial y} dydx + \frac{\partial h}{\partial y} dydz + \frac{\partial f}{\partial z} dzdx + \frac{\partial g}{\partial z} dzdy$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dxdz + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dydz$$

3. $d(fdxdy + gdxdz + hdydz) = \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) dxdydz$
4. *If* $\omega \in \Omega_p^k(U)$ $(p \geq 3)$, *then* $d\omega = 0$

Proposition 3.3.1. *Let U be an open set in $\mathbb{R}^n$, and let $\omega_1 \in \Omega_p^k(U)$ and $\omega_2 \in \Omega_q^k(U)$, then:*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$$

Lemma 3.3.1. *Let U be an open set in $\mathbb{R}^n$ and $f \in C^k(U)$ ($k \geq 2$), then:*

$$d(df) = 0$$

Lemma 3.3.2. *Let U be an open set in $\mathbb{R}^n$ and $f \in C^k(U)$ ($k \geq 2$), then:*

$$d\left(d(f dx_{i_1} \dots dx_{i_p})\right) = 0$$

Theorem 3.3.1. *Let U be an open set in $\mathbb{R}^n$ and $\omega \in \Omega_p^k(U)$ ($k \geq 2$), then:*

$$d(d\omega) = 0$$

Definition 3.3.2. *Let U be an open set in $\mathbb{R}^n$. A p-differential form $\omega \in \Omega_p^k(U)$ ($k \geq 2$) is said to be closed if*

$$d\omega = 0$$

.

Proposition 3.3.2. *Let U be an open set in $\mathbb{R}^n$. A 1-form ω is closed if and only if*

$$\frac{\partial \omega_i}{\partial x_i} = \frac{\partial \omega_j}{\partial x_j} \quad \forall i, j$$

3.4 Exact differential forms

Definition 3.4.1. *Let U be an open set in $\mathbb{R}^n$, $\omega \in \Omega_{p+1}^{k-1}(U)$ ($k \geq 1$), and $\overline{\omega} \in \Omega_p^k(U)$ ($k \geq 2$). ω is said to be a exact if*

$$d\overline{\omega} = \omega.$$

Proposition 3.4.1. *Every exact differential form is a closed differential form.*

Example 3.4.1. *1.* ***Let the 1-differential form on*** $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{xdx + ydy}{x^2 + y^2}$$

We have:

$$d\omega = \frac{-2yx}{(x^2+y^2)^2} dydx + \frac{-2xy}{(x^2+y^2)^2} dxdy = 0$$

$$\omega = d\left(\ln \sqrt{x^2 + y^2}\right)$$

Thus, ω is an exact 1-differential form on $\mathbb{R}^2 \setminus \{0\}$.

2. ***Let the 1-differential form on*** $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{ydx - xdy}{x^2 + y^2}$$

We have:

$$d\omega = \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx - \frac{y^2 - x^2}{(x^2 + y^2)^2} dxdy = 0$$

ω is closed, locally exact, but it is not exact.

3.5 Interior derivative of a differential form

Definition 3.5.1. *Let U be an open subset of $\mathbb{R}^n$, $X \in \chi(U)$, and $\omega \in \Omega_1^k(U)$. We define the function $\omega(X)$ on U by:*

$$\forall p \in U : \omega(X)(x) = \omega_x(X_x)$$

or $\omega_x \in \mathcal{L}_1(\mathbb{R}^n)$. If

$$X = \sum_i X^i \frac{\partial}{\partial x_i}, \quad \omega = \sum_i \omega_i dx_i$$

then

$$\omega(X) = \sum_i X^i \omega_i$$

The function $\omega(X)$ is called the interior derivative of ω by X, denoted as:

$$i_X(\omega)$$

Lemma 3.5.1. *If $\omega \in \Omega_1^k(U)$, then*

$$\begin{aligned} \omega : \chi(U) &\to C^k(U) = \Omega_0^k(U), \\ X &\mapsto \omega(X) \end{aligned}$$

is a $C^k(U)$-linear application, i.e., for all $X, Y \in \chi(U)$ and $f \in C^k(U)$, we have:

1. $\omega(X + Y) = \omega(X) + \omega(Y)$.
2. $\omega(fX) = f\omega(X)$.

Definition 3.5.2. *Let U be an open subset of $\mathbb{R}^n$ and $X \in \chi(U)$. The interior derivative i_X is an application defined by:*

$$\begin{aligned} i_X : \Omega_q^k(U) &\longrightarrow \Omega_{q-1}^k(U) \\ \omega &\longmapsto i_X(\omega) \end{aligned}$$

such that $i_X(\omega)$ is a $(q-1)$-linear form given by:

$$i_X\omega(X_1, \ldots, X_{q-1}) = \omega(X, X_1, \ldots, X_{q-1})$$

$$\omega(X, X_1, \ldots, X_{q-1})(x) = \omega_x(X(x), X_1(x), \ldots, X_{q-1}(x)), \quad \forall x \in U$$

Proposition 3.5.1. *Let $X \in \chi(U)$, $\omega \in \Omega_p^k(U)$, and $\alpha \in \Omega_q^k(U)$. Then:*

$$i_X(\omega \wedge \alpha) = i_X\omega \wedge \alpha + (-1)^p \omega \wedge i_X\alpha$$

Remark 3.5.1. *Let U be an open subset of $\mathbb{R}^n$ and $X = \sum_i^n X^i \frac{\partial}{\partial x^i}$ a vector field of class C^k on U. If $\omega = dx^1 \wedge \cdots \wedge dx^n$ is a volume form on U, then:*

$$i_X(\omega) = \sum_i^n X^i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $\widehat{dx_i}$ means that dx^i is omitted.

3.6 Pullback of a differential form

Definition 3.6.1. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, and $\varphi : U \to V$ a C^{k+1} map ($k \geq 1$). We define the pullback of a differential q-form:*

$$\begin{aligned} \varphi^* : \Omega_q^k(V) &\to \Omega_q^k(U), \\ \omega &\mapsto \varphi^*(\omega) \end{aligned}$$

which is defined by:

$$\varphi^*(\omega)_x(z_1, \ldots, z_q) = (d_x\varphi)^*(\omega_{\varphi(x)})(z_1, \ldots, z_q) = \omega_{\varphi(x)}(d_x\varphi(z_1), \ldots, d_x\varphi(z_q))$$

for $x \in U$ and $z_1, \ldots, z_q \in \mathbb{R}^n$(see Definition 2.3.1).

Proposition 3.6.1. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, and W an open subset of $\mathbb{R}^r$. Let $\varphi : U \to V$ and $\psi : V \to W$ be two C^{k+1} maps ($k \geq 1$). Then:*

1. $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
2. $\varphi^*(\omega_1 + \omega_2) = \varphi^*(\omega_1) + \varphi^*(\omega_2)$.
3. $\varphi^*(\lambda\omega) = \lambda\varphi^*(\omega)$ *for* $\lambda \in \mathbb{R}$.
4. $\varphi^*(\omega \wedge \alpha) = \varphi^*(\omega) \wedge \varphi^*(\alpha)$.
5. $\varphi^*(f) = f \circ \varphi$ *for* $f \in C^k(V)$.
6. *If* $k > n$, *then* $\varphi^*(\omega) = 0$.
   Let $f \in C^k(V)$, $\omega, \omega_1, \omega_2 \in \Omega_p^k(V)$, and $\lambda \in \mathbb{R}$, $\alpha \in \Omega_q^k(V)$.

Proposition 3.6.2. *Let U be an open subset of $\mathbb{R}^p$, V an open subset of $\mathbb{R}^n$, and $\varphi : U \to V$ a C^k function ($p \leq n$). If $\omega \in \Omega_p(V)$, then:*

$$\varphi^*(d\omega) = d\varphi^*(\omega)$$

Corollary 3.6.1. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, and $\varphi : U \to V$ a C^k function ($n \leq m$). If $\omega \in \Omega_n^k(V)$, then:*

$$\begin{aligned} \varphi^*(\omega) &= f \circ \varphi \, (d\varphi_1 \wedge \cdots \wedge d\varphi_n) \\ &= f(\varphi(x)) \, (d\varphi_1(x) \wedge \cdots \wedge d\varphi_n(x)) \\ &= \varphi^*(f(x)) \, (d\varphi_1 \wedge \cdots \wedge d\varphi_n) \end{aligned}$$

where $f : U \to \mathbb{R}$ is a C^{k-1} function and $x \in U$.

Corollary 3.6.2. *Let (U, φ) be a local chart of M, and let $\varphi_1, \ldots, \varphi_n$ be a C^∞ function on U. Then:*

$$d\varphi_1 \wedge \cdots \wedge d\varphi_n = \det\left[\frac{\partial \varphi_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n$$

Corollary 3.6.3. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, and $\varphi : U \to V$ a C^k function ($p \leq \min(n, m)$). Let $\omega \in \Omega_p^k(V)$, then:*

1. *If ω is closed, then $\varphi^*(\omega)$ is closed.*
2. *If ω is exact (i.e., $\omega = d\alpha$), then $\varphi^*(\omega) = d\varphi^*(\alpha)$.*

3.7 Differential forms on a differentiable manifold

Let M be a differentiable manifold of dimension n, we define the application:

$$\begin{aligned}\Pi : \wedge^q TM &\to M \\ (p, \omega_p) &\mapsto p\end{aligned}$$

as the canonical projection which associates $\omega_p \in \wedge^q(T_p^* M)$ with $\Pi(\omega_p) = p \in M$.

Definition 3.7.1. *(q-Differential form)*
A q-differential form of class C^∞ on M is a application:

$$\begin{aligned}\omega : M &\to \wedge^q T^* M := A_q(T_p M) \\ p &\mapsto \omega_p\end{aligned}$$

of class C^∞ that satisfies $\Pi \circ \omega = Id_M$.

Remark 3.7.1.

1. *If (U, φ) is a local chart of M, then a q-differential form is expressed in this chart as:*

$$\omega_p = \sum_{1 \leq i_1 < \cdots < i_q \leq n} f_{i_1 \ldots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

where $\{dx_{i_1} \wedge \cdots \wedge dx_{i_q}\}_{1 \leq i_1 < \cdots < i_q \leq n}$ forms the basis of $\wedge^q T_p^* M$.

2. *A 1-differential form of class C^k on M is a application:*

$$\omega : M \to T^* M$$

which associates to each point $p \in M$:

$$\omega_p = \sum_{i=1}^{n} f_i(p) dx_i$$

where $\{dx_i|_p\}_{i=1,\ldots,n}$ forms the basis of $T_p^* M$, the dual space of the tangent space of M, and $f_i \in C^\infty(M)$.

Theorem 3.7.1. *Let M be a differentiable manifold. For any $q \in \mathbb{N}$, there exists an exterior differentiation operator:*

$$d : \Omega^q(M) \to \Omega^{q+1}(M)$$

such that:

1. *For $q = 0$, $d : C^\infty \to \Omega^1$ is the usual differential of functions.*
2. *For all $\omega \in \Omega^q(M)$:*

$$d\omega(X_0, \dots, X_q) = \sum_{i=1}^{n} (-1)^i X_i \cdot \omega(X_0, \dots, \widehat{X}_i, \dots, X_q)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_q)$$

   where $\widehat{X}_i$ means omitting X_i.

3. *For $\omega \in \Omega^q(M)$, we have $d(d\omega) = 0$.*
4. *For $\omega_1 \in \Omega^q(M)$ and $\omega_2 \in \Omega^{q'}(M)$:*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^q \omega_1 \wedge d\omega_2$$

ee

4

Manifolds and tensor fields

*The references for this chapter are [**55**], [**7**], [**25**] .*

Let M be a topological space. We assume that M satisfies the Hausdorff separation axian which states that any two different points in M can be separated by disjoint open sets. An open chart on M is a pair (U, ϕ) where U is an open subset of M and ϕ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional Euclidean space.

4.1 Definition of manifolds

Definition 4.1.1. *Let M be a Hausdorff space. A differentiable structure on M of dimension n is a collection of open charts $(U_i, \phi_i)_{i\in\Lambda}$ on M where $\phi_i(U_i)$ is an open subset of $\mathbb{R}^n$, such that the following conditions are satisfied:*

1. $M = \bigcup_{i\in\Lambda} U_i$.
2. *For each pair $i, j \in \Lambda$, the mapping $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_j \cap U_i)$ is a differentiable mapping.*
3. *The collection $(U_i, \phi_i)_{i\in\Lambda}$ is a maximal family of open charts for which conditions 1 and 2 hold.*

4.2 Local coordinate system

Definition 4.2.1. *A differentiable manifold (or C^∞-manifold, or simply a manifold) of dimension n is a Hausdorff space with a differentiable structure of dimension n.*

- *If M is a manifold, a local coordinate system (or local chart) on M is by definition a pair (ϕ_i, U_i).*
- *If p is a point in U_i and $\phi_i(p) = (x^1(p), ..., x^n(p))$, then U_i is called a coordinate neighborhood of p, and the numbers $x_j(p)$ are called local coordinates of p.*

  The mapping $\phi_i \;:\; q \;\mapsto (x^1(q), ..., x^n(q)), \quad q \in U_i$ is often denoted by $\{x^1, \ldots, x^n\}$.
- *We notice that the condition 3 is not essential in the definition of a manifold.*
- *In fact, if only 1 and 2 are satisfied, the structure is still well-defined.*

4.3 Differentiable curve

Definition 4.3.1. *By a differentiable curve in a manifold M, it shall mean a differentiable mapping of a interval $[a, b]$ of $\mathbb{R}$ into M.*
We shall now define a tangent vector (or singly a vector) at a point p of M.

4.3.1 First definition: Tangent to a curve

Consider now a differentiable manifold M and a point p of M of class C^k. We are interested in differentiable curves in M that pass through p:

$$c : [-\varepsilon, \varepsilon] \to M$$
$$t \mapsto c(t), \quad c(0) = p$$

Definition 4.3.2. *Two curves c_1 and c_2 are tangent at point p if $c_1(0) = c_2(0) = p$ and there exists a local chart (U, φ) such that $p \in U$ and*

$$\frac{d}{dt}(\varphi \circ c_1)(0) = \frac{d}{dt}(\varphi \circ c_2)(0)$$

This defines an equivalence relation (i.e., a relation that is transitive, symmetric, and reflexive) on the set of curves passing through p: $c_1 \sim c_2$ if they are tangent at p.

4.3.2 Second definition: Derivation

Definition 4.3.3. *A tangent vector to M at p is an equivalence class of curves tangent at p.*
The tangent space to M at p, denoted T_pM, is the set of tangent vectors to M at p.

Definition 4.3.4. *Let $U \subset M$ be an open subset such that $p \in U$. We define the set :*

$$C^\infty(p) = \{f : U \to \mathbb{R},\ f \in C^\infty \mid f = g \Leftrightarrow \exists V_p \in \mathcal{V}(p) \text{ such that } f(x) = g(x),\ \forall x \in V_p \subset U\}$$

That is, we consider the set of real-valued C^∞ functions defined on open subsets of M containing a neighborhood of p, and we identify functions that are equal on some neighborhood of p. The set is denoted by $C^\infty(p)$.

Definition 4.3.5. *Let $\mathcal{C}^\infty(p)$ be the algebra of differentiable functions defined in a neighborhood of p. Let $c(t)$ $(a \le t \le b)$ be a curve such that $c(t_0) =: p$.*
The vector tangent to the curve $c(t)$ at p is a mapping: $D_p \ : \mathcal{C}^\infty(p) \ \to \ \mathbb{R}$ defined by

$$D_p = (df(c(t))/dt)_{t_0}$$

In other words, D_pf is the derivative of $\mathcal{C}^\infty(p)$ in the direction of the curve $c(t)$ at $t = t_0$. The vector X satisfies the following conditions:

1. *D_p is a linear mapping of $\mathcal{C}^\infty(p)$ into $\mathbb{R}$;*
2. *$D_p(fg) = (D_pf)g(p) + f(p)(D_pg)$ for $f, g \in \mathcal{C}^\infty(p)$.*
3. *If f is constant, then $D_p(f) = 0$.*

   The set of all derivations at p is called the tangent space of M at p, denoted T_pM. By definition, a tangent vector to M at p is an element of T_pM.

The set of mappings D_p of $\mathcal{C}^{\infty}(p)$ into $\mathbb{R}$ satisfying the preceding two conditions forms a real vector space.
Let $x^1, \ldots, x^n$ be local conditions in a coordinate neighborhood U of p .
For each i, $(\partial/\partial x^i)_p$ *is a mapping of $\mathcal{C}^{\infty}(p)$ into $\mathbb{R}$ which satisfies 1 and 2. Given any curve $c(t)$ with $p = c(t_0)$, let $x^i = c^i(t), i = 1, \ldots, n$, be its equations in terms of the local coordinates $x^1, \ldots, x^n$, Then*

$$(df(c(t))/dt)_{t_0} = \sum_i \left(\partial f/\partial x^i\right)_p \left(dc^i(t)/dt\right)_{t_0}$$

which proves that every vector at p is a linear combination of $(\partial/\partial x^1)_{p_i}, \ldots, (\partial/\partial x^n)_p$. Conversely, given a linear combination $\sum \xi^i (\partial/\partial x^i)_P$, consider the curve defined by

$$x^i = x^i(p) + \xi^i t, \quad i = 1, \ldots, n$$

4.4 The vector tangent

Definition 4.4.1. *Then the vector tangent to this curve at $t = 0$ is $\Sigma\xi^i (\partial/\partial x^i)_p$.*
If we assume $\Sigma\xi^i (\partial/\partial x^i)_P = 0$, then $0 = \Sigma\xi^i (\partial x^j/\partial x^i)_p = \xi^j$ for $j = 1, \ldots, n$.
Therefore, $(\partial/\partial x^1)_p, \ldots, (\partial/\partial x^n)_p$ are linearly independent and hence these form a basis of the set of vectors at p.

4.5 Tangent and cotangent spaces

Definition 4.5.1. *The set of tangent voctors at p denoted by $\mathrm{T_p(M)}$, is called the tangent space of M at p.*
The n-tuple of numbers $\xi^1, \ldots, \xi^n$ are components of the vectors $\sum \xi^i (\partial/\partial x^i)_p$ with respect the local coordinates $x^1, \ldots, x^n$.

*We notice that oe a C^∞ differentiable manifold it the tangent space $\mathrm{T_p(M)}$ coincides with the space of $D_p : \mathcal{C}^{\infty}(p) \longrightarrow$ **R** satisfying the conditions 1 and 2 above.*

4.6 Tangent bundle

Definition 4.6.1. *Let M be a manifold of dimension n. The tangent bundle of M, denoted TM, is the union of the tangent spaces T_pM for all $p \in M$:*

$$TM = \bigcup_{p \in M} T_pM$$

4.7 Vector field

Definition 4.7.1. *A vector field X on a manifold M is anassignment of a vector X_p to each point p of M. If f is a differentiable function on M, then Xf is a function on M defined by*

$$(Xf)(p) = X_p f.$$

A vector field X is said to be differentlable if Xf is differentiable for every differentiable function f. In terms of local coordinates $x^1, \ldots, x^n$, X may be expressed by

$$X = \sum \xi^i \left(\partial / \partial x^i\right)$$

where ξ^i are functions defined in the coordinate neighborhood, called components of X with respect to $x^1, ..., x^n$. X is differentiable if and only if its components ξ^i are differentiable.

Definition 4.7.2. *Let M be a differentiable manifold. A vector field of class C^k on M is a mapping*

$$X : M \to TM$$
$$p \mapsto (p, X_p)$$

- *of class C^k that assigns to each point p of M a tangent vector X_p to M at point p.*
  Where :

$$X_p = \sum_{k=1}^{n} f_k(p) \frac{\partial}{\partial x^k}$$

- *where $p = (x_1, \ldots, x_n)$, and $\left\{\frac{\partial}{\partial x^k}\right\}_{k=1,\ldots,n}$ is the basis of T_pM, with functions $f_k \in C^k(M)$ and $f_k : M \to \mathbb{R}$.*
- *The set of vector fields of class C^∞ on M is denoted by $\mathcal{X}(M)$.*

Definition 4.7.3. *The canonical projection on TM is the projection*

$$\Pi : TM \to M$$
$$(p, X_p) \mapsto p$$

such that $X \circ \Pi = Id_M$.

Remark 4.7.1. *A vector field X is of class C^k in a local chart (Ω, φ) if and only if the functions f_k of X in (U, φ) are of class C^k on U.*

4.8 Lie bracket

Definition 4.8.1. *If X and Y are vector fields, define the bracket $[X, Y]$ as a mapping from the ring of functions on M into itself by*

$$[X, Y]f = X(Yf) - Y(Xf)$$

Let $X = \sum \xi^i \left(\partial / \partial x^i\right)$ and $Y = \eta^j \left(\partial / \partial x^j\right)$. Then

$$[X, Y]f = \sum_{i,j} \left(\xi^j \left(\partial \eta^i / \partial x^j\right) - \eta^j \left(\partial \xi^i / \partial x^j\right) \left(\partial f / \partial x^i\right)\right).$$

This means that $[X, Y]$ is a vector field with carponents $\sum_j \left(\xi^j \left(\partial \eta^i / \partial x^j\right) - \eta^j \left(\partial \xi^i / \partial x^j\right)\right)$ $i = 1, \ldots, n$. With respect to this bracket operation,
*$\mathfrak{X}(M)$ is a Lie algebra over **R**. For any vector rields X, Y and Z, we have the Jacobi identity:*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

We may also regard $\mathfrak{X}(M)$ as a module over the algebra $\mathcal{F}(M)$ of differentiable functions on M as follows. If is is function and X is a vectior field on M, then fX is a vector field on M defined by $(fX) = f(p)X$; for $p \in M$. We also have

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

Lemma 4.8.1. *Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for all $f \in D(M)$,*

$$Zf = (XY - YX)f,$$

where $D(M)$ is the set of all differentiable functions on M.

The vector field Z given by this lemma is called the bracket of X and Y, noted

$$[X, Y] = XY - YX.$$

Proposition 4.8.1. *If X, Y, and Z are differentiable vector fields on M, a and b are real numbers, and f, g are differentiable functions, then:*

(a) $[X, Y] = -[Y, X]$ *(anti-commutativity),*

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(linearity),*

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity),*

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$

Differential mapping

Let M^n and N^k be differentiable manifolds of dimensions n and k, respectively. Let $F : M \to N$ be a differentiable map.

Definition 4.8.2 (Pullback)**.** *Let $g : N \to \mathbb{R}$. The pullback of g by F is the function:*

$$F^* : C^\infty(F(p)) \to C^\infty(p), \quad g \mapsto F^*g := g \circ F$$

Definition 4.8.3 (Differential)**.** *The differential of F at $p \in M$ is the linear map:*

$$dF_p : T_pM \to T_{F(p)}N, \quad X_p \mapsto dF_p(X_p)$$

such that:

$$dF_p(X_p) \cdot g := X_p \cdot (F^*g), \quad \forall g \in C^\infty(F(p))$$

Theorem 4.8.1. *(Composition theorem) If $F : M \to N$ and $G : N \to W$ are differentiable maps, then $G \circ F$ is differentiable at $p \in M$, and:*

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p$$

Corollary 4.8.1. *If $F : M \to N$ is a diffeomorphism, then dF_p is an isomorphism at every point $p \in M$. The converse is only true locally.*

Theorem 4.8.2. *(Local diffeomorphism) A map $F : M \to N$ is a local diffeomorphism at $p \in M$ if there exist neighborhoods $U \subset M$, $V \subset N$ such that:*

$$F|_U : U \to V$$

is a diffeomorphism.

Theorem 4.8.3. *(Local inversion theorem) Let $F : M \to N$ be differentiable at $p \in M$, and suppose:*

$$dF_p : T_pM \to T_{F(p)}N$$

is an isomorphism. Then F is a local diffeomorphism at p, and:

$$d(F|_U)^{-1} = (dF_p)^{-1}$$

This provides a key result for defining local coordinates. A chart $\varphi : M \to \mathbb{R}^n$ defines local coordinates at p if and only if $d\varphi_p$ is an isomorphism.

4.9 Cotangent space

Definition 4.9.1. *Let M be a manifold of dimension n, p a point in M, and (U, φ) a chart of M at p with associated coordinates $(x^1, \dots, x^n)$.*
We denote by T_p^*M the dual space of T_pM, and for each $i = 1, \dots, n$, we denote by $dx^i|_p$ the differential 1-form at p in T_p^*M, which is defined by:

$$dx^i|_p \left(\frac{\partial}{\partial x^j} \right)_p = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. The family $\{dx^i|_p\}_{i=1,\dots,n}$ forms a basis of T_p^*M.

Remark 4.9.1. *From now on, we omit the letter p and simply denote the family $\{dx^i|_p\}_{i=1,\dots,n}$ as $\{dx^i\}_{i=1,\dots,n}$.*

Definition 4.9.2. *The cotangent bundle of M, denoted T^*M, is defined as the disjoint union of the cotangent spaces T_p^*M for all $p \in M$:*

$$T^*M = \bigcup_{p \in M} T_p^*M$$

Theorem 4.9.1. *Let M be a differentiable manifold of dimension n. The cotangent bundle T^*M has a natural structure as a differentiable manifold of dimension $2n$.*

Definition 4.9.3. *A 1-form (or covector) at $p \in M$ is a linear form on T_pM, i.e., a linear map:*

$$\omega_p : \begin{array}{ll} T_pM & \to \mathbb{R} \\ X_p & \mapsto \omega_p(X_p) \end{array}$$

We denote $\omega_p(X_p) = \langle \omega_p, X_p \rangle$, where the bracket denotes the duality pairing.

- *The cotangent space to M at p, denoted T_p^*M, is the vector space of 1-forms at p.*
- *It is the dual vector space of T_pM, i.e., $T_p^*M = (T_pM)^*$.*

Example 4.9.1. *1. The tangent bundle of $\mathbb{R}^n$ admits a global trivialization: $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$, via the canonical identification $T_p\mathbb{R}^n \approx \mathbb{R}^n$.*

2. *The tangent bundle of the circle S^1 admits a global trivialization since it is diffeomorphic to a cylinder: $TS^1 \approx S^1 \times \mathbb{R}$. However, the tangent bundle TS^2 does not admit a global trivialization.*

Remark 4.9.2. *Using differentials of charts, we will extend local coordinate calculations to the tangent space. We start with the case of $\mathbb{R}^n$.*

Case of the tangent space $T_x\mathbb{R}^n$

We have the property that $T_x\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$, and we can identify it with the set of partial derivatives at x, i.e.,

Let $x \in \mathbb{R}^n$, the partial derivatives at x are given by the derivations on $\mathbb{R}^n$:

$$\left.\frac{\partial}{\partial x^i}\right|_x : g \mapsto \frac{\partial g}{\partial x^i}(x).$$

Among all directional derivatives at x, these partial derivatives form a basis, which is also a basis of the tangent space $T_x\mathbb{R}^n$, called the canonical basis *or the* natural basis*. Thus, any tangent vector $v_x \in T_x\mathbb{R}^n$ can be written as:*

$$v_x = v^1 \left.\frac{\partial}{\partial x^1}\right|_x + \cdots + v^n \left.\frac{\partial}{\partial x^n}\right|_x .$$

Remark 4.9.3. *This vector is also the equivalence class of curves $c(t)$ passing through x such that*

$$\dot{c} = (v^1, \ldots, v^n).$$

Hence, we have the canonical identification $T_x\mathbb{R}^n \simeq \mathbb{R}^n$, given by:

$$v_x \mapsto (v^1, \ldots, v^n).$$

Case of the tangent space T_pM

In the case of abstract differentiable manifolds, everything is related to local charts and the fact that chart maps are diffeomorphisms in the domain of the chart.

Let $p \in M$ and (U, φ) be a chart of M such that $p \in U$ (i.e., U is a neighborhood of p). Then:

$$\varphi : U \to \varphi(U) \subset \mathbb{R}^n$$

is a diffeomorphism, hence:

$$d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$$

is invertible, and

$$(d\varphi_p)^{-1} = d(\varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^n \to T_pM$$

is an isomorphism.

Furthermore, if we let $x = \varphi(p) \in \mathbb{R}^n$, then:

$$\left.\frac{\partial}{\partial x^1}\right|_x, \ldots, \left.\frac{\partial}{\partial x^n}\right|_x$$

is the canonical basis of $T_x\mathbb{R}^n$. We compute the image of this basis under the isomorphism $(d\varphi_p)^{-1}$, which we denote using the same notation:

$$\left.\frac{\partial}{\partial x^i}\right|_p := d(\varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_x \right), \quad i = 1, \ldots, n.$$

With these tangent vectors, we construct a basis of T_pM called the natural basis associated with the local coordinates φ*:*

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right).$$

Remark 4.9.4. *If $g \in C^\infty(M)$, then for each $i = 1, \dots, n$, we have:*

$$\frac{\partial}{\partial x^i}\bigg|_p \cdot g = d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_x \right) \cdot g = \frac{\partial}{\partial x^i}\bigg|_x \cdot (g \circ \varphi^{-1}) = \frac{\partial (g \circ \varphi^{-1})}{\partial x^i}(x).$$

5

Integration

*The references for this chapter are [**55**, **25**, **52**, **7**].*

On a manifold one integrates not functions as in calculus on $\mathbb{R}^n$ but differential forms. There are actually two theories of integration on manifolds, one in which the integration is over a submanifold and the other in which the integration is over what is called a singular chain. Singular chains allow one to integrate over an object such as a closed rectangle in $\mathbb{R}^2$:

$$[a,b]\times[c,d] := \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

which is not a submanifold of $\mathbb{R}^2$ because of its corners For integration over a manifold to be well defined, the manifold needs to be oriented. We begin the chapter with a discussion of orientations on a manifold. We then enlarge the category of manifolds to include manifolds with boundary. Our treatment of integration culminates in Stokes's theorem for an n-dimensional manifold. Stokes's theorem for a surface with boundary in $\mathbb{R}^3$ was first published as a question in the Smith's Prize Exam that Stokes set at the University of Cambridge in 1854. It is not known whether any student solved the problem. According to the same theorem had appeared four years earlier in a letter of Lord Kelvin to Stokes, which only goes to confirm that the attribution of credit in mathematics is fraught with pitfalls. Stokes's theorem for a general manifold resulted from the work of many mathematicians, including Vito Volterra (1889), Henri Poincare (1899), Edouard Goursat (1917), and Elie Cartan (1899 and 1922). First there were many special cases, then a general statement in terms of coordinates, and finally a general statement in terms of differential forms. Cartan was the master of differential forms with excellence, and it was in his work that the differential form version of Stokes's theorem found its clearest expression.

5.1 Orientations on a manifolds

While the definition of an orientation on a manifold as a continuous pointwise orientation is geometrically intuitive, in practice it is easier to manipulate the nowherevanishing top forms that specify a pointwise orientation. In this section we show that the continuity condition on pointwise orientations translates to a C^∞ condition on nowhere-vanishing top forms.

If f is a real-valued function on a set M, we use the notation $f > 0$ to mean that f is everywhere positive on M.

Lemma 5.1.1. *A pointwise orientation $[(X_1, \ldots, X_n)]$ on a manifold M is continuous if and only if each point $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which the function $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n)$ is everywhere positive.*

Theorem 5.1.1. *A manifold M of dimension n is orientable if and only if there exists a C^∞ nowhere-vanishing n-form on M.*

Proposition 5.1.1. *Let U and V be open subsets of $\mathbb{R}^n$, both with the standard orientation inherited from $\mathbb{R}^n$.*

Definition 5.1.1. *A diffeomorphism $F \; : U \longrightarrow V$ is orientation-preserving if and only if the Jacobian determinant $\det[\partial F^i/\partial x^j]$ is everywhere positive on U.*

5.2 Oriented atlas

Using the characterization of an orientation-preserving diffeomorphism by the sign of its Jacobian determinant, we can describe orientability of manifolds in terms of atlases.

Definition 5.2.1. *An atlas on M is said to be oriented if for any two overlapping charts $(U, x^1, ..., x^n)$ and $(V, y^1, ..., y^n)$ of the atlas, the Jacobian determinant $\det[\partial F^i/\partial x^j]$ is everywhere positive non-null on $U \cap V$.*

Theorem 5.2.1. *A manifold M is orientable if and only if it has an oriented atlas.*

Definition 5.2.2. *Two oriented atlases $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$ on a manifold M are said to be equivalent if the transition functions*

$$\phi_i \circ \psi_j^{-1} \;\; : \psi_j(U_i \cap V_j) \longrightarrow \phi_i(U_i \cap V_j)$$

have positive Jacobian determinant for all i, j

make Stokes's theorem sign-free.

5.3 Manifolds with boundary

Example of a manifold with boundary is the closed upper half-space

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\},$$

with the subspace topology inherited from $\mathbb{R}^n$.
The points $(x^1, \ldots, x^n) \in H^n$ with $x^n > 0$ are called the interior points of H^n, and the points with $x^n = 0$ are called the boundary points of H^n. These two sets are denoted by $(H^n)^\circ$ and ∂H^n, respectively.

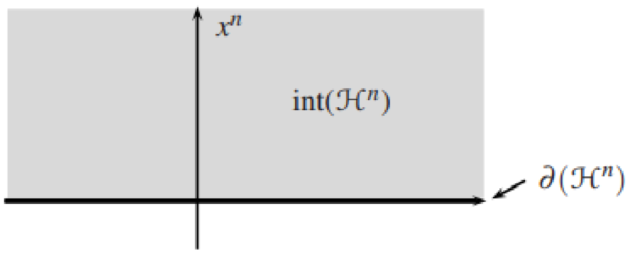


Figure 5.1: Upper half-space

In the literature, the upper half-space often means the open set

$$\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}.$$

We require that H^n include the boundary in order for it to serve as a model for manifolds with boundary.

If M is a manifold with boundary, then its boundary ∂M turns out to be a manifold of dimension one less without boundary.

Moreover, an orientation on M induces an orientation on ∂M. The choice of the induced orientation on the boundary is a matter of convention, guided by the desire to make Stokes's theorem sign-free.

Proposition 5.3.1. *Let U and V be open subsets of the upper half-space H^n and $F : U \longrightarrow V$ a diffeomorphism. Then f maps interior points to interior points and boundary points to boundary points.*

Manifolds with boundary

In the upper half-space H^n one may distinguish two kinds of open subsets, depending on whether the set is disjoint from the boundary or intersects the boundary(Figure 5.2) . Charts on a manifold are homeomorphic to only the first kind of open sets. A manifold with boundary generalizes the definition

Figure 5.2: Two types of open subsets of H^n

of a manifold by allowing both kinds of open sets. We say that a topological space M is locally H^n if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of H^n.

Definition 5.3.1. *A topological n-manifold with boundary is a second countable, Hausdorff topological space that is locally H^n. Let M be a topological n-manifold with boundary.*
For $n \geqslant 2$, a chart on M is defined to be a pair (U, ϕ) consisting of an open set U in M and a homeomorphism

$$\varphi : U \to \varphi(U) \subset H^n$$

of U with an open subset $\phi(U)$ of H^n.

5.4 The boundary of a manifold with boundary

Let M be a manifold of dimension n with boundary ∂M. If (U, φ) is a chart on M, we denote by

$$\varphi' = \varphi|_{U \cap \partial M}$$

the restriction of the coordinate map φ to the boundary. Since φ maps boundary points to boundary points,

$$\varphi' : U \cap \partial M \to \partial H^n = \mathbb{R}^{n-1}.$$

Moreover, if (U, φ) and (V, ψ) are two charts on M, then

$$\psi' \circ (\varphi')^{-1} : \varphi'(U \cap V \cap \partial M) \to \psi'(U \cap V \cap \partial M)$$

is C^∞. Thus, an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M induces an atlas

$$\{(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M})\}$$

for ∂M, making ∂M into a manifold of dimension $n-1$ without boundary.

5.5 Tangent vectors, differential forms, and orientations

Let M be a smooth manifold with boundary and let $p \in \partial M$. consider two smooth functions:

$$f : U \to \mathbb{R}, \quad g : V \to \mathbb{R}$$

defined on neighborhoods U and V of p in M. We say that f and g are equivalent *if there exists an open neighborhood $W \subseteq U \cap V$ of p such that:*

$$f|_W = g|_W.$$

Example 5.5.1. *Let $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ denote the upper half-plane, and let $p = (x, 0) \in \partial H^2$. Then both:*

$$\frac{\partial}{\partial x}\Big|_p, \quad \frac{\partial}{\partial y}\Big|_p$$

are point derivations on $C_p^\infty(H^2)$, and thus elements of $T_p(H^2)$.

The tangent space $T_p(H^2)$ is a 2-dimensional real vector space, visualized as being centered at the point p.
Since $\frac{\partial}{\partial y}\big|_p$ is a tangent vector at the boundary, its negative $-\frac{\partial}{\partial y}\big|_p$ is also a valid tangent vector at p, even though there is no smooth curve within H^2 passing through p with initial velocity $-\frac{\partial}{\partial y}\big|_p$.

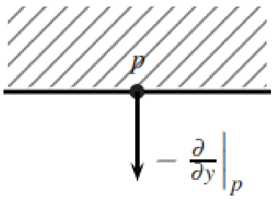


Figure 5.3: A tangent vector at the boundary.

Definition 5.5.1. *The **cotangent space** T_p^*M at $p \in M$ is defined as the dual vector space:*

$$T_p^*M := Hom(T_pM, \mathbb{R}),$$

consisting of all linear functionals from T_pM to $\mathbb{R}$.

*Differential k-forms on M are defined as sections of the bundle $\Lambda^k(T^*M)$, the k-th exterior power of the cotangent bundle. A differential k-form ω is said to be **smooth** if it is a smooth section of this vector bundle.*

Example 5.5.2. *On the upper half-plane H^2, the differential form:*

$$dx \wedge dy$$

is a smooth 2-form.

Definition 5.5.2. *An **orientation** on an n-manifold M with boundary is a continuous pointwise choice of orientation of the tangent space T_pM at each point $p \in M$. This means choosing a nowhere-vanishing section of the top exterior power $\Lambda^n(T^*M)$, up to positive scalar multiplication.*

Such orientations extend naturally to the boundary and play a crucial role in the formulation of Stokes's Theorem.

5.6 Outward-pointing vector fields

Let M be a smooth manifold with boundary, and let $p \in \partial M$. To understand how orientation behaves at the boundary particularly in the context of Stokes's Theorem we need to distinguish between inward-pointing *and* outward-pointing *tangent vectors at boundary points.*

*A tangent vector $X_p \in T_p(M)$ is called **inward-pointing** if:*

- *it is not tangent to the boundary, i.e., $X_p \notin T_p(\partial M)$, and*
- *there exists a smooth curve $c : [0, \varepsilon[\to M$ such that:*

$$c(0) = p, \quad c((0, \varepsilon[) \subset M^\circ, \quad c'(0) = X_p,$$

  meaning that the curve lies in the interior *of M for all $t > 0$, and its initial velocity at $t = 0$ is X_p.*

*Conversely, a vector X_p is called **outward-pointing** if $-X_p$ is inward-pointing. That is, an outward-pointing vector points* away *from the interior of M and toward the "outside" of the boundary.*

Example 5.6.1. *In the upper half-plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, the point $p = (x, 0)$ lies on the boundary (the x-axis). Then:*

- $\frac{\partial}{\partial y}\Big|_p$ *is* inward-pointing*,*
- $-\frac{\partial}{\partial y}\Big|_p$ *is* outward-pointing.

More generally, given a local coordinate system $(U, (x^1, \ldots, x^n))$ around a point $p \in \partial M$, a tangent vector $X_p \in T_pM$ can be expressed as:

$$X_p = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}\Big|_p.$$

In this setup, we can identify whether a vector is outward-pointing based on the sign of the component in the direction normal *to the boundary. If the coordinate x^n increases in the inward direction (as is conventional), then a vector is outward-pointing at p if and only if:*

$$a_n(p) < 0.$$

Definition 5.6.1. *A **vector field along the boundary** ∂M is a smooth assignment:*

$$X : \partial M \to TM,$$

such that $X_p \in T_pM$ for each $p \in \partial M$. Unlike a vector field on *∂M, this one is valued in the ambient manifold's tangent bundle , that is, it allows components normal to the boundary.*

Definition 5.6.2. *A vector field X along ∂M is said to be **smooth at** $p \in \partial M$ if there is a coordinate chart around p such that all coefficients a_i of the expression*

$$X_q = \sum_{i=1}^{n} a_i(q) \frac{\partial}{\partial x^i}\Big|_q$$

*are smooth functions of $q \in \partial M$. The field is called **smooth** if it is smooth at every point of the boundary.*

Proposition 5.6.1. *Every smooth manifold M with boundary admits a smooth outward-pointing vector field along ∂M.*

Such vector fields are crucial in defining a consistent boundary orientation and play a key role in expressing and proving Stokes's Theorem without sign ambiguity. Intuitively, they provide a "direction" for the boundary to face outward, helping to relate integrals over the boundary to those over the interior.

6

Integration on manifolds

*The references for this chapter are [**25**, **52**.]*

Theorem 6.0.1. *(Lebesgue's theorem)*
A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set Disc$(\tilde{f})$ of discontinuities of the extended function $\tilde{f}$ has measure zero.

Proposition 6.0.1. *If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.*

6.1 The integral of an n-form on $\mathbb{R}^n$

Once a set of coordinates $x^1, \ldots, x^n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n$$

for a unique function $f(x)$ on $\mathbb{R}^n$. In this way, the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 6.1.1. *Let $\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n$ be a smooth n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x^1, \ldots, x^n$. Its integral over a subset $A \subset U$ is defined to be the Riemann integral of $f(x)$:*

$$\int_A \omega = \int_A f(x)\, dx^1 \cdots dx^n,$$

if the Riemann integral exists.

If the n-form is written in the order $dx^1 \wedge \cdots \wedge dx^n$, to integrate, for example, $\tau = f(x)\, dx_2 \wedge dx_1$ over $A \subset \mathbb{R}^2$, we would write

$$\int_A \tau = -\int_A f(x)\, dx^1 \wedge dx^2 = -\int_A f(x)\, dx^1\, dx^2.$$

6.2 Transformation of n-forms under change of variables

Let $T : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$ be a diffeomorphism. Let $x^1, \ldots, x^n$ be the standard coordinates on U, and $y^1, \ldots, y^n$ the standard coordinates on V. Then $T^i := x^i \circ T$ is the i-th component of T. Denote by $J(T)$ the Jacobian matrix $\left[\frac{\partial T^i}{\partial y^j}\right]$.

$$dT^1 \wedge \cdots \wedge dT^n = \det(J(T))\, dy^1 \wedge \cdots \wedge dy^n.$$

Hence:

$$\begin{aligned} \int_V T^*\omega &= \int_V (T^*f)\, T^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= \int_V (f \circ T)\, dT^1 \wedge \cdots \wedge dT^n \qquad \textit{(because } T^*d = dT^*\textit{)} \\ &= \int_V (f \circ T)\, \det(JT)\, dy^1 \wedge \cdots \wedge dy^n \\ &= \int_V (f \circ T)\, \det(JT)\, dy^1 \cdots dy^n. \end{aligned} \tag{6.1}$$

On the other hand, the change-of-variables formula from advanced calculus gives:

$$\int_U \omega = \int_U f\, dx^1 \cdots dx^n = \int_V (f \circ T)|\det(J(T))|\, dy^1 \cdots dy^n.$$

Therefore, the relationship between the integrals is:

$$\int_V T^*\omega = \pm \int_U \omega,$$

depending on the sign of the Jacobian determinant $\det(J(T))$.
A diffeomorphism T is orientation-preserving if and only if $\det(J(T))$ *is positive everywhere on V.*

Definition 6.2.1. *The* support *of a differential form ω on M is the* closure *of the set of points $x \in M$ for which $\omega_x \neq 0$ in $\wedge^q(T^*_x M)$.*

We denote by $\Omega_c(M)$ the subalgebra of $\Omega(M)$ consisting of differential forms with compact support.

$$\operatorname{supp}(\omega) = \{x \in M \mid \omega_x \neq 0\}$$

6.3 Integral of a differential form over a manifold

Integration of an n-form over $\mathbb{R}^n$ is not so different from integration of a function. The integration over a manifold has several distinguishing features:

- *The manifold must be oriented (in fact, $\mathbb{R}^n$ has a standard orientation).*
- *On a manifold of dimension n, one can integrate only n-forms, not functions.*
- *The n-forms must have compact support.*

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$ giving the orientation of M. Denote by $\Omega^k_c(M)$ the vector space of smooth k-forms with compact support on M.

If $\omega \in \Omega^n_c(U)$ is an n-form with compact support on U, then because $\varphi : U \to \varphi(U)$ is a diffeomorphism, $(\varphi^{-1})^*\omega$ is an n-form with compact support on the open subset $\varphi(U) \subset \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\varphi \circ \psi^{-1} : \psi(U) \to \varphi(U)$ is an orientation-preserving diffeomorphism. Thus,

$$\int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_{\psi(U)} (\psi^{-1})^* \omega,$$

proving that the integral on a chart U is well-defined, independent of the choice of coordinates.

Finally, the integral of ω over M is defined by using a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. The integral is:

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.$$

Proposition 6.3.1. *Let ω be an n-form with compact support on an oriented manifold M of dimension n. If $-M$ denotes the same manifold but with the opposite orientation, then:*

$$\int_{-M} \omega = -\int_M \omega.$$

Thus, reversing the orientation of M reverses the sign of an integral over M.

Definition 6.3.1. *A parametrized set in an oriented n-manifold M is a subset A together with a C^∞ map $F : D \to M$ from a compact domain of integration $D \subset \mathbb{R}^n$ to M such that $A = F(D)$ and F restricts to an orientation-preserving diffeomorphism from int(D) to $F($int$(D))$. Note that by smooth invariance of domain for manifolds (Remark 22.5), $F($int$(D))$ is an open subset of M. The C^∞ map $F : D \to A$ is called a parametrization of A.*

If A is a parametrized set in M with parametrization $F : D \to A$ and ω is a C^∞ n-form on M, not necessarily with compact support, then we define

$$\int_A \omega \text{ to be } \int_D F^* \omega.$$

It can be shown that the definition of $\int_A \omega$ is independent of the parametrization and that in case A is a manifold, it agrees with the earlier definition of integration over a manifold. Subdividing an oriented manifold into a union of parametrized sets can be an effective method of calculating an integral over the manifold. We will not delve into this theory of integration .

6.4 Poincaré's lemma

Definition 6.4.1. *A set $U \subset \mathbb{R}^n$ is said to be **star-shaped** at $x_0 \in U$ if:*

$$\forall x \in U, \forall t \in [0, 1] : (1 - t)x_0 + tx \in U.$$

Lemma 6.4.1. *Let $U \subset \mathbb{R}^n$ be an open star-shaped set at 0, and let X be a vector field defined by:*

$$X_x = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$$

If we consider the application:

$$\begin{aligned} \xi : \Omega^{p+1}(U) &\longrightarrow \Omega^p(U) \\ \omega &\longmapsto \int_0^1 t^p \, i_X(\omega)(tx) \, dt \end{aligned}$$

Then ξ is a linear map such that:

$$d\xi(\omega) + \xi(d\omega) = \omega.$$

Proof 6.4.1. *It is sufficient to prove the lemma for $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}}$. We have:*

$$\xi(\omega) = \sum_{j=1}^{p+1} (-1)^{j-1} \left(\int_0^1 t^p f(tx) x^{i_j} dt \right) dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_{p+1}}.$$

By differentiating under the integral sign:

$$d\left(\int_0^1 t^p f(tx) x^{i_j} dt \right) = \left(\int_0^1 t^p f(tx) x^{i_j} dt \right) dx^{i_j} + \sum_{i=1}^{n} \left(\int_0^1 t^{p+1} \frac{\partial f}{\partial x^i}(tx) x^{i_j} dt \right) dx^i.$$

We deduce:

$$\begin{aligned} d\xi(\omega) =& \sum_{j=1}^{p+1} (-1)^{j-1} \left[\left(\int_0^1 t^p f(tx) dt \right) x^{i_j} + \sum_{i=1}^{n} \left(\int_0^1 t^p f(tx) x^{i_j} dt \right) dx^i \right] \\ & \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_{p+1}}. \\ =& (p+1) \left(\int_0^1 t^p f(tx) dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \\ & + \sum_{j=1}^{p+1} (-1)^{j-1} \sum_{i=1}^{n} \left(\int_0^1 t^p f(tx) x^{i_j} dt \right) dx^i \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_{p+1}}. \end{aligned}$$

On the other hand, we have:

$$d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}}.$$

$$i_X(dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}}) = x^i dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} + \sum_{j=1}^{p+1} (-1)^j x^{i_j} dx^i \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_{p+1}}.$$

Thus:

$$\begin{aligned} \xi(d\omega) &= \sum_{i=1}^{n} \xi \left(\frac{\partial f}{\partial x^i}(tx)\, dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \right) \\ &= \sum_{i=1}^{n} \int_0^1 t^{p+1} \frac{\partial f}{\partial x^i}(tx)\, i_X(dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}})\, dt \end{aligned}$$

$$= \sum_{i=1}^{n} \int_0^1 t^{p+1} \frac{\partial f}{\partial x^i}(tx) \left(x^i dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} + \sum_{j=1}^{p+1} (-1)^j x^{i_j} dx^i \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_{p+1}} \right) dt.$$

Summing the previous equations, we obtain:

$$\begin{aligned} d\xi(\omega) + \xi(d\omega) &= \int_0^1 \left((p+1) t^p f(tx) + \sum_{i=1}^{n} t^{p+1} \frac{\partial f}{\partial x^i}(tx)\, x^i \right) dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \\ &= \left[\int_0^1 \left(t^{p+1} f(tx) \right)' dt \right] dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \\ &= \left[t^{p+1} f(tx) \right]_0^1 dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \\ &= f(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} = \omega \end{aligned}$$

Lemma 6.4.2. *(Poincaré)*

Let $U \subset \mathbb{R}^n$ be an open star-shaped set. If ω is a closed p-differential form, then ω is an exact p-differential form.

Proof 6.4.2. *We can assume that U is star-shaped at 0. Since:*

$$d\xi(\omega) + \xi(d\omega) = \omega.$$

As ω is closed, i.e., $d\omega = 0$, we deduce:

$$d\xi(\omega) = \omega.$$

6.5 Stokes's theorem

Definition 6.5.1. *Let M be an oriented manifold of dimension n with boundary. We give its boundary ∂M the boundary orientation and let $i : \partial M \hookrightarrow M$ be the inclusion map. If ω is an $(n-1)$-form on M, it is customary to write $\int_{\partial M} \omega$ instead of $\int_{\partial M} i^*\omega$.*

Definition 6.5.2. *(Stokes's theorem) For any smooth $(n-1)$-form ω with compact support on the oriented n-dimensional manifold M,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof 6.5.1. *Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or H^n via an orientation-preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ and any half-disk containing its boundary diameter is diffeomorphic to H^n .*
Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$.
As we showed in the preceding section, the $(n-1)$-form $\rho_\alpha\omega$ has compact support in U_α.

Suppose Stokes's theorem holds for $\mathbb{R}^n$ and for H^n.
Then it holds for all the charts U_α in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or H^n. Also, note that

$$(\partial M) \cap U_\alpha = \partial U_\alpha.$$

Therefore,

$$\int_{\partial M} \omega = \int_{\partial M} \sum_\alpha \rho_\alpha\omega = \sum_\alpha \int_{\partial M} \rho_\alpha\omega \quad (\text{since } \sum_\alpha \rho_\alpha = 1).$$

The sum $\sum_\alpha \rho_\alpha\omega$ is finite. This becomes

$$\sum_\alpha \int_{\partial U_\alpha} \rho_\alpha\omega \quad (\text{since supp } \rho_\alpha\omega \text{ is contained in } U_\alpha).$$

Now apply Stokes's theorem to each U_α:

$$\sum_\alpha \int_{U_\alpha} d(\rho_\alpha\omega) = \int_M d\left(\sum_\alpha \rho_\alpha\omega\right).$$

Thus, we have shown that

$$\int_{\partial M} \omega = \int_M d\omega.$$

It suffices to prove Stokes's theorem for $\mathbb{R}^n$ and for H^n.

6.5.1 Proof of Stokes's theorem for the upper half-plane $\mathbf{H}^2$

Let x, y be the coordinates on H^2. The standard orientation on H^2 is given by $dx \wedge dy$, and the boundary orientation on ∂H^2 is given by $i_{\frac{\partial}{\partial y}}(dx \wedge dy) = dx$.

The form ω is a linear combination

$$\omega = f(x, y)dx + g(x, y)dy$$

for C^∞ functions f, g with compact support in H^2. Since the supports of f and g are compact, we may choose a real number $a > 0$ large enough that the supports of f and g are contained in the interior of the square $[-a, a] \times [0, a]$. We will use the notation f_x, f_y to denote the partial derivatives of f with respect to x and y, respectively. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = (g_x - f_y)dx \wedge dy.$$

Now calculate

$$\int_{H^2} d\omega = \int_{H^2} g_x \, dx \, dy - \int_{H^2} f_y \, dx \, dy.$$

We can express this as

$$\int_0^\infty \int_{-\infty}^\infty g_x \, dx \, dy - \int_{-\infty}^\infty \int_0^\infty f_y \, dy \, dx = \int_0^a \int_{-a}^a g_x \, dx \, dy - \int_{-a}^a \int_0^a f_y \, dy \, dx. \tag{6.2}$$

In this expression,

$$\int_{-a}^a g_x(x, y) \, dx = g(x, y)\Big|_{x=-a}^{x=a} = 0$$

because supp g lies in the interior of $[-a, a] \times [0, a]$. Similarly,

$$\int_0^a f_y(x, y) \, dy = f(x, 0)\Big|_{y=0}^{y=a} = -f(x, 0),$$

so this expression becomes

$$\int_{H^2} d\omega = \int_{-a}^a f(x, 0) \, dx.$$

On the other hand, ∂H^2 is the x-axis, and $dy = 0$ on ∂H^2. It follows from (6.2) *that $\omega = f(x, 0)dx$ when restricted to ∂H^2, and*

$$\int_{\partial H^2} \omega = \int_{-a}^a f(x, 0) \, dx.$$

This completes the proof of Stokes's theorem for the upper half-plane.

6.5.2 Stokes's Theorem on $\mathbb{R}^n$

By Fubini's theorem:

$$\begin{aligned}
\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} (-1)^{\alpha-1} \frac{\partial f_\alpha}{\partial x_\alpha} \, dx_1 \ldots dx_n \\
&= (-1)^{\alpha-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^\infty \frac{\partial f}{\partial x_\alpha} \, dx_\alpha \right) dx_1 \ldots dx_{\alpha-1} dx_{\alpha+1} \ldots dx_n \\
&= (-1)^{\alpha-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-a}^a \frac{\partial f}{\partial x_\alpha} \, dx_\alpha \right) dx_1 \ldots dx_{\alpha-1} dx_{\alpha+1} \ldots dx_n.
\end{aligned}$$

But,

$$\int_{-a}^{a} \frac{\partial f}{\partial x_\alpha} dx_\alpha = f(x_1, \dots, x_{\alpha-1}, a, \dots, x_n) - f(x_1, \dots, x_{\alpha-1}, -a, \dots, x_n) = 0 - 0 = 0.$$

Therefore,

$$\int_{\mathbb{R}^n} d\omega = 0.$$

On the other hand, we have:

$$\int_{\partial \mathbf{R}^n} \omega = \int_{\emptyset} \omega = 0,$$

since the boundary of $\mathbb{R}^n$ *is empty. This verifies Stokes' theorem for* $\mathbb{R}^n$.

6.5.3 Stokes's Theorem on $\mathbf{H}^n$

The case when $\alpha \neq n$:

$$\begin{aligned}
\int_{\mathbf{H}^n} d\omega &= (-1)^{\alpha-1} \int_{\mathbf{H}^n} \frac{\partial f}{\partial x_\alpha} \, dx_1 \dots dx_n \\
&= (-1)^{\alpha-1} \int_{\mathbf{H}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_\alpha} \, dx_\alpha \right) dx_1 \dots dx_{\alpha-1} dx_{\alpha+1} \dots dx_n \\
&= (-1)^{\alpha-1} \int_{\mathbf{H}^{n-1}} \left(\int_{-a}^{a} \frac{\partial f}{\partial x_\alpha} \, dx_\alpha \right) dx_1 \dots dx_{\alpha-1} dx_{\alpha+1} \dots dx_n \\
&= 0.
\end{aligned}$$

For the same reason as in the case of $\mathbb{R}^n$, *by definition of* $\partial \mathbf{H}^n$, *the* 1*-form* dx_n *is identically zero. Since* $\alpha \neq n$:

$$\omega = f \, dx_1 \wedge \dots \wedge dx_\alpha \wedge \dots \wedge dx_n \equiv 0 \textit{ on } \partial \mathbf{H}^n.$$

Thus,

$$\int_{\partial \mathbf{H}^n} \omega = 0.$$

The case $i = n$:

$$\int_{\mathbf{H}^n} d\omega = (-1)^{n-1} \int_{\mathbf{H}^n} \frac{\partial f}{\partial x_n} \, dx_1 \dots dx_n = (-1)^{n-1} \int_{\mathbf{R}^{n-1}} \left(\int_{0}^{\infty} \frac{\partial f}{\partial x_n} \, dx_n \right) dx_1 \dots dx_{n-1}.$$

On the other hand, we have:

$$\int_{0}^{\infty} \frac{\partial f}{\partial x_n} dx_n = \int_{0}^{a} \frac{\partial f}{\partial x_n} dx_n = f(x_1, \dots, x_{n-1}, a) - f(x_1, \dots, x_{n-1}, 0) = -f(x_1, \dots, x_{n-1}, 0).$$

$$\int_{\mathbf{H}^n} d\omega = (-1)^n \int_{\mathbf{R}^{n-1}} f(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{n-1} = \int_{\partial \mathbf{H}^n} \omega.$$

Since $(-1)^n \mathbb{R}^{n-1}$ *is precisely* $\partial \mathbf{H}^n$ *with its oriented boundary.*
Thus, Stokes' theorem also holds in this case.

Example 6.5.1. *Let* $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *and* $\omega = xdx + ydy$.
We have:

1. D *is a compact submanifold with boundary* $\partial D = \mathbf{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
2. *If* $\varphi : t \in [0, 2\pi] \to (\cos t, \sin t) \in \mathbb{R}^2$ *is a parameterization of the submanifold* $\mathbf{S}^1$*, then:*

$$\varphi^*(\omega) = 0 \quad \textit{and} \quad \int_{\mathbf{S}^1} \omega = \int_{\varphi} \omega = \int_{[0,2\pi]} \varphi^*(\omega) = 0.$$

3. *We have* $d\omega = 0$*. By applying Stokes' formula, we obtain:*

$$\int_{\mathbf{S}^1} \omega = \int_{\partial D} \omega = \int_D d\omega = 0.$$

Example 6.5.2. *On* $\mathbb{R}^2$*, we define:* $\omega = dxdy, \quad \eta = \frac{1}{2}(xdy - ydx), \quad \mathbf{D_1} = [0, 1] \times [0, 1]$*;*
$\mathbf{D_2} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$

The parameterizations are given by:

$$\begin{aligned} \boldsymbol{\varphi_1} : [0, R] \times [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ (r, \theta) &\longmapsto (r \cos\theta,\ r \sin\theta) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\varphi_2} : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto (R \cos\theta,\ R \sin\theta) \end{aligned}$$

We have:

1. $$\int_{\mathbf{D_1}} \omega = \int_0^1 \int_0^1 dxdy = 1.$$

2. $$\begin{aligned} \varphi_1^*(\omega) &= (\cos\theta\, dr - r \sin\theta\, d\theta)(\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\, dr\, d\theta - r\sin^2\theta\, d\theta\, dr \\ &= r(\cos^2\theta + \sin^2\theta)\, dr\, d\theta \\ &= r\, dr\, d\theta. \end{aligned}$$

$$\int_{\mathbf{D_2}} \omega = \int_{\varphi_1} \omega = \int_{[0,R]\times[0,2\pi]} \varphi_1^*(\omega) = \int_0^{2\pi} \left(\int_0^R r dr \right) d\theta = \pi R^2.$$

3. *Since* $\omega = d\eta$*, and* $\varphi_2^*(\eta) = \frac{1}{2}R^2 d\theta$*, Stokes' theorem gives:*

$$\int_{\mathbf{D_2}} \omega = \int_{\mathbf{D_2}} d\eta = \int_{\partial \mathbf{D_2}} \theta = \int_{\varphi_2} \theta = \int_0^{2\pi} \varphi_2^*(\eta) = \int_0^{2\pi} \frac{1}{2} R^2 d\theta = \pi R^2.$$

7

Variation of a mapping

*The references for this chapter are [**43**, **1**, **37**, **34**, **30**, **53**, **12**, **45**, **41**, **19**, **51**, **46**, **44**, **19**, **29**, **55**.]*

Definition 7.0.1. *(n-dimensional interval) A set of the form*

$$T = \prod_{i=1}^{n} [a_i, b_i)$$

Which is a cartesion product of n intervals in $\mathbb{R}^n$

* *In one dimension (n=1) , an interval is simply* $T = [a, b)$ *which is a segment on the real line .*
* *In two dimensions (n=2) the set* $T = [a_1, b_1) \times [a_2, b_2)$ *is a rectangle in* $\mathbb{R}^2$*,half-closed.*
* *In three dimensions (n=3) the set* $T = [a_1, b_1) \times [a_2, b_2) \times [a_3, b_3)$ *is paralleleped in* $\mathbb{R}^3$.
* *In general , fore any n , it is a parallelepiped in* $\mathbb{R}^n$ *thus , I used the erm□ n-dimensional interval.*

to mean a paralleleped with half-open sides in $\mathbb{R}^n$ *also call this a half-closed box.*

Definition 7.0.2. *(cell)*

A parallelepiped (n-dimensional interval) of the form

$$T = \prod_{i=1}^{n} [a_i, b_i)$$

Where $a_i, b_i \in \mathbb{R}$
and $a_i < b_i$ *This means that T is an open-closed set (half-closed in each coordinite) called cell.*

Decomposition of a cell into compact sets

Let G *be an open set in* $\mathbb{R}^n$*, and let* $f : G \to \mathbb{R}^n$ *be a continuous mapping. A cell* $P \subset G$ *is a parallelepiped of the form:*

$$P = \prod_{i=1}^{n} [a_i, b_i) \tag{7.1}$$

where each interval $[a_i, b_i)$ is half-closed.
where the K_j are compact parallelepipeds , can be represented in the in the form

$$P = \bigcup_{j=1}^{\infty} K_j.$$

The set P can be written as a contable union of compact parallelepipeds $\{K_j\}_{j=1}^{\infty}$ such that each is compact (ie: closed bounded)
and the sets $f(K_j)$ are compact. Hence, the set

$$f(P) = \bigcup_{j=1}^{\infty} f(K_j). \tag{7.2}$$

is Lebesgue measurable.
*** We will prove:*

Any half-closed parallelepiped

$$P = \prod_{i=1}^{n} [a_i, b_i) \subset \mathbb{R}^n$$

can be expressed as a countable increasing union of compact parallelepipeds:

$$P = \bigcup_{j=1}^{\infty} K_j,$$

with $K_j \subset K_{j+1}$, and each K_j compact in $\mathbb{R}^n$.

Key concepts used

- *A half-closed cell (or half-closed parallelepiped) in $\mathbb{R}^n$ has the form:*

$$P = \prod_{i=1}^{n} [a_i, b_i).$$

- *A set is compact in $\mathbb{R}^n$ if and only if it is closed and bounded (Heine-Borel theorem).*
- *If f is continuous on a compact set, then the image is also compact:*

If $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^n$ is continuous, then $f(K)$ is compact.

Strategy

We construct an increasing sequence (K_j) of compact subsets inside P such that their union equals P. We define:

$$K_j = \prod_{i=1}^{n} [a_i, b_i - \tfrac{1}{j}]$$

for j large enough so that $b_i - \frac{1}{j} > a_i$ for all i. This ensures:

- *Each K_j is compact (closed and bounded),*
- *$K_j \subset K_{j+1} \subset P$,*
- *$\bigcup_{j=1}^{\infty} K_j = P$.*

Proof 7.0.1. *Let $P = \prod_{i=1}^{n} [a_i, b_i) \subset \mathbb{R}^n$ be a half-open parallelepiped.*

Step 1: Define the compact sets

Choose an integer N such that for all $j \geq N$, $b_i - \frac{1}{j} > a_i$ for all i. Define:

$$K_j = \prod_{i=1}^{n} [a_i, b_i - \tfrac{1}{j}].$$

Each K_j is a closed and bounded parallelepiped, hence compact in $\mathbb{R}^n$.

Step 2: Show inclusion $K_j \subset K_{j+1} \subset P$

Since $\frac{1}{j+1} < \frac{1}{j}$, we have:

$$b_i - \tfrac{1}{j+1} > b_i - \tfrac{1}{j} \quad \Rightarrow \quad [a_i, b_i - \tfrac{1}{j}] \subset [a_i, b_i - \tfrac{1}{j+1}],$$

so $K_j \subset K_{j+1}$. Also, since $b_i - \frac{1}{j} < b_i$, clearly $K_j \subset P$. Thus,

$$K_1 \subset K_2 \subset \cdots \subset P.$$

Step 3: Union covers P

Let $x = (x_1, \dots, x_n) \in P$. Then $x_i \in [a_i, b_i)$, so $x_i < b_i$. For each i, there exists j_i such that:

$$x_i \leq b_i - \tfrac{1}{j_i}.$$

Let $j = \max(j_1, \dots, j_n)$. Then for all i,

$$x_i \leq b_i - \tfrac{1}{j} \quad \Rightarrow \quad x \in K_j.$$

Hence,

$$P \subset \bigcup_{j=1}^{\infty} K_j.$$

The reverse inclusion is immediate: $K_j \subset P$ for all j, so

$$\bigcup_{j=1}^{\infty} K_j \subset P.$$

Therefore,

$$P = \bigcup_{j=1}^{\infty} K_j.$$

Step 4: Compactness of $f(K_j)$

Suppose $f : G \to \mathbb{R}^n$ is continuous, with $P \subset G$. Then since each $K_j \subset G$ is compact and f is continuous,

$$f(K_j) \text{ is compact in } \mathbb{R}^n.$$

Conclusion

Any half-open parallelepiped $P = \prod_{i=1}^{n}[a_i, b_i)$ in $\mathbb{R}^n$ can be written as a countable union of compact sets:

$$P = \bigcup_{j=1}^{\infty} K_j,$$

where each K_j is a compact parallelepiped contained in P. Moreover, if $f : G \to \mathbb{R}^n$ is continuous and $P \subset G$, then:

$$f(P) = \bigcup_{j=1}^{\infty} f(K_j),$$

with each $f(K_j)$ compact.

7.0.1 Compact subdivision K_j

Ech K_j is compact , meaning it must be closed and bounded.
A natural way to construt k_j is to take fully closed parallelepipeds inside P

$$K_j = \prod_{i=1}^{n}[c_i, d_i],$$

Where $c_i, d_i \in [a_i, b_i)$ and $c_i < d_i$
Since K_j is closed and bounded , it is compact .

7.0.2 Exhaustion by compact sets

A common way to construct such a sequence is

$$K_j = \prod_{i=1}^{n}[a_i, b_i - \frac{1}{J_l}], \quad 1/J_l \in \mathbb{Q}. \tag{7.3}$$

Where $b_i - 1/J_l$ ensures that each k_j is strictly inside P but gets closer to including all of it as $J_l \to +\infty$

* *Each k_j is compact because it is closed and bounded parallelepiped*
* *The set sequence satifies $k_1 \subseteq k_2 \subseteq ....$ and $\bigcup_{J_l=k}^{\infty} K_j = P$.*

(This is called an increasing exhaustion of P by compact set)

Why $b_i - 1/j$:

Take any point $x = (x_1, \ldots, x_n) \in P$. Then for each coordinate:

$$x_i \in [a_i, b_i) \Rightarrow x_i < b_i \Rightarrow \exists j \text{ large enough so that } x_i \leq b_i - \frac{1}{j}$$

So eventually, $x \in K_j$ for some finite j, which means:

$$x \in \bigcup_{j=1}^{\infty} K_j \Rightarrow P \subseteq \bigcup_{j=1}^{\infty} K_j$$

Also clearly $K_j \subseteq P$ for all j, so:

$$\bigcup_{j=1}^{\infty} K_j = P$$

7.0.3 Decompositin of P

The key idea is that we can approximate P using a contable of compact parallelepipeds k_j.
One standard way is to take a nested sequence of compact sets that exhaust P .
for exemple :
define :

$$K_j = \prod_{i=1}^{n}[a_i, b_i - \frac{1}{J_l}], \quad 1/J_l \in \mathbb{Q}. \tag{7.4}$$

Which are fully closed parallelepipeds and satisfy :

$$P = \bigcup_{j=1}^{\infty} K_j. \tag{7.5}$$

7.0.4 Lebesgue measurability of $f(P)$

* *P is an n-dimensional Paralleleped in $\mathbb{R}^n$.*
* *$\{K_j\}_{j=1}^{\infty}$ is a nested squence of compact Parallelepiped That exahaust P. meaning*

$$P = \bigcup_{j=1}^{\infty} K_j. \tag{7.6}$$

* *$f : G \to \mathbb{R}^n$ is continous function since f is continuous image of a compact set under f is compact meaning*

$$\boxed{f(K_j) \text{ is compact for each } j.}$$

Theorem 7.0.1. *Every compact set $F \subseteq \mathbb{R}^d$ is Lebesgue measurable*

* *Since Compact set are Lebesgue measurable , their contanle union*

$$f(P) = \bigcup_{j=1}^{\infty} f(K_j). \tag{7.7}$$

Theorem 7.0.2 (Contable unions of the measurable sets are measurable)**.** *Let $\{E_n | n \in \mathbb{N}\}$ be a contable collection of mesurable setsandlet $E = \bigcup_{n=1}^{\infty} E_n$*
The lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^n)$ is closed under unions meaning if E_1, E_2.... are Lebesgue measurable set
Then

$$E = \bigcup_{n=1}^{\infty} E_n$$

is also lebesgue mesurable
*** Since each $f(K_j)$ is compact and hence mesurable. Their contanble union :*

$$f(P) = \bigcup_{j=1}^{\infty} f(K_j). \tag{7.8}$$

remains lebesgue mesurable.

7.1 Variation of the mapping

The variation function $V(h)$ of a function $f(x)$ at $x = a$ is defined by

$$V(h) = f(a+h) - f(a).$$

This is a measure of how much the function change when x change frome a to $a+h$ or, in other words when x start frome a and change by a variable amount h.

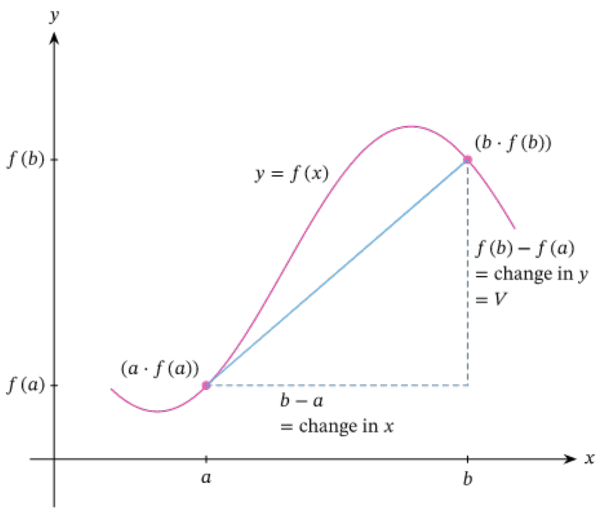


Figure 7.1: Variation of the Mapping f

Variation of the mapping f on a cell P

The variation of a function f over a set P measures how much f stretches or distorts the volume of P defined as:

$$V_f(P) = \sup \left\{ \sum_{k=1}^{m} \lambda_n \left(f \left(P_k \right) \right) \mid \{P_k\}_{k=1}^{m} \text{ is a decomposition of the set } P \text{ into cells} \right\},$$

where the supremum is taken over all possible decompositions of P into finitely many disjoint subcells P_k, and $\lambda_n(f(P_k))$ denotes the Lebesgue measure of the set $f(P_k)$.

7.1.1 Decomposing P into smaller cells

A finite decomposition of P is a paratition :

$$P = \bigcup_{j=1}^{m} P_k. \qquad P_i \cap P_j = \emptyset \quad \textit{for} \quad i \neq j$$

Each P_k is smaller parallelepiped (a cell) contained in P.

7.1.2 Measuring the image of each partition

For each P_k , we compute the Lebesgue measure of its image under f ie $\lambda_n(f(P_k))$ hich represents the volume in $\mathbb{R}^n$. (λ_n is the Lebesgue measure, that is, the volume in $\mathbb{R}^n$)
Summing these over all k we get :

$$\sum_{k=1}^{m} \lambda_n\left(f\left(P_k\right)\right) \qquad [\textit{This gives a total measure}]$$

$f(P_k)$ is the image of the cell P_k under f, and $\lambda_n(f(P_k))$ is the volume of this image in $\mathbb{R}^n$. The supremum is taken over all possible partitions of P.

7.2 The Banach indicatrix

Definition 7.2.1. *Let $f(x)$ be a continuous function defined on the interval $[a, b]$,*
where $a \leq x \leq b$. An integer-valued function $N(y, f)$, defined for $-\infty < y < \infty$, is equal to the number of roots of the equation $f(x) = y$.
If, for a given value of y, this equation has an infinite number of roots, then

$$N(y, f) = +\infty,$$

and if it has no roots, then

$$N(y, f) = 0.$$

*The function $N(y, f)$ was defined by Banach [**1**] (see also [**37**]). He proved that the indicatrix $N(y, f)$ of any continuous function $f(x)$ in the interval $[a, b]$ is a function of Baire class no higher than 2, and*

$$V_a^b(f) = \int_{-\infty}^{+\infty} N_f([a, b], y)\, dy, \tag{7.9}$$

where $V_a^b(f)$ is the variation of $f(x)$ on $[a, b]$. Thus, equation (7.9) can be considered as the definition of the variation of a continuous function $f(x)$.
*The Banach indicatrix is also defined (preserving equation (7.9)) for functions with discontinuities of the first kind [**34**].*
*The concept of a Banach indicatrix was employed to define the variation of functions in several variables [**30**], [**53**].*

Definition 7.2.2. *For a set $E \subset G$ and a point $y \in \mathbb{R}^n$, denote*

$$N_f(E, y) = \begin{cases} \textit{card}(f^{-1}(y) \cap E) & \textit{if the set } f^{-1}(y) \cap E \textit{ is finite,} \\ +\infty & \textit{if the set } f^{-1}(y) \cap E \textit{ is infinite.} \end{cases}$$

*The function $N_f(E)$ defined on $\mathbb{R}^n$ is called the Banach indicatrix see [[**38**], Chapter VII, §5]*

* *If the set $f^{-1}(y) \cap E$ infinite the card $f^{-1}(y) \cap E$ is just the number of the points in E that map to y under f.*
* *If the set set $f^{-1}(y) \cap E$ infinite , then the function $N_f(E, y)$ is defined to be $+\infty$ indicating that there are infinitely many points in E that map to y ($f(x) = y$ has an infinite numbre of roots).*

Theorem 7.2.1. *Let $P \subset G$ be a cell (a Parallelepiped in $\mathbb{R}^n$) and f be continous mapping and then Banach indicatrix $N_f(P)$ is Lebesgue measurable in $\mathbb{R}^n$, and satisfaies the integral fourmule*

$$\int_{\mathbb{R}^n} N_f(P)\, d\lambda_n = V_f(P). \tag{7.10}$$

Proof 7.2.1. Partition of the Cell P:

A cell $P \in G$ is Parallelepiped into disjoint cells P_k forming a decomposition $\tau = \{P_k\}_{k=1}^m$,such that:

$$P = \bigcup_{j=1}^{m} P_k. \qquad P_i \cap P_k = \emptyset \quad \text{for} \quad i \neq j$$

Indicator function** $X_f(P)(y)$ **: *This is the characteristic function of $f(P)$ defined by*

$$X_f(P)(y) = \begin{cases} 1 & \text{if } y \in f(P) \\ 0 & \text{otherwise.} \end{cases}$$

**The function $X_f(P)$ is indicator function of the lebesgue measurable set $f(P)$. Since $f(P)$ is lebesgue measurable its indicator function $X_f(P)$ is also measurable .*

Decomposition into disjoint cells:

Let $\tau = \{P_k\}_{k=1}^m$ be a decomposition of P into pairwaise disjoint cell P_k.
Define:

$$N_\tau(y) = \sum_{k=1}^{m} \chi_{f(P_k)}(y).$$

$N_\tau(y)$ **is measurable:**

Thus, $N_\tau(y)$ counts how many of the sets $f(P_k)$ (the images of the sub-cells under f) contain the point y.

Since each $\chi_{f(P_k)}(y)$ is measurable,because $f(P_k)$ is lebesgue measurable A finite sum of meaurable function is also measurable so sum $N_\tau(y)$ is measurable.

Now consider the integral of $N_\tau(y)$ over $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} N_\tau(y)\, d\lambda_n = \int_{\mathbb{R}^n} \sum_{k=1}^{m} \chi_{f(P_k)}(y)\, d\lambda_n.$$

Using the linearity of the integral, this becomes:

$$\int_{\mathbb{R}^n} N_\tau(y)\, d\lambda_n = \sum_{k=1}^{m} \int_{\mathbb{R}^n} \chi_{f(P_k)}(y)\, d\lambda_n.$$

The integral of $\chi_{f(P_k)}(y)$ is simply the n-dimensional Lebesgue measure of $f(P_k)$, denoted $\lambda_n(f(P_k))$. In particular, it follows directly from the definition of the Lebesgue integral:

$$\int_{\mathbb{R}^n} \chi_E(y)\, d\lambda_n(y) = \lambda_n(E)$$

for any measurable set $E \subset \lambda_n$.
So, for measurable $f(P_k) \subset \lambda_n$, you get

$$\int_{\mathbb{R}^n} \chi_{f(P_k)}(y)\, d\lambda_n(y) = \lambda_n(f(P_k)).$$

$$\int_{\mathbb{R}^n} X_f(y)\, d\lambda_n = \lambda_n(f(P_k)).$$

$P = \bigcup_{j=1}^{m} P_k$. P_k *are disjoint . Summing over all paratitions* P_k *we obtain:*

$$\int_{\mathbb{R}^n} N_\tau(y)\, d\lambda_n = \sum_{k=1}^{m} \lambda_n(f(P_k)).$$

Remark: *The integral of* $\chi_{f(P_k)}(y)$ *over* $\mathbb{R}^n$ *gives the measure (volume) of* $f(P_k)$ *.*
The function N_τ *is measurable, and*

$$\int_{\mathbb{R}^n} N_\tau\, d\lambda_n = \sum_{k=1}^{m} \int_{\mathbb{R}^n} \chi_{f(P_k)}\, d\lambda_n = \sum_{k=1}^{m} \lambda_n(f(P_k)), \tag{7.11}$$

Decompositions:

Indeed any cell $P_k \in \tau$ *can be represented as*

$$P_k = \bigcup_{i=1}^{m_k} P_{ki},$$

If $\tau' = P_{ki}$ *is a refinement of* $\tau = P_k$*. This means that each cell* $P_k \in \tau$ *can be written as a disjoint union Then:*

$$P_k = \bigcup_{i=1}^{m_k} P_{ki},$$

- *Since* $f(P_k)$ *is also a union of its refined images so*

$$f(P_k) = \bigcup_{i=1}^{m_k} f(P_{ki}),$$

where the union is disjoint. Consequently, the indicator function satisfies:

$$\chi_{f(P_k)}(y) \le \sum_{i=1}^{m_k} \chi_{f(P_{ki})}(y).$$

Summing over all k*, we find:*

$$N_\tau(y) = \sum_{k=1}^{m} \chi_{f(P_k)}(y) \le \sum_{k=1}^{m} \sum_{i=1}^{m_k} \chi_{f(P_{ki})}(y) = N_{\tau'}(y).$$

so

$$N_\tau(y) \le N_{\tau'}(y)$$

Thus, refining the decomposition increases $N_\tau(y)$*.*

Decomposition τ_s:

This represents a specific way of dividing the cell P *into smaller subregions. The subscript* s *indicates the level of subdivision, where* s *is a parameter that controls how fine the decomposition is.*

Division into 2^{sn} equal parts:

The decomposition divides P into 2^{sn} smaller regions. Here:

- 2^s*: Each side of the cell P is divided into 2^s equal segments.*
  This means the number of divisions along each dimension grows exponentially with s.
- n*: This could represents the additional factors, such as the number of calls in a higher-dimensional setting or a multiplier related to the geometry of P. of dimensions of P, meaning the total number of subregions is 2^{sn}.*

Dividing any side of P into 2^s equal parts:

This implies that each side (or edge) of the cell P is split into 2^s segments of equal length.

Define τ_s as the decomposition of P into 2^{sn} equal parts by dividing each side of P into 2^s equal intervals. The sequence of decompositions $\{\tau_s\}_{s=1}^{\infty}$ is nested (each τ_{s+1} refines τ_s), so:

$$N_{\tau_s}(y) \leq N_{\tau_{s+1}}(y) \quad \textit{for all } y \in \mathbb{R}^n.$$

By monotonicity, the limit

$$\lim_{s\to\infty} N_{\tau_s}(y) \quad (\textit{finite or equal to } +\infty).$$

exists.

Relation between $N_f(P, y)$ and the preimage:

We claim that $N_f(P, y) = \lim_{s\to\infty} N_{\tau_s}(y)$. For any fixed paratition $\tau = \{P_k\}_{k=1}^m$, the equality $P = \bigcup_{k=1}^m P_k$ implies:

$$f^{-1}(y) \cap P = \bigcup_{k=1}^{m} (f^{-1}(y) \cap P_k),$$

Since the set $(f^{-1}(y) \cap P_k)$ are disjoint we sum over ther cardinalities Therefore:

$$N_f(P, y) = \sum_{k=1}^{m} \textit{card}(f^{-1}(y) \cap P_k).$$

because

$$\textit{card}(f^{-1}(y) \cap P) = \sum_{k=1}^{m} \textit{card}(f^{-1}(y) \cap P_k).$$

***Cas 01: If** $f^{-1}(y) \cap P_k = \emptyset$*

Then $y \notin f(P_k)$, so $\chi_{f(P_k)}(y) = 0$. Which implies :

$$\textit{card}(f^{-1}(y) \cap P_k) = 0 = \chi_{f(P_k)}(y)$$

***Cas 02: If** $f^{-1}(y) \cap P_k \neq \emptyset$*

Then $y \in f(P_k)$, so $\chi_{f(P_k)}(y) = 1$. Since $\textit{card}(f^{-1}(y) \cap P_k) \geq 1$ we get :

$$\textit{card}(f^{-1}(y) \cap P_k) \geq \chi_{f(P_k)}(y),$$

Summing over all k, we obtain:

$$card(f^{-1}(y) \cap P) \geq \sum_{k=1}^{m} \chi_{f(P_k)}(y) = N_\tau(y).$$

$$N_f(P, y) \geq N_\tau(y).$$

Since $N_\tau(y)$ increases with finer partition, we conclude:

$$N_f(P, y) \geq \lim_{s\to\infty} N_{\tau_s}(y). \tag{7.12}$$

If $N_f(P, y) \geq q \in \mathbb{N}$, then the set P contains at least q different roots of the equation $f(x) = y$. $N_f(P, y) \geq q \in \mathbb{N}$ means that there are q distinct points $x_1, x_2, \ldots, x_q \in P$ such that:

$$f(x_i) = y, \quad \text{for } i = 1, 2, \ldots, q,$$

and these points are pairwise distinct.

Pairwise distinctness:

Since the roots $x_1, x_2, \ldots, x_q$ are distinct, their pairwise distances (measured using a metric ρ on $\mathbb{R}^n$) are strictly positive:

$$\rho_0 = \min_{1\leq i<j\leq q} \rho(x_i, x_j) > 0.$$

This minimum distance ρ_0 ensures that no two roots can be arbitrarily close to each other.

Behavior of the decomposition τ_s

The decomposition τ_s divides the cell P into smaller subregions (cells) $P_{s,i}$, where:

- *Each side of P is divided into 2^s equal parts.*
- *The total number of subregions is 2^{sn} for an n-dimensional cell P.*
- *The diameter of each subregion $P_{s,i}$ decreases as $s \to \infty$:*

$$diam(P_{s,i}) = \frac{diam(P)}{2^s}.$$

As $s \to \infty$, the diameter of each subregion approaches zero:

$$\lim_{s\to\infty} diam(P_{s,i}) = 0.$$

For sufficiently large s, the diameter of each subregion $P_{s,i}$ becomes smaller than the minimum distance ρ_0 between any two roots:

$$diam(P_{s,i}) < \rho_0.$$

This implies that no single subregion $P_{s,i}$ can contain more than one of the roots $x_1, x_2, \ldots, x_q$. Consequently:

- *Each root x_i must lie in its own unique subregion $P_{s,i}$.*
- *At most q subregions can contain any of the roots $x_1, x_2, \ldots, x_q$.*

Fix roots $x_1, \ldots, x_q \in P$. Since these roots are distinct, define

$$\rho_0 = \min_{1 \le i < j \le q} \rho(x_i, x_j) > 0$$

where ρ is a metric in $\mathbb{R}^n$. If P_{si} are the cells of the decomposition τ_s, then

$$\operatorname{diam} P_{si} = \frac{1}{2^s} \operatorname{diam} P \to 0 \quad as \quad s \to \infty.$$

Thus, there exists s such that $\operatorname{diam} P_{si} < \rho_0$.
In this case, each cell P_{si} contains not more than one point of the set $\{x_1, \ldots, x_q\}$, hence the existe not more than q cells P_{si} satisfy $y \in f(P_{si})$, for each subregion $P_{s,i}$, the indicator function $\chi_{f(P_{s,i})}(y)$ is defined as:

$$\chi_{f(P_{s,i})}(y) = \begin{cases} 1, & \text{if } y \in f(P_{s,i}), \\ 0, & \text{otherwise}. \end{cases}$$

The value $N_{\tau_s}(y)$ is the sum of these indicator function over all subregions :

$$N_{\tau_s}(y) = \sum_{i=1}^{2^{sn}} \chi_{f(P_{si})}(y).$$

As $s \to \infty$, the decomposition τ_s becomes arbitrarily fine, and the subregions $P_{s,i}$ shrink to individual points. In this limit:

- *Each root x_i contributes one subregion $P_{s,i}$ such that $y \in f(P_{s,i})$.*
- *Therefore, the count $N_{\tau_s}(y)$ converges to the number of roots q.*

Thus,

$$\lim_{s\to\infty} N_{\tau_s}(y) \ge q. \tag{7.13}$$

We have shown that if $N_f(P, y) = q$, then:

$$\lim_{s\to\infty} N_{\tau_s}(y) \ge q. \tag{7.14}$$

If $N_f(P, y) = q \in \mathbb{N}$, then

$$\lim_{s\to\infty} N_{\tau_s}(y) \le q,$$

which implies $N_f(P, y) \ge \lim_{s\to\infty} N_{\tau_s}(y)$.
If $N_f(P, y) = +\infty$, then for any q, the inequality $N_f(P, y) \ge q$ holds.
In this case,

$$\lim_{s\to\infty} N_{\tau_s}(y) \ge q \quad \text{for any } q,$$

which means $\lim_{s\to\infty} N_{\tau_s}(y) = +\infty$. Thus, we conclude:

$$N_f(P, y) \ge \lim_{s\to\infty} N_{\tau_s}(y).$$

From inequality (7.12), we deduce:

$$N_f(P, y) = \lim_{s\to\infty} N_{\tau_s}(y).$$

Conclusion:

In both cases, we have:

$$N_f(P, y) = \lim_{s \to \infty} N_{\tau_s}(y).$$

This confirms that $N_f(P)$ is measurable. By Lévy's theorem and inequality (7.11), we obtain:

$$\int_{\mathbb{R}^n} N_f(P) d\lambda_n = \lim_{s \to \infty} \int_{\mathbb{R}^n} N_{\tau_s} d\lambda_n. \tag{7.15}$$

Since

$$\lim_{s \to \infty} \sum_{i=1}^{2^{sn}} \lambda_n(f(P_{s,i})) \leq \sup_{\{P_k\}_{k=1}^m} \sum_{k=1}^{m} \lambda_n(f(P_k)), \tag{7.16}$$

where $\{P_k\}_{k=1}^m$ is a decomposition of the cell P, we conclude:

$$V_f(P) = \sup_{\tau} \sum_{k=1}^{m} \lambda_n(f(P_k)) \leq \int_{\mathbb{R}^n} N_f(P) d\lambda_n. \tag{7.17}$$

Inequalities and limits:

On the other hand, since $N_f(P, y) \geq N_\tau(y)$ for any decomposition τ, it follows that:

$$\sum_{k=1}^{m} \lambda_n(f(P_k)) = \int_{\mathbb{R}^n} N_\tau d\lambda_n \leq \int_{\mathbb{R}^n} N_f(P) d\lambda_n. \tag{7.18}$$

Thus,

$$V_f(P) = \sup_{\tau} \sum_{k=1}^{m} \lambda_n(f(P_k)) \leq \int_{\mathbb{R}^n} N_f(P) d\lambda_n. \tag{7.19}$$

For any decomposition τ, the inequality

$$N_f(P, y) \geq N_\tau(y) \tag{7.20}$$

holds. Integrating both sides with respect to y over $\mathbb{R}^n$, we get:

$$\int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y) \geq \int_{\mathbb{R}^n} N_\tau(y)\, d\lambda_n(y). \tag{7.21}$$

For the specific decomposition τ_s, we have:

$$\int_{\mathbb{R}^n} N_{\tau_s}(y)\, d\lambda_n(y) = \sum_{i=1}^{2^{sn}} \lambda_n(f(P_{s,i})). \tag{7.22}$$

Taking the limit as $s \to \infty$, we obtain:

$$\lim_{s \to \infty} \int_{\mathbb{R}^n} N_{\tau_s}(y)\, d\lambda_n(y) = \int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y). \tag{7.23}$$

Thus:

$$\int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y) = V_f(P). \tag{7.24}$$

Supremum over all decompositions:

For any decomposition $\tau = \{P_k\}_{k=1}^m$, we have:

$$\sum_{k=1}^{m} \lambda_n(f(P_k)) = \int_{\mathbb{R}^n} N_\tau(y)\, d\lambda_n(y). \tag{7.25}$$

Since $N_f(P, y) \geq N_\tau(y)$, it follows that:

$$\sum_{k=1}^{m} \lambda_n(f(P_k)) \leq \int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y). \tag{7.26}$$

Taking the supremum over all decompositions τ, we conclude:

$$V_f(P) = \sup_\tau \sum_{k=1}^{m} \lambda_n(f(P_k)) \leq \int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y). \tag{7.27}$$

On the other hand, since $N_f(P, y) = \lim_{s\to\infty} N_{\tau_s}(y)$, and

$$\int_{\mathbb{R}^n} N_{\tau_s}(y)\, d\lambda_n(y) \to V_f(P), \tag{7.28}$$

we also have:

$$\int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y) \leq V_f(P). \tag{7.29}$$

Combining these inequalities, we conclude:

$$V_f(P) = \int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y). \tag{7.30}$$

Definition 7.2.3. *A continuous mapping $f : G \to \mathbb{R}^n$ is called a mapping with locally-finite variation if the variation $V_f(P)$ is finite for any cell P such that $\overline{P} \subset G$.*

The variation $V_f(P)$ measures how much the image of P under f "spreads out" in $\mathbb{R}^n$.
It quantifies the "size" of the image of P under f, taking into account the multiplicity of preimages.Formally, $V_f(P)$ is defined as:

$$V_f(P) = \sup_\tau \sum_{k=1}^{m} \lambda_n(f(P_k)), \tag{7.31}$$

where:

- *$\tau = \{P_k\}_{k=1}^m$ is a decomposition of P into disjoint subregions P_k,*
- *λ_n is the n-dimensional Lebesgue measure,*
- *The supremum is taken over all possible decompositions τ of P.*

Closure condition:

$\overline{P} \subset G$

The condition $\overline{P} \subset G$ ensures that not only the interior of P but also its boundary lies entirely within the domain G of the mapping f.
This is stricter than requiring $P \subset G$, as it guarantees that f is well-defined and continuous on the entire boundary of P.

Examples

(a) Mapping with locally-finite variation

Let $f : [0,1] \to \mathbb{R}^2$ *be defined as:*

$$f(x) = (x, x^2).$$

For any subinterval $[a,b] \subset [0,1]$*, the image of* $[a,b]$ *under* f *is a curve in* $\mathbb{R}^2$ *with finite length. Thus,* $V_f([a,b])$ *is finite for all* $[a,b] \subset [0,1]$*, and* f *has locally-finite variation.*

(b) Mapping without locally-finite variation

Let $f : (0,1) \to \mathbb{R}$ *be defined as:*

$$f(x) = \sin\left(\frac{1}{x}\right) \quad \textit{for } x > 0.$$

Near $x = 0$*,* $f(x)$ *oscillates infinitely often, causing the variation* $V_f([a,b])$ *to become infinite for any interval* $[a,b]$ *containing points arbitrarily close to* 0*.*
Therefore, f *does not have locally-finite variation.*

Corollary 7.2.1. *If* $f : G \to \mathbb{R}^n$ *is a mapping with locally-finite variation and* P *is a cell such that* $P \subset G$*, then the set*

$$f^{-1}(y) \cap P \qquad \textit{is finite for almost every point } y \in \mathbb{R}^n$$

(with respect to the Lebesgue measure).

Moreover, the function $N_f(P)$ *is summable, hence this function is finite almost everywhere.*

Proof 7.2.2. Step 1: Finite variation implies finiteness of $f^{-1}(y) \cap P$ almost everywhere:

Since f *has locally-finite variation, the variation* $V_f(P)$ *is finite for any cell* P *such that* $\overline{P} \subset G$*. By definition of* $V_f(P)$*, the total "size" of the image of* P *under* f *is finite:*

$$V_f(P) = \sup_{\tau} \sum_{k=1}^{m} \lambda_n(f(P_k)) < \infty, \tag{7.32}$$

where $\tau = \{P_k\}_{k=1}^m$ *is a decomposition of* P *into disjoint subregions.*

If $f^{-1}(y) \cap P$ *were infinite for some* $y \in \mathbb{R}^n$*, then the contribution of* y *to the variation* $V_f(P)$ *would also be infinite, contradicting the finiteness of* $V_f(P)$*. Thus,* $f^{-1}(y) \cap P$ *must be finite for almost every* $y \in \mathbb{R}^n$ *(with respect to the Lebesgue measure).*

Step 2: Summability of $N_f(P, y)$:

The function $N_f(P, y)$ *counts the number of points in* $f^{-1}(y) \cap P$*. Since* $f^{-1}(y) \cap P$ *is finite for almost every* y*,* $N_f(P, y)$ *is finite almost everywhere. Moreover, the integral of* $N_f(P, y)$ *over* $\mathbb{R}^n$ *is equal to the variation* $V_f(P)$*:*

$$\int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y) = V_f(P). \tag{7.33}$$

Since $V_f(P) < \infty$ *by assumption, it follows that* $N_f(P, y)$ *is summable (i.e., integrable).*

Step 3: Conclusion

The summability of $N_f(P, y)$ implies that $N_f(P, y)$ is finite almost everywhere. Thus, the set $f^{-1}(y) \cap P$ is finite for almost every $y \in \mathbb{R}^n$.

Remak:Intuition behind the result

Why is $f^{-1}(y) \cap P$ finite almost everywhere?

The condition of locally-finite variation ensures that the image of P under f does not spread out infinitely in $\mathbb{R}^n$, preventing $f^{-1}(y) \cap P$ from being infinite for too many values of y.

Intuitively, if $f^{-1}(y) \cap P$ were infinite for many y, the total "size" of the image of P would become infinite, contradicting the finiteness of $V_f(P)$.

Why is $N_f(P, y)$ summable?

The summability of $N_f(P, y)$ reflects the fact that the total number of preimages of all points $y \in \mathbb{R}^n$ is controlled by the finite variation $V_f(P)$.
This ensures that $N_f(P, y)$ does not grow too large over any significant subset of $\mathbb{R}^n$.

8

Topological degree of a mapping with locally-finite variation

*The references for this chapter are [**43**, **35**, **27**, **47**, **9** , **2**.]*

Compact manifold $K \subset \mathbb{R}^n$

K is an n-dimensional compact manifold with a piecewise-smooth boundary ∂K. The boundary ∂K is a union of finitely many smooth manifolds whose dimensions do not exceed $n-1$. A simple example of such a manifold is a compact parallelepiped (e.g., a closed rectangle or cube).

Definition 8.0.1. *The boundary ∂K is a union of a finite family of smooth manifolds whose dimensions do not exceed $n-1$:*

$$\partial K = \bigcup_{i=1}^{m} M_i, \quad \text{where each } M_i \text{ is a smooth manifold with } \dim M_i \leq n-1.$$

Mapping $g : K \to \mathbb{R}^n$

g is a C^1-mapping (continuously differentiable) from K to $\mathbb{R}^n$. The point $y \in \mathbb{R}^n$ is chosen such that $y \notin g(\partial K)$, meaning y does not lie in the image of the boundary of K under g.

Definition of piecewise-smooth boundary

A piecewise-smooth boundary of the set $K \subset \mathbb{R}^n$ means that the boundary ∂K is composed of a finite union of smooth $(n-1)$-dimensional manifolds .That is ∂K consists of several smooth pieces that may meet at their boundarics but are otherwise smooth within their individual regions.
Formally ,∂K is piecewise-smooth if :

1. *There exists a finite number of smooth $(n-1)$-dimensional manifolds $M_1, M_2, \ldots, M_m$ such that :*

$$\partial K = \bigcup_{i=1}^{m} M_i,$$

2. *Each M_i is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ meaning it has continous partial derivatives up to a certain order*

3. *The manifolds M_i may intersect along lower-dimensional subsets (such as edges or corners in the cas of a polyhedron)*

Degree of a smooth map between manifolds

Let M and N be oriented n-dimensional manifolds without boundary, and let

$$S : M \to N$$

be a smooth map. If M is compact and N is connected, then the ***degree*** *of S is defined as follows:*

Let $x \in M$ be a regular point of S, so that the differential

$$dS_x : T_x M \to T_{S(x)} N$$

is a linear isomorphism between oriented vector spaces.

Define the ***sign of*** *dS_x to be $+1$ or -1 according as dS_x preserves or reverses orientation.*

For any regular value $y \in N$, define:

$$\deg(S; y) = \sum_{x \in S^{-1}(y)} \operatorname{sign}(dS_x).$$

*this integer $\deg(S; y)$ is a locally constant function of y. It is defined on a dense open subset of N see ([**35**]).*

Definition of the topological degree

*Let $g : K \to \mathbb{R}^n$ be a C^1 mapping, and let $y \notin g(\partial K)$.The topological degree of the mapping g at the point y is defined as the index of the cycle $g|_{\partial K}$ (see [[**47**], Chapter VI,§8] and see [**43**])with respect to y, i.e., the integral:*

$$\deg g(K, y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K)} \omega_y,$$

where:

- *$\mu(S^{n-1})$ is the surface area of the unit sphere in $\mathbb{R}^n$.*
- *ω_y is the $(n-1)$-form:*

$$\omega_y = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i - y_i}{|x - y^n|} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $|x - y| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$ and $\widehat{dx_i}$ indicates omission of dx_i.

Interpretation of ω_y:

- *The differential form ω_y represents a normalized volume form on the sphere centered at y. It captures orientation and density of $g(\partial K)$ around y.*
- *The wedge product $dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i-1} \cdots \wedge dx_n$, represent the exterior product of all variables except x_i.*

Integral over $g(\partial K)$

The integral $\int_{g(\partial K)} \omega_y$ computes the index of the cycle $g|_{\partial K}$ with respect to y.
This index reflects how $g(\partial K)$ wraps around y in $\mathbb{R}^n$.

Normalization

- *Dividing by $\mu(S^{n-1})$ ensures that the topological degree is an integer.*
- *If f is continous mapping of a set $G \subset \mathbb{R}^n$ into $\mathbb{R}^n$ $f : G \to \mathbb{R}^n$ Assume , $K \subset G$ is compact subset, and $y \notin f(\partial K)$,meaning y deoes not lie in the image of the boundary of K under f.*

Approximation by C^1-mappings

- *Since f is continuous but not necessarily differentiable we approximate f by a sequence of C^1-mappings g_k such that $g_k \to f$ uniformly on K.*
- *For each g_k ,the topological degree $\deg g_k(K, y)$ is well-defined using the formula above.*

The topological degree is stable under small perturbations of the mapping. Therefore, the degree $\deg f(K, y)$ can be defined as the limit:

$$\deg f(K, y) = \lim_{k\to\infty} \deg g_k(K, y).$$

Compactness and distance condition

(a) Since ∂K is compact and f is continuous, the image $f(\partial K)$ is also compact in $\mathbb{R}^n$.

*(b) **Positive distance to** y:*
The condition $y \notin f(\partial K)$ implies that there exists a positive minimum distance between y and $f(\partial K)$:

$$\rho(y, f(\partial K)) = \inf\{\rho(y, z) \mid z \in f(\partial K)\} > 0.$$

This ensures that y is sufficiently far from $f(\partial K)$, so the integral defining the degree is well-defined.

8.1 Definitions: paths and homotopy of paths

Let us assume the closed unit interval $I := [0, 1]$ and a topological space X are given.

1. ***Path :** A continuous function $p : I \to X$ from a point $x_0 \in X$ to a point $x_1 \in X$ is called a **path** if it satisfies:*

$$p(0) = x_0, \quad p(1) = x_1.$$

2. ***Homotopy of paths :** Let $f \equiv h_0$ and $g \equiv h_1$ be two paths in X from x_0 to x_1. A family of paths $\{h_t : I \to X\}_{t\in[0,1]}$ is called a **homotopy of paths** (relative to endpoints) if:*

(i) For all $t \in [0, 1]$, the paths satisfy the endpoint conditions:

$$h_t(0) = x_0, \quad h_t(1) = x_1.$$

(ii) The map $H : I \times I \to X$, defined by

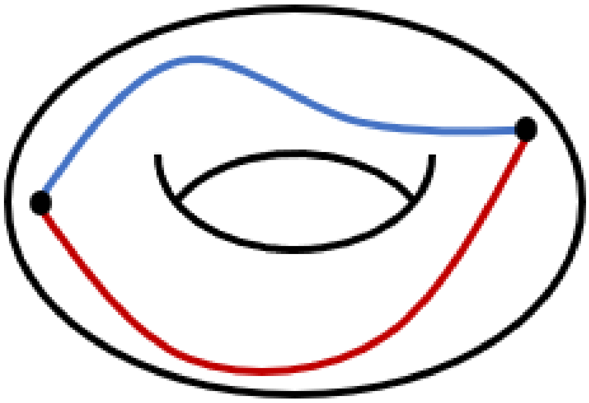
$$H(s,t) := h_t(s),$$

is continuous.

Depending on the context, either the function H or the family $\{h_t\}_{t\in[0,1]}$ itself is referred to as a homotopy.

8.1.1 The meaning of homotopy

*As mentioned before, the idea that two paths are homotopic can be understood to mean that by slightly altering one path, it can be turned into the other. However, in topology, exploring such 'practical sameness' is to see the 'true difference' see([**27**]).*



8.2 Definition of C^0-homotopy

(a) General definition

Let X and Y be topological spaces, and let $f, g : X \to Y$ be two continuous mappings.
A C^0-homotopy between f and g is a continuous mapping:

$$H : X \times [0,1] \to Y,$$

such that:

$$H(x,0) = f(x), \quad H(x,1) = g(x) \quad \text{for all } x \in X,$$

and for each fixed $t \in [0,1]$, the map $H_t(x) = H(x,t)$ is continuous.

In simpler terms, H provides a continuous "path" of mappings from f to g.

(b) Homotopy in $\mathbb{R}^n \setminus \{y\}$

If $f, g : \partial K \to \mathbb{R}^n \setminus \{y\}$, then $f|_{\partial K}$ and $g|_{\partial K}$ are said to be C^0-homotopic in $\mathbb{R}^n \setminus \{y\}$ if there exists a C^0-homotopy:

$$H : \partial K \times [0,1] \to \mathbb{R}^n \setminus \{y\}$$

such that:

$$H(x,0) = f(x), \quad H(x,1) = g(x) \quad \text{for all } x \in \partial K,$$

and

$$H(x,t) \neq y \quad \text{for all } x \in \partial K \text{ and } t \in [0,1].$$

*This ensures that the homotopy avoids the point y throughout the deformation see([**35**]).*

8.2.1 Properties of C^0-homotopy

- ***Continuity:*** *The homotopy H is continuous as a function of both $x \in \partial K$ and $t \in [0,1]$.*
- ***Avoidance of*** y***:*** *Since $H(x,t) \neq y$ for all $x \in \partial K$ and $t \in [0,1]$, the homotopy stays entirely within $\mathbb{R}^n \setminus \{y\}$.*
- ***Equivalence Relation:*** *C^0-homotopy defines an equivalence relation on the set of continuous mappings from ∂K to $\mathbb{R}^n \setminus \{y\}$:*
  - *Reflexivity: $f|_{\partial K}$ is C^0-homotopic to itself.*
  - *Symmetry: If $f|_{\partial K}$ is C^0-homotopic to $g|_{\partial K}$, then $g|_{\partial K}$ is C^0-homotopic to $f|_{\partial K}$.*
  - *Transitivity: If $f|_{\partial K}$ is C^0-homotopic to $g|_{\partial K}$, and $g|_{\partial K}$ is C^0-homotopic to $h|_{\partial K}$, then $f|_{\partial K}$ is C^0-homotopic to $h|_{\partial K}$.*

Lemma 8.2.1. *Let $f : G \to \mathbb{R}^n$ be a continuous mapping and suppose $y \notin f(\partial K)$.*
Then there exists a mapping $g : K \to \mathbb{R}^n$ of class C^1 such that:

$$\|f - g\| = \max_{x \in K} |f(x) - g(x)| < \rho(y, f(\partial K)),$$

and the restriction $g|_{\partial K}$ is C^0-homotopic to the restriction $f|_{\partial K}$ in $\mathbb{R}^n \setminus \{y\}$ for any such mapping y.
Moreover,

$$\deg f(K, y) = \deg g(K, y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K)} \omega_y.$$

Proof

Step 1: Approximation by C^1-mappings:

*By Theorem 52 in [**47**], any continuous mapping $f : K \to \mathbb{R}^n$ can be approximated by a C^1-mapping $g : K \to \mathbb{R}^n$ such that:*

$$\|f - g\| = \max_{x \in K} |f(x) - g(x)| < \rho(y, f(\partial K)).$$

This ensures that $g(x) \neq y$ for all $x \in \partial K$, because:

$$|g(x) - y| \geq |f(x) - y| - |f(x) - g(x)| > 0 \quad \text{for all } x \in \partial K.$$

Distance condition:

The condition $\|f - g\| < \rho(y, f(\partial K))$ guarantees $g(x) \neq y$ for all $x \in \partial K$. Hence, $g(\partial K)$ avoids y, making the integral formula valid.

Step 2: Homotopy between restrictions:

*By item 2 of Theorem 51 in [**47**], the restriction $g|_{\partial K}$ is C^0-homotopic to $f|_{\partial K}$ in $\mathbb{R}^n \setminus \{y\}$. That is, there exists a continuous family of mappings $H_t : \partial K \to \mathbb{R}^n \setminus \{y\}$, $t \in [0,1]$, such that:*

$$H_0 = f|_{\partial K}, \quad H_1 = g|_{\partial K}, \quad \text{and } H_t(x) \neq y \text{ for all } x \in \partial K,\ t \in [0,1].$$

Homotopy avoiding y:

Since $\|f-g\| < \rho(y, f(\partial K))$, we have for all $x \in \partial K$ and $t \in [0,1]$:

$$|H_t(x) - y| \geq |f(x) - y| - |f(x) - g(x)| > 0,$$

ensuring $H_t(x) \neq y$ throughout the homotopy.

Step 3: Invariance of the degree under homotopy:

*By Theorem 62 in [**47**], the topological degree is invariant under C^0-homotopies that avoid y. Therefore:*

$$\deg f(K, y) = \deg g(K, y).$$

Homotopy invariance:

The degree is invariant under C^0-homotopies avoiding y.

Step 4: Integral formula for the degree:

For the C^1-mapping g, the degree $\deg g(K, y)$ can be computed via the integral formula:

$$\deg g(K, y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K)} \omega_y,$$

Smooth Approximation:

The smooth approximation g is constructed using standard techniques such as convolution or partitions of unity. These ensure that g is C^1 and remains uniformly close to f where ω_y is the differential form defined previously. Since $\deg f(K, y) = \deg g(K, y)$, we conclude:

$$\deg f(K, y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K)} \omega_y.$$

8.3 Topological degree on a cell

Let $P = \prod_{i=1}^{n} [a_i, b_i)$, and for $\varepsilon > 0$, define the shifted cell

$$K_\varepsilon(P) = \prod_{i=1}^{n} [a_i - \varepsilon, b_i - \varepsilon].$$

$K_\varepsilon(P)$ is used to construct neighborhoods of P that remain within G, ensuring that the approximation g can be extended smoothly to the boundary of P.

Definition 8.3.1. *If there exists $\varepsilon_y > 0$ such that the degree $\deg f(K_\varepsilon(P), y)$ is defined for all $0 < \varepsilon < \varepsilon_y$, and the finite limit*

$$\lim_{\varepsilon \to 0^+} \deg f(K_\varepsilon(P), y) \quad \textit{exists} \quad \textit{for any} \quad y \notin f(\partial K_\varepsilon)$$

*Then this limit is called the **topological degree** of the mapping f on the cell P at the point y, and is denoted by $\deg f(P, y)$.*

Theorem 8.3.1. *Assume that $f : G \to \mathbb{R}^n$ is a mapping with locally-finite variation, and let $P = \prod_{i=1}^{n} [a_i, b_i)$ be such that $\overline{P} \subset G$. Then the degree $\deg f(P, y)$ exists for almost all $y \in \mathbb{R}^n$, and the function $\deg f(P, y)$ is Lebesgue measurable.*

Proof 8.3.1. Step (1): Points not on the boundary $\partial K_\varepsilon(P)$

We first show that for any $x \in \mathbb{R}^n$, there exists $\varepsilon_x > 0$ such that $x \notin \partial K_\varepsilon(P)$ for all $\varepsilon \in (0, \varepsilon_x)$.

Case (a): $x \in P$.

Let $x = (x_1, \ldots, x_n) \in P$. Then :

$$a_i \le x_i < b_i \text{ for all } i = 1, \ldots, n.$$

Define:

$$\varepsilon_x = \min_{1 \le i \le n} \{b_i - x_i\} > 0.$$

Then for all $\varepsilon < \varepsilon_x$, we have

$$a_i \le x_i < b_i \quad (i = 1, ..., n)$$

$$x_i < b_i - \varepsilon \quad \text{and} \quad x_i > a_i - \varepsilon \quad \text{for all} \quad i = 1, \ldots, n \quad \text{and} \quad 0 < \varepsilon < \varepsilon_x.$$

Hence, $x \notin \partial K_\varepsilon(P)$.

Case (b): $x \notin P$.

If $x \notin P$ .Then there exists an index i_0 such that either $x_{i_0} \ge b_{i_0}$ or $x_{i_0} < a_{i_0}$.
In the first case ($x_{i_0} \ge b_{i_0}$):

$$x_{i_0} > b_{i_0} - \varepsilon \quad \text{for all} \quad \varepsilon > 0$$

So $x \notin \partial K_\varepsilon(P)$ for all $\varepsilon > 0$.
In the second case $x_{i_0} < a_{i_0}$:
define

$$\varepsilon_x = a_{i_0} - x_{i_0} > 0$$

For $\varepsilon < \varepsilon_x$, we have

$$x_{i_0} < a_{i_0} - \varepsilon.$$

So $x \notin \partial K_\varepsilon(P)$

Conclusion of Step (1):

For any finite set $T \subset \mathbb{R}^n$, there exists $\varepsilon_T > 0$ such that define:

$$T \cap \partial K_\varepsilon(P) = \emptyset \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_T).$$

This follows by setting :

$$\varepsilon_T := \min\{\varepsilon_x \mid x \in T\} \ne 0.$$

Step (2): Avoidance of y in $f(\partial K_\varepsilon(P))$

We now claim that for almost every $y \in \mathbb{R}^n$, there exists $\varepsilon(y) > 0$ such that:

$$y \notin f(\partial K_\varepsilon(P)) \quad \textit{for all} \quad \varepsilon \in (0, \varepsilon(y)).$$

Since $\overline{P} \subset G$ and $\overline{P}$ is compact,the distance frome $\overline{P}$ to $\mathbb{R}^n \setminus G$is positive :

$$\inf\{\rho(x,z) \mid x \in P, z \in \mathbb{R}^n \setminus G\} > 0.$$

Hence, there exists $\varepsilon_0 > 0$ such that

$$P_0 := \prod_{i=1}^{n} [a_i - \varepsilon_0, b_i + \varepsilon_0) \subset \overline{P_0} \subset G.$$

By the corollary 7.2.1 to Theorem 7.2.1, the set $T = f^{-1}(y) \cap P_0$ is finite for almost all $y \in \mathbb{R}^n$. For the finite set T, by Step (1), there exists $\varepsilon_T > 0$ such that:

$$f^{-1}(y) \cap P_0 \cap \partial K_\varepsilon(P) = \emptyset \quad \textit{for all } \varepsilon \in (0, \varepsilon_T).$$

Since $\partial K_\varepsilon(P) \subset P_0$ for $\varepsilon \in (0, \varepsilon_0)$,it follows that :

$$f^{-1}(y) \cap P_0 \cap \partial K_\varepsilon(P) = \emptyset \quad \textit{for all } \varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_T\}).$$

Where for almost all y, there exists $\varepsilon(y) := \min\{\varepsilon_0, \varepsilon_T\}$.
In this case $x \in f^{-1}(y)$ for $x \in f(\partial K_\varepsilon(P)$ ie $f(x) \neq y$ thus $y \notin f(\partial K_\varepsilon(P))$ for all $\varepsilon \in (0, \varepsilon(y))$.
Then $\deg f(P, y)$ existe

Step (3): Existence of the limit

To establish the existence of the limite :

$$\lim_{\varepsilon \to 0^+} \deg f(K_\varepsilon(P), y)$$

exists for almost all y.
We use the mappings $\varphi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ define as(Translation diffeo C^∞):

$$\varphi_\varepsilon(x_1, \ldots, x_n) = (x_1 - \varepsilon, \ldots, x_n - \varepsilon),$$

φ_ε shifts the coordinates of x by $-\varepsilon$, effectively shrinking the domain P to $K_\varepsilon(P)$.

For small $\varepsilon > 0$, φ_ε ensures that $K_\varepsilon(P) \cap P_0$ and $\partial K_\varepsilon(P) \cap P_0$.

Then

$$K_\varepsilon(P) = \varphi_\varepsilon(\overline{P}).$$

Boundary transformation under φ_ε:

The boundary of the shifted parallelepiped $K_\varepsilon(P)$ is given by:

$$\partial K_\varepsilon(P) = \varphi_\varepsilon(\partial P),$$

- *where $\varphi_\varepsilon(x_1, \ldots, x_n) = (x_1 - \varepsilon, \ldots, x_n - \varepsilon)$.*
- *This means that $\partial K_\varepsilon(P)$ is a translated of $\partial\overline{P}$, shifled inward by ε*

Finite preimage condition :

Assume that the set $f^{-1} \cap P_0$ is finite for some $y \in \mathbb{R}^n$.
Where:

$$P_0 = \prod_{i=1}^{n} [a_i - \varepsilon_0, b_i + \varepsilon_0] \subset G$$

- *Since $\partial K_\varepsilon(P) \subset P_0 \subset G$ for $\forall \varepsilon$, $0 < \varepsilon < \varepsilon(y)$, the mappings $\varphi_\varepsilon|_{\partial \overline{P}}$ are homotopic on G.*
- *Consequently, the compositions $f \circ \varphi_\varepsilon|_{\partial \overline{P}}$ are C^0-homotopic on $\mathbb{R}^n \backslash \{y\}$, because $y \notin f(\partial K_\varepsilon(P))$.*

Homotopy between $f \circ \varphi_{\varepsilon_1}|_{\partial P}$ and $f \circ \varphi_{\varepsilon_2}|_{\partial P}$:

- *If $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon(y)$, then the mappings $f \circ \varphi_{\varepsilon_1}|_{\partial P}$ and $f \circ \varphi_{\varepsilon_2}|_{\partial P}$ are C^0-homotopic.*
- *This implies that the topological degree is invariant under this homotopy.*

Approximation by C^1-mappings:

Construction of g_1 and g_2:

- *Let $g_1 : K_{\varepsilon_1}(P) \to \mathbb{R}^n$ and $g_2 : K_{\varepsilon_2}(P) \to \mathbb{R}^n$ be C^1-mappings approximating f such that:*

$$\|f - g_1\| = \max_{x \in K_{\varepsilon_1}(P)} |f(x) - g_1(x)| < \rho(y, f(\partial K_{\varepsilon_1}(P))),$$

and

$$\|f - g_2\| = \max_{x \in K_{\varepsilon_2}(P)} |f(x) - g_2(x)| < \rho(y, f(\partial K_{\varepsilon_2}(P))).$$

Homotopy between restrictions:

- *By Lemma 8.2.1, the restrictions $g_1|_{\partial K_{\varepsilon_1}(P)}$ and $f|_{\partial K_{\varepsilon_1}(P)}$ are C^0-homotopic in $\mathbb{R}^n \setminus \{y\}$, similarly for $g_2|_{\partial K_{\varepsilon_2}(P)}$ and $f|_{\partial K_{\varepsilon_2}(P)}$ are C^0-homotopic in $\mathbb{R}^n \setminus \{y\}$.*

Transitivity of homotopy:

- *Since $f \circ \varphi_{\varepsilon_1}|_{\partial P}$ is C^0-homotopic to $g_1 \circ \varphi_{\varepsilon_1}|_{\partial P}$, and $f \circ \varphi_{\varepsilon_2}|_{\partial P}$ is C^0-homotopic to $g_2 \circ \varphi_{\varepsilon_2}|_{\partial P}$.*
- *By transitivity ,$g_1 \circ \varphi_{\varepsilon_1}|_{\partial P}$ is C^0-homotopic to $g_2 \circ \varphi_{\varepsilon_2}|_{\partial P}$.*

C^1-Homotopy:

- *By Theorem 53 in [**47**], the mappings $g_1 \circ \varphi_{\varepsilon_1}|_{\partial P}$ and $g_2 \circ \varphi_{\varepsilon_2}|_{\partial P}$ are C^1-homotopic in $\mathbb{R}^n \setminus \{y\}$.*

Integral invariance under homotopy:

Equality of integrals:

If ω_y is a closed differential form in $\mathbb{R}^n \setminus \{y\}$, then the integrals of ω_y over homotopic cycles are equal specifically:

$$\int_{g_1 \circ \varphi_{\varepsilon_1}(\partial \overline{P})} \omega_y = \int_{g_2 \circ \varphi_{\varepsilon_2}(\partial \overline{P})} \omega_y.$$

Change of variables:

- *Since the determinant of the Jacobi matrix of a mapping* $\det(D\varphi_\varepsilon) = 1$ *at the faces of the parallelepiped* $\overline{P}$*, we can change variables in the integrals to show:*

$$\int_{g_1(\partial K_{\varepsilon_1}(P))} \omega_y = \int_{g_1 \circ \varphi_{\varepsilon_1}(\partial \overline{P})} \omega_y,$$

and

$$\int_{g_2(\partial K_{\varepsilon_2}(P))} \omega_y = \int_{g_2 \circ \varphi_{\varepsilon_2}(\partial \overline{P})} \omega_y.$$

Combinig these results , we have :

$$\int_{g_1(\partial K_{\varepsilon_1}(P))} \omega_y = \int_{g_2(\partial K_{\varepsilon_2}(P))} \omega_y.$$

Constancy of the degree function

- *From Lemma 8.2.1 the degrees of* f*,*g_1 *and* g_2 *are related as follows:*

$$\deg f(K_{\varepsilon_1}(P), y) = \deg g_1(K_{\varepsilon_1}(P), y),$$

and

$$\deg f(K_{\varepsilon_2}(P), y) = \deg g_2(K_{\varepsilon_2}(P), y).$$

- *Using the integral formula for the degree:*

$$\deg g_1(K_{\varepsilon_1}(P), y) = \frac{1}{\mu(S^{n-1})} \int_{g_1(\partial K_{\varepsilon_1}(P))} \omega_y,$$

$$\deg g_2(K_{\varepsilon_2}(P), y) = \frac{1}{\mu(S^{n-1})} \int_{g_2(\partial K_{\varepsilon_2}(P))} \omega_y.$$

- *Since the integrals are equal,it follows that :*

$$\deg f(K_{\varepsilon_1}(P), y) = \deg f(K_{\varepsilon_2}(P), y).$$

Thus, the function $\deg f(K_\varepsilon(P), y)$ *is constant for* $0 < \varepsilon < \varepsilon(y)$.

Existence of the limit

- *The constancy of* $\deg f(K_\varepsilon(P), y)$ *for* $0 < \varepsilon < \varepsilon(y)$ *implies that the limit:*

$$\lim_{\varepsilon \to 0^+} \deg f(K_\varepsilon(P), y)$$

exists and is finite.

- *By the corollary 7.2.1 to theorem 7.2.1 ,the set* $f^{-1}(y) \cap P_0$ *is finite for almost all* $y \in \mathbb{R}^n$.
- *Therefore ,the above limit exists for almost all* $y \in \mathbb{R}^n$.

Step (4): Measurability of $\deg f(P, y)$

Compact parallelepiped $K \subset G$**:**

- *If K is a compact parallelepiped such that $K \subset G$, then* $\deg f(K, y)$ *is defined for* $y \in \mathbb{R}^n \setminus f(\partial K)$*. The function:*

$$d(K, y) = \begin{cases} \deg f(K, y), & \text{if } y \in \mathbb{R}^n \setminus f(\partial K), \\ 0, & \text{if } y \in f(\partial K), \end{cases}$$

- *is Lebesgue measurable. This follows because $d(K, y)$ is constant on connected open components of $\mathbb{R}^n \setminus f(\partial K)$, and $f(\partial K)$ is compact (hence measurable).*

Sequence of shrunken parallelepipeds $K_{1/m}(P)$**:**

- *For $m > 1/\varepsilon_0$:*

$$K_{1/m}(P) \subset \overline{P_0} \subset G$$

(see the proof of the step(02))

- *Thus , the function $d(K_{1/m}(P), y)$ is defined and measurable for $y \in \mathbb{R}^n$.*

Limit of the degree function

- *If $1/m < \varepsilon(y)$:*

$$d(K_{1/m}(P), y) = \deg f(K_{1/m}(P), y).$$

- *Taking the limit as $m \to \infty$ (or equivalently $\varepsilon \to 0^+$):*

$$\begin{aligned} \lim_{m\to\infty} d\left(K_{\frac{1}{m}}(P), y\right) &= \lim_{\varepsilon\to 0^+} \deg f(K_\varepsilon(P), y) \\ &= \deg f(P, y) \end{aligned}$$

This limit exists for almost all $y \in \mathbb{R}^n$, and the function $\deg f(P, y)$ *is measurable as the pointwise limit of measurable functions.*

Lemma 8.3.1. *Let $f : G \to \mathbb{R}^n$ be a mapping with locally-finite variation. If P is a cell such that $P \subset G$ and*

$$P = \bigcup_{k=1}^{m} P_k,$$

where the P_k are disjoint cells, then:

$$\deg f(P, y) = \sum_{k=1}^{m} \deg f(P_k, y)$$

for almost all $y \in \mathbb{R}^n$.

Proof 8.3.2. Step (1): Additivity for cells sharing a common face:

Let P and Q be two cells such that:

- $\overline{P}$ *and Q share a common $(n-1)$-dimensional face.*
- $\overline{P \cup Q} \subset G$*, where G is the domain of f.*

Claim:
For almost all $y \in \mathbb{R}^n$:

$$\deg f(P \cup Q, y) = \deg f(P, y) + \deg f(Q, y).$$

Finite preimage set:

- *Since $P \cup Q \subset G$, there exists $\varepsilon_0 > 0$ such that $\overline{P \cup Q_0} \subset G$, where $\overline{P \cup Q_0}$ is a slightly expanded version of $\overline{P \cup Q}$.*
- *By the Theorem 8.3.1, the set $f^{-1}(y) \cap (P \cup Q)_0$ is finite for almost all $y \in \mathbb{R}^n$.*
- *Consequently, the sets $f^{-1}(y) \cap P_0$ and $f^{-1}(y) \cap Q_0$ are also finite.*

Existence of small ε:

- *For sufficiently small $\varepsilon > 0$, the degrees $\deg f(P, y)$ and $\deg f(Q, y)$ can be expressed as: There exist ε_1 and ε_2 such that:*

$$0 < \varepsilon < \varepsilon_1 \Rightarrow \deg f(P, y) = \deg f(K_\varepsilon(P), y),$$

and

$$0 < \varepsilon < \varepsilon_2 \Rightarrow \deg f(Q, y) = \deg f(K_\varepsilon(Q), y),$$

where $K_\varepsilon(P)$ and $K_\varepsilon(Q)$ are shrunken versions of P and Q, respectively.

- *If $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, then :*

$$\deg f(P, y) + \deg f(Q, y) = \deg f(K_\epsilon(P), y) + \deg f(K_\epsilon(Q), y)$$

Approximation by C^1-mappings:

- *Let $g : K_\varepsilon(P \cup Q) \to \mathbb{R}^n$ be a C^1-mapping approximating f such that:*

$$\|f - g\| < \rho\left(y, f(\partial K_\varepsilon(P)) \cup f(\partial K_\varepsilon(Q))\right).$$

Additivity of the degree for g:

- *By Lemma 8.2.1, the degree satisfies:*

$$\deg f(K_\varepsilon(P), y) = \deg g(K_\varepsilon(P), y),$$

$$\deg f(K_\varepsilon(Q), y) = \deg g(K_\varepsilon(Q), y).$$

- *The mappings $g|_{\partial K_\varepsilon(P)}$ and $g|_{\partial K_\varepsilon(Q)}$ contribute to the boundary integral in a way that respects the shared face D_ε between P and Q.*

Orientation of shared faces:

- *The boundaries $\partial K_\varepsilon(P)$ and $\partial K_\varepsilon(Q)$ intersect along a common face D_ε.*
- *The orientation of D_ε induced from $K_\varepsilon(P)$ is opposite to the orientation induced from $K_\varepsilon(Q)$. This ensures that contributions from D_ε cancel out in the integral formula for the degree.*

Integral formula:

- *Using the integral formula for the degree:*

$$\deg g(K_\varepsilon(P), y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K_\varepsilon(P))} \omega_y,$$
$$\deg g(K_\varepsilon(Q), y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K_\varepsilon(Q))} \omega_y.$$

Adding these contributions:

$$\deg g(K_\varepsilon(P), y) + \deg g(K_\varepsilon(Q), y) = \frac{1}{\mu(S^{n-1})} \int_{g(\partial K_\varepsilon(P \cup Q))} \omega_y.$$

- *Since $\overline{P}$ and $\overline{Q}$ have a common $(n-1)$-dimensional face, the sets $\partial K_\varepsilon(P)$ and $\partial K_\varepsilon(Q)$ have a common face D_ε, and the orientation of D_ε induced from $K_\varepsilon(P)$ is opposite to the orientation induced from $K_\varepsilon(Q)$.*
- *In this case,*

$$g|_{\partial K_\varepsilon(P \cup Q)} = g|_{\partial K_\varepsilon(P)} + g|_{\partial K_\varepsilon(Q)}$$

(the sum of cycles), and

$$g\left(\partial K_\varepsilon(P)\right) \omega_y + g\left(\partial K_\varepsilon(Q)\right) \omega_y = g\left(\partial K_\varepsilon(P \cup Q)\right) \omega_y.$$

- *Thus, if $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, then :*

$$\deg f(P, y) + \deg f(Q, y) = \frac{1}{\mu(S^{n-1})} g\left(\partial K_\varepsilon(P \cup Q)\right) \omega_y$$

$$\begin{aligned} \deg f(P, y) + \deg f(Q, y) &= \deg g\left(K_\varepsilon(P \cup Q), y\right), \\ &= \deg f(P \cup Q, y). \end{aligned}$$

Step (2): Decomposition of intervals:

Partitioning each interval $[a_i, b_i)$:

- *We start by decomposing each interval $[a_i, b_i)$ into smaller disjoint subintervals:*

$$[a_i, b_i) = \bigcup_{j_i=0}^{k_i-1} [a_i^{(j_i)}, a_i^{(j_i+1)}), \tag{8.1}$$

where:

$$a_i^{(0)} = a_i, \quad a_i(k_i) = b_i, \quad a_i^{(j_i)} < a_i^{(j_i+1)} \quad \textit{for all } j_i.$$

This partition ensures that:

- *The intervals $[a_i^{(j_i)}, a_i^{(j_i+1)})$ are disjoint.*
- *Any point $x_i \in [a_i, b_i)$ belongs to exactly one subinterval.*

Notation for subcells:

Using the partition points, we define subcells of P as follows:

- *For $m < n$, let:*

$$P_{j_1 j_2 \cdots j_m} = \prod_{i=1}^{m} [a_i^{(j_i)}, a_i^{(j_i+1)}) \times \prod_{i=m+1}^{n} [a_i, b_i).$$

- *For $m = n$, let:*

$$P_{j_1 j_2 \cdots j_n} = \prod_{i=1}^{n} [a_i^{(j_i)}, a_i^{(j_i+1)}).$$

These subcells form a decomposition of P into smaller disjoint cells.

Recursive decomposition of P:

The cell P can be expressed as:

$$P = \bigcup_{j_1=0}^{k_1-1} \bigcup_{j_2=0}^{k_2-1} \cdots \bigcup_{j_m=0}^{k_m-1} P_{j_1 j_2 \cdots j_m}, \quad m = 1, \ldots, n \tag{8.2}$$

where:

- *The union is over all indices $j_1, j_2, \ldots, j_m$,*
- *The subcells $P_{j_1 j_2 \cdots j_m}$ are disjoint.*

For example:

- *When $m = 1$, P is decomposed into subcells along the first coordinate:*

$$P = \bigcup_{j_1=0}^{k_1-1} P_{j_1},$$

  where $P_{j_1} = [a_1^{(j_1)}, a_1^{(j_1+1)}) \times \prod_{i=2}^{n} [a_i, b_i)$.

- *When $m = n$, P is fully decomposed into subcells:*

$$P = \bigcup_{j_1=0}^{k_1-1} \bigcup_{j_2=0}^{k_2-1} \cdots \bigcup_{j_n=0}^{k_n-1} P_{j_1 j_2 \cdots j_n}.$$

Shared faces between neighboring cells:

- *From equality (8.1), neighboring subcells in the decomposition share common $(n-1)$-dimensional faces. For example:*
- *If $P_{j_1 j_2 \cdots j_m}$ and $P_{j'_1 j'_2 \cdots j'_m}$ differ only in one index (e.g., $j_1 \neq j'_1$ but $j_2 = j'_2$, …, $j_m = j'_m$), then their boundaries intersect along an $(n-1)$-dimensional face.*
  This structure ensures that the degree is additive across neighboring subcells, as shown in Step (1).

Induction hypothesis:

- *Assume that for any cell P decomposed into subcells up to dimension $m = n-1$, the degree satisfies:*

$$\deg f(P,y) = \sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}\cdots\sum_{j_m=0}^{k_m-1} \deg f(P_{j_1j_2\cdots j_m}, y) \tag{8.3}$$

for almost all $y \in \mathbb{R}^n$.

Base case ($m = 1$):

- *When $m = 1$, the cell P is decomposed along the first coordinate:*

$$P = \bigcup_{j_1=0}^{k_1-1} P_{j_1},$$

where:

$$P_{j_1} = [a_1^{(j_1)}, a_1^{(j_1+1)}) \times \prod_{i=2}^{n} [a_i, b_i).$$

- *By step (1), the degree satisfies:*

$$\deg f(P,y) = \sum_{j_1=0}^{k_1-1} \deg f(P_{j_1}, y),$$

for almost all $y \in \mathbb{R}^n$.

Base case ($m = 1$):

- *Assume the hypothesis holds for $m = n-1$. We now prove it for $m = n$.*

Decompose P:

- *Using equality (8.2), write:*

$$P = \bigcup_{j_1=0}^{k_1-1}\bigcup_{j_2=0}^{k_2-1}\cdots\bigcup_{j_n=0}^{k_n-1} P_{j_1j_2\cdots j_n}.$$

Decompose intermediate cells:

- *Each intermediate cell $P_{j_1j_2\cdots j_m}$ can be further decomposed along the $(m+1)$-th coordinate:*

$$P_{j_1j_2\cdots j_m} = \bigcup_{j_{m+1}=0}^{k_{m+1}-1} P_{j_1j_2\cdots j_m j_{m+1}}.$$

- *By Step (1), the degree satisfies:*

$$\deg f(P_{j_1j_2\cdots j_m}, y) = \sum_{j_{m+1}=0}^{k_{m+1}-1} \deg f(P_{j_1j_2\cdots j_m j_{m+1}}, y),$$

for almost all $y \in \mathbb{R}^n$.

- *Summing over all indices $j_1, j_2, \ldots, j_m$, we obtain:*

$$\deg f(P, y) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_m=0}^{k_m-1} \deg f(P_{j_1 j_2 \ldots j_m}, y).$$

- *Replacing $P_{j_1 j_2 \ldots j_m}$ with its decomposition along the $(m+1)$-th coordinate, we get:*

$$\deg f(P, y) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_n=0}^{k_n-1} \deg f(P_{j_1 j_2 \ldots j_n}, y),$$

for almost all $y \in \mathbb{R}^n$.

Step (3) : General decomposition into disjoint cells

- *Let $P = \prod_{i=1}^n [a_i, b_i)$ and $P_k = \prod_{i=1}^n [c_i^{(k)}, d_i^{(k)}]$ ($k = 1, \ldots, m$) such that:*

$$P = \bigcup_{k=1}^{m} P_k,$$

and the P_k are disjoint.

Refinement of partitions:

- *In this case, $[c_i^{(k)}, d_i^{(k)}) \subset [a_i, b_i)$. For each coordinate i, order all of the numbers $c_i^{(k)}$ and $d_i(k)$ ($k = 1, 2, \ldots, m$) in the form of an increasing sequence:*

$$a_i = a_i^{(0)} < a_i^{(1)} < \cdots < a_i^{(k_i)} = b_i. \tag{8.4}$$

- *This refinement partitions P into subcells $P_{j_1 j_2 \ldots j_n}$, where:*

$$P_{j_1 j_2 \ldots j_n} = \prod_{i=1}^{n} [a_i^{(j_i)}, a_i^{(j_i+1)}].$$

Family of subcells:

- *Denote by T_k the family of subcells $P_{j_1 j_2 \ldots j_n}$ that belong to P_k ($k = 1, \ldots, m$).*
- *Since the P_k are disjoint, each subcell $P_{j_1 j_2 \ldots j_n}$ belongs to exactly one family T_k.*

Applying formula :

- *Using formula (8.3) with $m = n$, we have:*

$$\deg f(P, y) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \cdots \sum_{j_n=0}^{k_n-1} \deg f(P_{j_1 j_2 \ldots j_n}, y).$$

for almost all $y \in \mathbb{R}^n$.

- *Since each $P_{j_1 j_2 \dots j_n}$ belongs to exactly one P_k , we can group terms according to families T_k:*

$$\begin{aligned}\deg f(P,y) &= \sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}\cdots\sum_{j_n=0}^{k_n-1}\deg f(P_{j_1 j_2\dots j_n},y)\\ &= \sum_{k=1}^{m}\sum_{P_{j_1 j_2\dots j_n}\in T_k}\deg f(P_{j_1 j_2\dots j_n},y)\\ &= \sum_{k=1}^{m}\deg f(P_k,y)\end{aligned}$$

for almost all $y \in \mathbb{R}^n$.

Compact manifold with boundary:

Let K be a compact n-dimensional manifold with boundary such that $K \neq \emptyset$ and $K \subset G$. The interior $\operatorname{Int} K = K \setminus \partial K$ is a boundaryless manifold.

8.3.1 Definition : Degree of a mapping over a compact set

Let $f : M'^n \to M^n$ be a continuous map between oriented n-manifolds, and let $K \subset M$ be a compact connected subset ($K \neq \emptyset$) such that $f^{-1}(K)$ is compact.
Then the induced map on homology:

$$f_* : H_n(M', M' \setminus f^{-1}(K)) \to H_n(M, M \setminus K)$$

*sends the fundamental class $o_{f^{-1}(K)}$ to an integral multiple of o_K. That is, there exists an integer called the **degree of** f **over** K, denoted $\deg_K f$, such that:*

$$f_*(o_{f^{-1}(K)}) = (\deg_K f) \cdot o_K.$$

Special cases

- ***If** $K = \emptyset$**:***
  The degree $\deg_K f$ is not defined. By convention, we can agree that:

$$\deg_\emptyset f = \mathbb{Z} \quad \textit{(the set of all integers)}.$$

- ***If** $K = \{y\}$ **is a single point and** M, M' **are open subsets of** $\mathbb{S}^n$**:***
  The definition of $\deg_K f$ reduces to the classical degree of a map at a point (see Section IV.5).
  - *If $f^{-1}(K) = \emptyset$ (e.g., if $y \notin \operatorname{im}(f)$), then:*

$$\deg_K f = 0.$$

  - *If f is the inclusion map of an open subset $M' \subset M$, with orientations agreeing, then:*

$$\deg_K f = 1 \quad \textit{for every } K \subset M'.$$

  - *More generally, if f is a homeomorphism from M' onto an open subset of M, then:*

$$\deg_K f = \pm 1 \quad \textit{for every } K \subset \operatorname{im}(f).$$

8.3.2 Generalization to larger compact sets

It is sometimes convenient to replace $f^{-1}(K)$ with a larger compact set $C \supset f^{-1}(K)$, provided C satisfies certain conditions:

- ***Compact support:*** *Let $C \subset M'$ be a compact set such that $f^{-1}(K) \subset \operatorname{Int}(C)$. Then the degree $\deg_K f$ can be computed using C instead of $f^{-1}(K)$, because:*
$$H_n(M', M' \setminus C) \cong H_n(M', M' \setminus f^{-1}(K)).$$
- ***Homotopy invariance:*** *If f is homotopic to another map g with $g^{-1}(K)$ compact, then:*
$$\deg_K f = \deg_K g.$$
- ***Additivity over disjoint sets:*** *If $K_1, K_2 \subset M$ are disjoint compact sets, and $f^{-1}(K_1 \cup K_2)$ is compact, then:*
$$\deg_{K_1 \cup K_2} f = \deg_{K_1} f + \deg_{K_2} f.$$

8.3.3 Proposition (Additivity)

Let $f : M' \to M$ be a continuous map between oriented n-manifolds and $K \subset M$ a compact set as in Definition 4.2. Suppose that M' is a finite union of open sets M'_λ ($\lambda = 1, 2, \ldots, r$), such that the sets
$$K'_\lambda = f^{-1}(K) \cap M'_\lambda$$
are mutually disjoint. Then:
$$\deg_K f = \sum_{\lambda=1}^{r} \deg_K f^\lambda,$$
where $f^\lambda = f|_{M'_\lambda}$ is the restriction of f to M'_λ.

Key observations

1. *Each K'_λ is compact, since $f^{-1}(K)$ is a topological sum of the K'_λ.*
2. *The additivity reflects that the global degree is the sum of the degrees over disjoint parts of the preimage.*

Proof 8.3.3. **Step 1: Decomposition of homology groups**

Consider the maps:
$$\bigoplus_{\lambda=1}^{r} H_n(M'_\lambda, M'_\lambda - K'_\lambda) \xrightarrow{(i^*_\lambda)} H_n(M', M' - f^{-1}(K)) \xrightarrow{i^*_\infty} H_n(M', M' - Q),$$
where i^*_λ are the inclusion-induced maps, and $Q \in f^{-1}(K)$ is an arbitrary point.

Step 2: Behavior of fundamental classes

Applying $i^*_\infty \circ i^*_\lambda$ to the fundamental classes $o_{K'_\lambda}$ gives:
$$i^Q_* \left(i^\lambda_*(o_{K'_\lambda})\right) = \begin{cases} o_Q & \text{if } Q \in K'_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$
This implies:
$$i^*_N(\{o_{K'_\lambda}\}) = o_{f^{-1}(K)},$$
where $o_{f^{-1}(K)}$ is the fundamental class of the preimage.

Step 3: Global contribution and degree

Now consider:

$$\begin{aligned}(\deg_K f)\, o_K &= f_*\left(o_{f^{-1}(K)}\right) \\ &= f_*\left\{i_*^\lambda(o_{K'_\lambda})\right\} \\ &= \sum_{\lambda=1}^{r} f_*^\lambda\left(o_{K'_\lambda}\right) \\ &= \left(\sum_{\lambda=1}^{r} \deg_K f^\lambda\right) o_K.\end{aligned}$$

By comparing both sides:

$$\boxed{\deg_K f = \sum_{\lambda=1}^{r} \deg_K f^\lambda.}$$

Interpretation of $\deg_p f$

Number of points in $f^{-1}(p)$ counted with multiplicities

$$\deg_p f = \sum_{x\in f^{-1}(p)} k(x),$$

where $k(x)$ is the local degree at x, determined by the orientation:

$$k(x) = \begin{cases} +1 & \text{if } x \text{ is positively oriented}, \\ -1 & \text{if } x \text{ is negatively oriented}. \end{cases}$$

Detailed explanation of the proof

Step 1: Direct sum of homology

$$H_n(M', M' - f^{-1}(K)) \cong \bigoplus_{\lambda=1}^{r} H_n(M'_\lambda, M'_\lambda - K'_\lambda).$$

Step 2: Pushforward of fundamental classes

$$o_{f^{-1}(K)} = \sum_{\lambda} i_*^\lambda(o_{K'_\lambda}),$$

$$f_*\left(o_{f^{-1}(K)}\right) = \sum_{\lambda} f_*^\lambda(o_{K'_\lambda}).$$

Step 3: Additivity of contributions

$$f_*^\lambda(o_{K'_\lambda}) = (\deg_K f^\lambda) o_K.$$

Step 4: Conclude

$$f_*(o_{f^{-1}(K)}) = (\deg_K f)o_K = \left(\sum_\lambda \deg_K f^\lambda\right) o_K.$$

Examples

Example 1: Map between spheres

Let $f : \mathbb{S}^n \to \mathbb{S}^n$, and $K = \{p\}$. If $f^{-1}(p) = \{x_1, \dots, x_q\}$, then:

$$\deg_p f = \sum_{i=1}^{q} k(x_i).$$

Example 2: Disjoint subsets

If $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$, then:

$$\deg_K f = \deg_{K_1} f + \deg_{K_2} f.$$

Remark 8.3.1.

1. ***Homotopy invariance:*** *If $f \simeq g$ and both have compact preimages of K, then $\deg_K f = \deg_K g$.*
2. ***Locally-finite variation:*** *If $f : G \to \mathbb{R}^n$ has locally finite variation, then $f^{-1}(y) \cap P$ is finite for almost every y.*
3. ***Non-oriented manifolds:*** *The mod 2 degree:*

$$\deg_K f \mod 2 = \left|f^{-1}(K)\right| \mod 2.$$

Summary

$$\boxed{\deg_K f = \sum_{\lambda=1}^{r} \deg_K f^\lambda,}$$

with $f^\lambda = f|_{M'_\lambda}$.
This expresses the degree as a sum of local contributions from disjoint subsets of the preimage.

Isolated preimage point:

A point $x \in G$ is said to be isolated in the set $f^{-1}(f(x))$ if there exists a neighborhood $V(x)$ such that:

$$f^{-1}(f(x)) \cap V(x) = \{x\}.$$

8.3.4 Multiplicity $k(x)$

The multiplicity $k(x)$ of a point x is defined as the degree $\deg f(K, f(x))$ for any compact n-dimensional manifold K with $x \in \operatorname{Int} K$ and $K \subset V(x)$, where $V(x)$ is a sufficiently small neighborhood of x.

8.3.5 Additivity of degree

The degree satisfies the additivity property:

$$\deg f(K_1 \cup K_2, y) = \deg f(K_1, y) + \deg f(K_2, y),$$

provided K_1 and K_2 are disjoint compact manifolds with boundary and $y \notin f(\partial K_1 \cup \partial K_2)$.

Lemma 8.3.2. *If $x \in G$ is an isolated point of the set $f^{-1}(f(x))$, then there exists a neighborhood $V(x)$ of x such that for any n-dimensional compact C^0-manifold K with boundary satisfying $x \in \operatorname{Int} K$ and $K \subset V(x)$, the degree $\deg f(K, f(x))$ is defined and does not depend on K.*

Proof 8.3.4. Compact parallelepiped K_0:

- *Construct a compact parallelepiped $K_0 \subset G$ such that $x \in \operatorname{Int} K_0$ and K_0 contains no other points of $f^{-1}(f(x))$ except x.*
- *This ensures that if $x' \in K_0$ and $x' \neq x$, then $f(x') \neq f(x)$.*

***Claim** $V(x) = \operatorname{Int} K_0$:*

- *Let K be any compact n-dimensional manifold with boundary such that $K \subset \operatorname{Int} K_0$ and $x \in \operatorname{Int} K$.*
- *Decompose $\operatorname{Int} K_0$ as:*
  $$\operatorname{Int} K_0 = \operatorname{Int} K \cup (\operatorname{Int} K_0 \setminus \{x\}),$$
  where both sets on the right-hand side are open.

Degree over** $\operatorname{Int} K_0 \setminus \{x\}$**:

- *Since $f(x) \notin f(\operatorname{Int} K_0 \setminus \{x\})$ (by construction of K_0), the degree:*
  $$\deg f(\operatorname{Int} K_0 \setminus \{x\}, f(x)) = 0.$$

Additivity of degree:

- *By the additivity property:*
  $$\deg f(K_0, f(x)) = \deg f(\operatorname{Int} K, f(x)) + \deg f(\operatorname{Int} K_0 \setminus \{x\}, f(x)).$$
- *Substituting $\deg f(\operatorname{Int} K_0 \setminus \{x\}, f(x)) = 0$, we get:*
  $$\deg f(K_0, f(x)) = \deg f(\operatorname{Int} K, f(x)).$$

Conclusion:

- *The degree $\deg f(K, f(x))$ is independent of the choice of K, as long as $K \subset \operatorname{Int} K_0$ and $x \in \operatorname{Int} K$.*
- *Define $k(x) = \deg f(K_0, f(x))$, which serves as the multiplicity of x.*

Lemma 8.3.3. *Let P be a cell (e.g., a parallelepiped). Assume that the degree $\deg f(P, f(x))$ is defined and the set $f^{-1}(f(x)) \cap P$ consists of a single point x.*
Then x is an isolated point of $f^{-1}(f(x)) \cap P$, and:

$$\deg f(P, f(x)) = k(x).$$

Proof 8.3.5. Definition of $\deg f(P, f(x))$**:**

- *By assumption, the degree* $\deg f(P, f(x))$ *is defined, meaning there exists* $\varepsilon_{f(x)} > 0$ *such that:*

$$f(x) \notin f(\partial K_\varepsilon(P)) \quad \textit{for all } 0 < \varepsilon < \varepsilon_{f(x)}.$$

- **Isolation of** x**:**

  Suppose $x' \neq x$ *and* $x' \in f^{-1}(f(x))$.
  Then $f(x') = f(x)$.

- *For sufficiently small* $\varepsilon > 0$, $x' \notin K_\varepsilon(P)$ *because* $x' \notin P$ *or* x' *lies outside* $K_\varepsilon(P)$.

Contradiction argument for $x' \notin \operatorname{Int} K_\varepsilon(P)$

Assume $x' \neq x$ *and* $x' \in f^{-1}(f(x))$. *To show that* $x' \notin \operatorname{Int} K_\varepsilon(P)$, *we proceed as follows:*

Assumption:

Suppose $x' \in \operatorname{Int} K_{\varepsilon_0}(P)$ *for some* ε_0 *such that* $0 < \varepsilon_0 < \varepsilon_{f(x)}$.
This implies:

$$a_i - \varepsilon_0 < x'_i < b_i - \varepsilon_0, \quad \textit{for all i = 1, ..., n .}$$

Contradiction with $x' \notin P$**:**

- *By the assumptions of Lemma 8.3.3, we have* $x' \notin P$, *thus, at least one of the inequalities* $a_i \leq x'_i < b_i$ *does not hold.*
- *Since* $x'_i < b_i - \varepsilon_0 < b_i$ *for all* i, *it must be that* $x'_i < a_i$ *for some index* i_0. *Specifically,*

$$a_{i_0} > x'_{i_0}.$$

Define ε_1**:**

- *Define*

$$\varepsilon_1 = \max_{1 \leq i \leq n} \{a_i - x'_i\} > 0.$$

- *By construction,* $\varepsilon_1 > 0$, *and the following chain of implications holds:*

$$a_i - \varepsilon_0 < x'_i \quad \textit{for all } i \quad \Rightarrow \quad \varepsilon_0 > a_i - x'_i \quad \textit{for all } i \quad \Rightarrow \quad \varepsilon_0 > \varepsilon_1.$$

Verification of $x' \in \partial K_{\varepsilon_1}(P)$:

- *For* ε_1, *consider the shrunken parallelepiped*

$$K_{\varepsilon_1}(P) = \prod_{i=1}^{n} [a_i - \varepsilon_1, b_i - \varepsilon_1].$$

- *By definition of* ε_1, *we have:*

$$a_i - \varepsilon_1 \leq x'_i < b_i - \varepsilon_1 \quad \textit{for all } i,$$

  with equality holding for some index i_0*:*

$$a_{i_0} - \varepsilon_1 = x'_{i_0}.$$

  This implies $x' \in \partial K_{\varepsilon_1}(P)$.

Contradiction with $x' \notin \partial K_{\varepsilon_1}(P)$:

- *Since* $0 < \varepsilon_1 < \varepsilon_0 < \varepsilon_{f(x)}$*, it follows that* $f(x') \neq f(x) \quad$ *for* $x' \in \partial K_{\varepsilon_1}(P)$.
- *Therefore,* $x' \notin \partial K_{\varepsilon_1}(P)$*, leading to a contradiction.*

$$\boxed{x' \notin \operatorname{Int} K_\varepsilon(P) \quad \textit{for any} \quad \varepsilon > 0.}$$

Localization of x in $K_\varepsilon(P)$:

Behavior of x***:***

- *Since* $x \in P$*, for sufficiently small* $\varepsilon > 0$*, we have:*

$$a_i - \varepsilon < x_i < b_i - \varepsilon \quad \textit{for all } i = 1, \ldots, n.$$

- *This implies* $x \in \operatorname{Int} K_\varepsilon(P)$ *for* $0 < \varepsilon < \varepsilon_{f(x)}$.

Exclusion of other points:

- *Any point* $x' \neq x$ *satisfying* $f(x') = f(x)$ *lies outside* $K_\varepsilon(P)$ *for sufficiently small* $\varepsilon > 0$.
- *Specifically,*

$$x' \notin \partial K_\varepsilon(P) \cup \operatorname{Int} K_\varepsilon(P) = K_\varepsilon(P).$$

Isolation of x**:**

- *For* $0 < \varepsilon < \varepsilon_{f(x)}$*, the set* $f^{-1}(f(x)) \cap K_\varepsilon(P)$ *consists only of* x.
- *Thus,* x *is isolated in set* $f^{-1}(f(x))$ *and single point of the mentionel set inside the parallepiped* $K_\varepsilon(P)$.

Obviously, any set $\operatorname{Int} K_\varepsilon(P)$ *with* $0 < \varepsilon < \varepsilon_{f(x)}$ *has the desired properties of the neighborhood* $V(x)$ *in Lemma 8.3.2 .*
In addition,

$$\deg f(P, f(x)) = \deg f(K_\varepsilon(P), f(x)) = k(x).$$

Corollary 8.3.1. *If* x *is an isolated point of the set* $f^{-1}(f(x))$*, then there exists a neighborhood* $V(x)$ *of the point* x *such that if* $P \subset V(x)$ *is a cell and* $x \in P$*, then:*

$$\deg f(P, f(x)) = k(x).$$

Obviously, it is enough to take the neighborhood $V(x)$ *given by Lemma 8.3.2; in this case, the set* $f^{-1}(f(x)) \cap P$ *consists of the single point* x.

9

Measures of oriented parts of the image of a set

*The references for this chapter are [**43**, **15**, **24**, **55**, **13**, **25**, **50**, **27**, **2**, **49**].*

Definition 9.0.1. *Let x be an isolated point of the set $f^{-1}(f(x))$. The mapping f is called positively (negatively) oriented at the point x if $k(x) > 0$ ($k(x) < 0$, respectively).*

- *For $y \in \mathbb{R}^n$, denote:*

$$f_+^{-1}(y) = \left\{x \in f^{-1}(y) \mid k(x) > 0\right\} \quad \text{and} \quad f_-^{-1}(y) = \left\{x \in f^{-1}(y) \mid k(x) < 0\right\}.$$

- *For a set $E \subset G$, denote*

$$N_f^+(E, y) = \begin{cases} \operatorname{card}(f_+^{-1}(y) \cap E) & \text{if the set } f_+^{-1}(y) \cap E \text{ is finite}, \\ +\infty & \text{if the set } f_+^{-1}(y) \cap E \text{ is infinite}; \end{cases}$$

  and

$$N_f^-(E, y) = \begin{cases} \operatorname{card}(f_-^{-1}(y) \cap E) & \text{if the set } f_-^{-1}(y) \cap E \text{ is finite}, \\ +\infty & \text{if the set } f_-^{-1}(y) \cap E \text{ is infinite}. \end{cases}$$

- *Since the sets $f_+^{-1}(y)$ and $f_-^{-1}(y)$ are disjoint and $f_+^{-1}(y) \cup f_-^{-1}(y) \subset f^{-1}(y)$, the inequality*

$$N_f^+(E, y) + N_f^-(E, y) \le N_f(E, y) \quad \text{holds.}$$

Theorem 9.0.1. *Let $f : G \to \mathbb{R}^n$ be a mapping with locally-finite variation. If P is a cell such that $\overline{P} \subset G$, then the functions $N_f^+(P, y)$ and $N_f^-(P, y)$ are summable in $\mathbb{R}^n$ with respect to the Lebesgue measure.*

Proof 9.0.1. *Let $\tau = \{P_k\}_{k=1}^m$ be a decomposition of P into disjoint cells.*

- *Define:*

$$N_\tau^+(y) = \sum_{k=1}^{m} \chi_{f+}(P_k)(y).$$

- *Where $\chi_{f+}(P_k)(y)$ is the indicator function of the set :*

$$f_+(P_k) = \{y \in \mathbb{R}^n : \deg f(P_k, y) > 0\}.$$

- *By Theorem 8.3.1, the function* $\deg f(P)$ *is measurable; hence, the set above is measurable as well. (here $\chi_{f+}(P_k)$ is the indicator function of the set $f_+(P_k)$).*
  Then obviously the function N_τ^+ is measurable.

Step (1): Monotonicity under refinement

Claim:

If a decomposition τ' refines τ, then:

$$N^+_{\tau'}(y) \geq N^+_{\tau}(y) \quad \textit{for almost all } y \in \mathbb{R}^n.$$

Refinement property:

- *If τ' refines τ, then any cell $P_k \in \tau$ can be written as:*

$$P_k = \bigcup_{i=1}^{m_k} P_{k,i},$$

  where $P_{k,i} \in \tau'$ are disjoint subcells of P_k.

Additivity of degree:

- *By additivity of the degree :*

$$\deg f(P_k, y) = \sum_{i=1}^{m_k} \deg f(P_{k,i}, y) \quad \textit{for almost all } y \in \mathbb{R}^n.$$

  (in particular, for almost all $y \in f_+(P_k)$)

- *If $\deg f(P_k, y) > 0$, then at least one of the terms (for the latter points y), there exists an index i, $1 \leq i \leq m_k$, such that $\deg f(P_{k,i}, y) > 0$.*
  This implies:

$$y \in f_+(P_k) \Rightarrow y \in \bigcup_{i=1}^{m_k} f_+(P_{k,i}).$$

Indicator function inequality:

- *The above implication leads to:*

$$\chi_{f+}(P_k)(y) \leq \sum_{i=1}^{m_k} \chi_{f+}(P_{k,i})(y).$$

- *Summing over all k, we get:*

$$N^+_{\tau}(y) = \sum_{k=1}^{m} \chi_{f+}(P_k)(y) \leq \sum_{k=1}^{m} \sum_{i=1}^{m_k} \chi_{f+}(P_{k,i})(y) = N^+_{\tau'}(y).$$

  for almost all $y \in \mathbb{R}^n$

Conclusion:

The sequence $\{N^+_{\tau_s}(y)\}_{s=1}^{\infty}$ increases monotonically for almost all $y \in \mathbb{R}^n$.

Step (2): Indicator function and counting function

Claim:

For almost all $y \in \mathbb{R}^n$:

$$\chi_{f+}(P_k)(y) \le N_f^+(P, y).$$

(a) Indicator function $\chi_{f^+}(P)(y)$***:***

- *The function $\chi_{f^+}(P)(y)$ is defined as:*

$$\chi_{f_+}(P)(y) = \begin{cases} 1, & \text{if } y \in f_+(P), \\ 0, & \text{otherwise}. \end{cases}$$

- *Here,* $f_+(P) = \{y \in \mathbb{R}^n : \deg f(P, y) > 0\}$.

(b) Counting function $N_f^+(P, y)$***:***

- *The function $N_f^+(P, y)$ counts the number of positively oriented preimages of y in P:*

$$N_f^+(P, y) = \begin{cases} \text{card}(f_+^{-1}(y) \cap P), & \text{if } f_+^{-1}(y) \cap P \text{ is finite}, \\ +\infty, & \text{if } f_+^{-1}(y) \cap P \text{ is infinite}. \end{cases}$$

- *Here,* $f_+^{-1}(y) = \{x \in f^{-1}(y) : k(x) > 0\}$.

We aim to prove that:

$$\chi_{f^+}(P)(y) \le N_f^+(P, y) \quad \text{for almost all } y \in \mathbb{R}^n.$$

Existence of a cell P_0:

- *From Theorem 8.3.1 in (Step (2)), there exists a cell $P_0 = \prod_{i=1}^n [a_i - \varepsilon_0, b_i + \varepsilon_0)$ such that:*

  1. $P_0 \subset G \Rightarrow P_0 \subset \overline{P_0} \subset G$
  2. *The set $f^{-1}(y) \cap P_0$ is finite for almost all $y \in \mathbb{R}^n$.*

  Fix a point $y \in \mathbb{R}^n$ satisfying this property and such that $\chi_{f_+}(P)(y) = 1$. This implies:

$$y \in f_+(P), \quad \text{i.e., } \deg f(P, y) > 0.$$

Isolation of preimages:

- *Since $f^{-1}(y) \cap P_0$ is finite, any point $x \in f^{-1}(y) \cap P$ is isolated. By isolating these points, we can decompose P into a finite union of disjoint cells $\{P_k\}_{k=1}^m$ such that:*

  1. *Each cell P_k contains at most one point of $f^{-1}(y) \cap P$.*
  2. *If $x \in f^{-1}(y) \cap P_k$, then $k(x) = \deg f(P_k, y)$ (by Lemma 8.3.3).*

Additivity of degree:

- *Using the additivity property of the degree :*

$$\deg f(P, y) = \sum_{k=1}^{m} \deg f(P_k, y).$$

- *Since* $\deg f(P, y) > 0$*, at least one term* $\deg f(P_k, y) > 0$*. For such a* P_k*, we have:*

$$y \in f^+(P_k).$$

Contribution to $f_+^{-1}(y) \cap P$**:**

- *For each* P_k *containing a positively oriented preimage* $x \in f_+^{-1}(y) \cap P$*, we know:*

$$\deg f(P_k, y) = k(x) > 0.$$

- *Thus,* $x \in f_+^{-1}(y) \cap P_k$*, and the counting function satisfies:*

$$N_f^+(P, y) = \mathit{card}(f_+^{-1}(y) \cap P).$$

Inequality for $\chi_{f_+}(P)(y)$**:**

- *If* $\chi_{f_+}(P)(y) = 1$*, then* $y \in f_+(P)$*, meaning* $\deg f(P, y) > 0$*.*
  From the decomposition $P = \bigcup_{k=1}^{m} P_k$*, it follows that:*

$$\deg f(P, y) = \sum_{k=1}^{m} \deg f(P_k, y),$$

  and at least one term $\deg f(P_k, y) > 0$*. Consequently:*

$$N_f^+(P, y) = \mathit{card}(f_+^{-1}(y) \cap P) \geq 1 = \chi_{f^+}(P)(y).$$

- *If* $\chi_{f_+}(P)(y) = 0$*, then* $y \notin f_+(P)$*, meaning* $\deg f(P, y) \leq 0$*.*
  In this case:

$$N_f^+(P, y) \geq 0 = \chi_{f_+}(P)(y).$$

Conclusion: *For almost all* $y \in \mathbb{R}^n$*:*

$$\chi_{f^+}(P)(y) \leq N_f^+(P, y).$$

Step (3): Lower bound on the limit

Claim:

$$\lim_{s \to \infty} N_{\tau_s}^+(y) \leq N_f^+(P, y) \quad \textit{for almost all } y \in \mathbb{R}^n.$$

Decomposition τ_s**:**

- *Divide* P *into* 2^{sn} *smaller cells* $\{P_{s,i}\}_{i=1}^{2^{sn}}$ *with* $\operatorname{diam}(P_{s,i}) = 2^{-s} \operatorname{diam}(P)$*.*

Monotonicity:

- *From Step (1), the sequence* $\{N^+_{\tau_s}(y)\}_{s=1}^{\infty}$ *increases monotonically:*

$$N^+_{\tau_s}(y) \leq N^+_{\tau_{s+1}}(y).$$

Limit exists:

- *Since* $N^+_{\tau_s}(y)$ *is monotonic non-decreasing and bounded above by* $N^+_f(P,y)$*, the limit*

$$\lim_{s\to\infty} N^+_{\tau_s}(y)$$

exists (finite or infinite).

Inequality:

- *For any decomposition of the set* P *into disjoint cells* $\tau = \{P_k\}_{k=1}^m$*, we have :*

$$\text{card}(f_+^{-1}(y) \cap P) = \sum_{k=1}^{m} \text{card}(f_+^{-1}(y) \cap P_k), \quad \text{i.e.,} \quad N^+_f(P,y) = \sum_{k=1}^{m} N^+_f(P_k,y).$$

- *By Step (2), the inequality* $\chi_{f^+}(P_k)(y) \leq N_{f+}(P_k,y)$ *holds for all* P_k *and almost all* $y \in \mathbb{R}^n$*.Hence:*

$$N^+_{\tau}(y) = \sum_{k=1}^{m} \chi_{f^+}(P_k)(y) \leq \sum_{k=1}^{m} N_{f+}(P_k,y) = N^+_f(P,y)$$

for almost all $y \in \mathbb{R}^n$*.*

- *Taking the limit as* $s \to \infty$*, it follows that :*

$$\lim_{s\to\infty} N^+_{\tau_s}(y) \leq N^+_f(P,y).$$

Step (4): Upper bound on the limit

Claim:

$$\lim_{s\to\infty} N^+_{\tau_s}(y) \geq N^+_f(P,y) \quad \text{for almost all } y \in \mathbb{R}^n.$$

Finite case ($N^+_f(P,y) = q$ **(**q **value is finite)):**

- *Suppose* $N^+_f(P,y) = q$*,meaning* $f_+^{-1}(y) \cap P = \{x_1, \ldots, x_q\}$.

- *By a corollary 8.3.1 to Lemma 8.3.3 , there exist neighborhoods* $\{V_m(x_m)\}_{m=1}^q$ *around each* x_m *such that:*

$$\deg f(P_m,y) = k(x_m) > 0$$

for any cell $P_m \subset V_m(x_m)$ *with* $x_m \in P_m$*.*

Neighborhoods:

- *By the definition of the value $k(x_m)$, the neighborhoods $V_m(x_m)$ may be chosen pairwise disjoint.*

- *Assume that balls $B(x_m; r_m)$ are contained in $V_m(x_m)$.*

  (Choose $r_0 = \min_{1 \le m \le q}\{r_m\}$, where $B(x_m; r_m) \subset V_m(x_m)$.)

- *For sufficiently large s, the diameter of each subcell $P_{s,i} < r_0$.*

$$(P_{s,i_m} = \frac{1}{2^s} \operatorname{diam} P \to 0 \quad \textit{as } s \to \infty.)$$

- *Hence, there exists an integer s_0 such that the diameter of $P_{s,i}$ is less than r_0 for all $s > s_0$. In this case, if $x_m \in P_{s,i_m}$, then $P_{s,i_m} \subset B(x_m; r_m) \subset V_m(x_m)$, and* $\deg f(P_{s,i}, y) = k(x_m) > 0$.

- *Thus, $y \in f_+(P_s, i_m)$,and the indicator function satisfies:*

$$\chi_{f_+(P_{s,i_m})}(y) = 1.$$

- *Each x_m is in some $P_{s,i_m} \subset B(x_m; r_m) \subset V_m(x_m)$, and:*

$$\deg f(P_{s,i_m}, y) = k(x_m) > 0 \Rightarrow y \in f_+(P_{s,i_m}) \Rightarrow \chi_{f+}(P_{s,i_m})(y) = 1.$$

Counting cells:

- *Since the neighborhoods $V_m(x_m)$ are pairwise disjoint, the corresponding subcells P_{s,i_m} are distinct.*

- *The number of such subcells equals $q = N_f^+(P, y)$.*

Inequality:

- *For sufficiently large s:*

$$N_{\tau_s}^+(y) = \sum_{i=1}^{2^s n} \chi_{f^+(\Psi)}(y) \quad \textit{for } s > s_0, \quad \textit{i.e.,} \quad \lim_{s \to \infty} N_{\tau_s}^+(y) = N_f^+(P, y)$$

  Then

$$N_{\tau_s}^+(y) \ge N_f^+(P, y).$$

- *Taking the limit as $s \to \infty$, it follows that:*

$$\lim_{s \to \infty} N_{\tau_s}^+(y) \ge N_f^+(P, y).$$

Infinite case ($N_f^+(P, y) = +\infty$):

- *If $N_f^+(P, y) = +\infty$, then $N_f(P, y) = +\infty$ as well, which occurs on a set of measure zero corollary 7.2.1 to Theorem 7.2.1.*

Conclusion:

For almost all $y \in \mathbb{R}^n$:

$$\boxed{\lim_{s\to\infty} N^+_{\tau_s}(y) \geq N^+_f(P, y).}$$

- *From Steps (3) and (4), we have:*

$$\lim_{s\to\infty} N^+_{\tau_s}(y) = N^+_f(P, y) \quad \text{for almost all } y \in \mathbb{R}^n.$$

- *Similarly, for the negatively oriented case:*

$$\lim_{s\to\infty} N^-_{\tau_s}(y) = N^-_f(P, y) \quad \text{for almost all } y \in \mathbb{R}^n.$$

- *Since $N^+_{\tau_s}(y)$ is measurable for each s (as a sum of indicator functions), and $N^+_f(P, y) = \lim_{s\to\infty} N^+_{\tau_s}(y)$.*
- *It follows that $N^+_f(P, y)$ is measurable.*

Summability of $N^+_f(P, y)$:

- *The summability of $N^+_f(P, y)$ follows from its relationship to $N_f(P, y)$:*

$$N^+_f(P, y) + N^-_f(P, y) \leq N_f(P, y).$$

- *Since $N_f(P, y)$ is summable (by assumption of locally-finite variation), the functions $N^+_f(P, y)$ and $N^-_f(P, y)$ are also summable.*

Lemma 9.0.1. *Let $f : G \to \mathbb{R}^n$ be a continuous mapping. Then the following statements hold:*

1. *If $E = \bigcup_k E_k \subset G$ (union of an at most countable family of pairwise disjoint sets), then :*

$$N_f(E) = \sum_k N_f(E_k).$$

2. *If $E_1 \subset E_2$, then $N_f(E_1) \leq N_f(E_2)$.*
3. *If $E_1 \subset E_2 \subset \cdots \subset E_k \subset E_{k+1} \subset \ldots$ and $E = \bigcup_{k=1}^{\infty} E_k$, then :*

$$N_f(E, y) = \lim_{k\to\infty} N_f(E_k, y) \quad \text{for any } y \in \mathbb{R}^n;$$

4. *If $E_1 \supset E_2 \supset \cdots \supset E_k \supset E_{k+1} \supset \ldots$, $E = \bigcap_{k=1}^{\infty} E_k$, and $N_f(E_1, y) < \infty$, then*

$$N_f(E, y) = \lim_{k\to\infty} N_f(E_k, y).$$

- *If, in addition, f is a mapping with locally-finite variation, then similar statements are valid for the functions $N^+_f(E)$ and $N^-_f(E)$.*

Proof 9.0.2. [1]: Countable additivity of $N_f(E, y)$:

- *statement (1), Let $E = \bigcup_k E_k \subset G$ where $\{E_k\}_{k=1}^{\infty}$ is an at most countable family of pairwise disjoint sets, then:*
- *For any $y \in \mathbb{R}^n$ the equality*

$$f^{-1}(y) \cap E = \cup(f^{-1}(y) \cap E_k) \qquad \textit{holds.}$$

- *Since the sets ets $f^{-1}(y) \cap E_k$ are disjoint, and their cardinalities add up:*

$$\operatorname{card}\left(f^{-1}(y) \cap E\right) = \sum_k \operatorname{card}\left(f^{-1}(y) \cap E_k\right)$$

- *Thus:*

$$N_f(E, y) = \sum_k N_f(E_k, y).$$

- *Similarly, the same argument applies to $N_f^+(E, y)$ and $N_f^-(E, y)$:*

$$N_f^+(E, y) = \sum_k N_f^+(E_k, y), \quad N_f^-(E, y) = \sum_k N_f^-(E_k, y).$$

[2]: Monotonicity of $N_f(E, y)$:

- *statement (2), if $E_1 \subset E_2$, then:*

$$f^{-1}(y) \cap E_1 \subset f^{-1}(y) \cap E_2$$

- *This implies:*

$$N_f(E_1, y) \leq N_f(E_2, y).$$

- *The same applies to $N_f^+(E, y)$ and $N_f^-(E, y)$:*

$$N_f^+(E_1, y) \leq N_f^+(E_2, y), \quad N_f^-(E_1, y) \leq N_f^-(E_2, y).$$

[3]: Limit properties of $N_f(E, y)$:

- *The statements (3) :*
- *Let $E_1 \subset E_2 \subset \cdots$ and $E = \bigcup_{k=1}^{\infty} E_k$, then:*

$$f^{-1}(y) \cap E = \cup_{k=1}^{\infty}(f^{-1}(y) \cap E_k)$$

- *By the monotone continuity of $N_f(E, y)$,we have :*

$$N_f(E, y) = \lim_{k \to \infty} N_f(E_k, y).$$

[4]: Limit properties of $N_f(E, y)$:

- *The statements (4) :*
- *If $E_1 \supset E_2 \supset \cdots$ and $E = \bigcap_{k=1}^{\infty} E_k$ with $N_f(E_1, y) < \infty$, then:*

$$f^{-1}(y) \cap E = \cup_{k=1}^{\infty}(f^{-1}(y) \cap E_k)$$

- *By the monotone continuity of $N_f(E, y)$,we have :*

$$N_f(E, y) = \lim_{k \to \infty} N_f(E_k, y).$$

- *The same properties hold for $N_f^+(E, y)$ and $N_f^-(E, y)$.*

Extension to $N_f^+(E, y)$ and $N_f^-(E, y)$:

The same arguments apply to $N_f^+(E, y)$ and $N_f^-(E, y)$, since these functions also satisfy countable additivity and monotone continuity.

9.0.1 Measures on the semiring of cells:

(a) Definitions of ν_f^+, ν_f^-, and V_f:

Let f be mapping with locally-finite variation and let P a cell such that $\overline{P} \subset G$, define:

$$\nu_f^+(P) = \int_{\mathbb{R}^n} N_f^+(P, y)\, d\lambda_n(y) \quad , \quad \nu_f^-(P) = \int_{\mathbb{R}^n} N_f^-(P, y)\, d\lambda_n(y),$$

$$V_f(P) = \nu_f^+(P) + \nu_f^-(P) = \int_{\mathbb{R}^n} N_f(P, y)\, d\lambda_n(y).$$

(these values are the measures of the positively and negatively oriented parts of the image of the cell P, counted with multiplicities).

(b) Countable additivity

From Lemma 9.0.1, statements (1)--(4), the functions $N_f^+(E, y)$, $N_f^-(E, y)$, and $N_f(E, y)$ are countably additive on the semiring of cells. Therefore,

$$\nu_f^+\left(\bigcup_k P_k\right) = \sum_k \nu_f^+(P_k), \quad \nu_f^-\left(\bigcup_k P_k\right) = \sum_k \nu_f^-(P_k), \quad V_f\left(\bigcup_k P_k\right) = \sum_k V_f(P_k),$$

where $\{P_k\}_{k=1}^{\infty}$ is a disjoint family of cells.

(c) Lebesgue integral properties

The countable additivity of ν_f^+, ν_f^-, and V_f follows from the properties of the Lebesgue integral:

$$\int_{\mathbb{R}^n} \sum_k N_f^+(P_k, y)\, d\lambda_n(y) = \sum_k \int_{\mathbb{R}_n} N_f^+(P_k, y)\, d\lambda_n(y).$$

9.0.2 Extension to σ-algebras via caratheodory process:

Theorem 9.0.2 (Caratheodory extension theorem). *Let $\mathcal{A}$ be an algebra of subsets of a set X, and let $\mu_0 : \mathcal{A} \to [0, \infty]$ be a countably additive pre-measure. Then there exists a measure μ on the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ such that:*

1. *μ extends μ_0, i.e., $\mu|_{\mathcal{A}} = \mu_0$;*
2. *the extension is unique if μ_0 is σ-finite.*

(a): Caratheodory extension theorem:

The measures ν_f^+, ν_f^-, and V_f are initially defined on the semiring of cells $\{P \mid \overline{P} \subset G\}$. To extend these measures to more general subsets of G, we use the Carathéodory extension theorem, which states:

1. *Any countably additive measure defined on a semiring can be uniquely extended to a σ-algebra.*
2. *The extended measure satisfies the same countable additivity and monotone continuity properties.*

(b): Outer measure

We denote the corresponding σ-algebras by $\mathfrak{A}_f^+$, $\mathfrak{A}_f^-$, and $\mathfrak{A}_f$,, respectively, and preserve the notation ν_f^+, ν_f^-, and V_f for the extended measures (we note that the σ-algebra $\mathfrak{A}_f$, and the measure V_f are defined for a continuous mapping f).

Define the outer measure $V_f^*(E)$ for any subset $E \subset G$ as:

$$V_f^*(E) = \inf\{V_f(Q) \mid Q \text{ is open and } E \subset Q \subset G\}.$$

A set $E \subset G$ is measurable with respect to V_f if and only if:

$$V_f^*(E) + V_f^*(G \setminus E) = V_f(G).$$

(c): Extended measures

Using the Carathéodory process, we extend ν_f^+, ν_f^-, and V_f to σ-algebras $\mathfrak{A}_f^+$, $\mathfrak{A}_f^-$, and $\mathfrak{A}_f$, respectively. These extensions preserve the notation ν_f^+, ν_f^-, and V_f.

9.0.3 Regularity of the measures

(a) Regularity of V_f:

The measure V_f is regular, meaning:
- For any subset $E \subset G$, the outer measure $V_f^*(E)$ can be approximated by open sets:

$$V_f^*(E) = \inf\{V_f(Q) \mid Q \text{ is open and } E \subset Q \subset G\}.$$

- A set $E \subset G$ is measurable if and only if there exist G_δ-subsets $K, H \subset G$ such that:

$$E \subset K, \quad K \setminus E \subset H, \quad V_f(H) = 0.$$

In this case:

$$V_f(E) = V_f(K).$$

(b) Regularity of ν_f^+ and ν_f^-:

The same regularity properties hold for ν_f^+ and ν_f^-, since they are defined similarly using $N_f^+(E, y)$ and $N_f^-(E, y)$.

9.0.4 Locally-finite variation :

- *The assumption that f has locally-finite variation ensures that $N_f(P, y)$, $N_f^+(P, y)$, and $N_f^-(P, y)$ are finite for almost all $y \in \mathbb{R}^n$ when $P \subset G$ is a cell.*

- *This finiteness allows us to integrate these functions over $\mathbb{R}^n$, defining the measures ν_f^+, ν_f^-, and V_f.*

9.0.5 Countable additivity :

The measures ν_f^+, ν_f^-, and V_f are countably additive because the counting functions $N_f^+(E, y)$, $N_f^-(E, y)$, and $N_f(E, y)$ are countably additive.

9.0.6 Monotone continuity:

The measures ν_f^+, ν_f^-, and V_f satisfy monotone continuity:

- *For increasing sequences $E_1 \subset E_2 \subset \cdots$, $V_f(E) = \lim_{k\to\infty} V_f(E_k)$,*
- *For decreasing sequences $E_1 \supset E_2 \supset \cdots$ with $V_f(E_1) < \infty$, $V_f(E) = \lim_{k\to\infty} V_f(E_k)$.*

Lemma 9.0.2. *Let $f : G \to \mathbb{R}^n$ be a mapping with locally-finite variation. Then the following statements hold:*

1. *the measures V_f, ν_f^+, and ν_f^- are σ-finite;*

2. *if $K \subset G$ is a Borel set, then K is measurable with respect to the measures V_f, ν_f^+, and ν_f^-;*

3. *the measure V_f is regular, i.e., for any subset $E \subset G$, the outer measure of E is given by the formula*
$$V_f^*(E) = \inf \{V_f(Q) \mid Q \text{ is open and } E \subset Q \subset G\},$$

and a set $E \subset G$ is measurable with respect to the measure V_f if and only if there exist G_δ-subsets $K, H \subset G$ such that $E \subset K$, $K \setminus E \subset H$, and $V_f(H) = 0$ (obviously, $V_f(E) = V_f(K)$ in this case). The measures ν_f^+ and ν_f^- are regular in the same sense.

Proof 9.0.3. Step (1): Decomposition of open sets

*Let $Q \subset G$ be an open set. By standard results in topology (see [Chapter V] [**54**]), there exists a decomposition:*

$$Q = \bigcup_{k=1}^{\infty} P_k,$$

where:

- *The P_k are disjoint cells,*

- *The closures $\overline{P_k} \subset Q$.*

Since f has locally-finite variation, the values $V_f(P_k)$, $\nu_f^+(P_k)$, and $\nu_f^-(P_k)$ are finite for each k. Thus:

$$V_f(Q) = \sum_{k=1}^{\infty} V_f(P_k), \quad \nu_f^+(Q) = \sum_{k=1}^{\infty} \nu_f^+(P_k), \quad \nu_f^-(Q) = \sum_{k=1}^{\infty} \nu_f^-(P_k).$$

This shows that open sets are measurable with respect to V_f, ν_f^+, and ν_f^-.

Step (2): σ-Finiteness of V_f, ν_f^+, and ν_f^-

- *Since G is an open set, it can be written as a countable union of disjoint cells $\{P_k\}_{k=1}^{\infty}$ such that $P_k \subset G$.*
- *For each cell P_k, the measures $V_f(P_k)$, $\nu_f^+(P_k)$, and $\nu_f^-(P_k)$ are finite (by the definition of locally-finite variation and Theorem 9.0.1).*
- *Thus:*

  $$V_f(G) = \sum_{k=1}^{\infty} V_f(P_k), \quad \nu_f^+(G) = \sum_{k=1}^{\infty} \nu_f^+(P_k), \quad \nu_f^-(G) = \sum_{k=1}^{\infty} \nu_f^-(P_k).$$

  This implies that V_f, ν_f^+, and ν_f^- are σ-finite.
- *Since open sets are measurable, Borel sets are measurable as well (in particular, compact sets, G_δ-sets, and F_σ-sets are measurable).*

Step (3): Regularity of V_f

- *Assume that $P = \prod_{i=1}^{n}[a_i, b_i)$ is a cell such that $P \subset G$. In the proof of Theorem 8.3.1, it was shown that there exists $\varepsilon_0 > 0$ such that:*

  $$P_0 = \prod_{i=1}^{n}[a_i - \varepsilon_0, b_i + \varepsilon_0) \subset G.$$

- *Define a sequence of open parallelepipeds:*

  $$\Delta_k = \prod_{i=1}^{n}\left[a_i - \frac{\varepsilon_0}{2^k}, b_i\right).$$

  Then:

  $$\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_k \supset \cdots, P = \bigcap_{k=1}^{\infty} \Delta_k, \quad \Delta_k \subset \overline{P} \quad \textit{then} \quad V_f(\Delta_k) \le V_f(P_0) < \infty \textit{ for all } k.$$

- *By the monotone continuity of V_f, we have:*

  $$V_f(P) = \lim_{k\to\infty} V_f(\Delta_k).$$

  For any $P \subset G$ and any $\varepsilon_0 > 0$, there exists an open parallelepiped Δ such that:

  $$\Delta \subset P \subset G \quad \textit{and} \quad V_f(\Delta) < V_f(P) + \varepsilon.$$

- *For any subset $E \subset G$, the outer measure $V_f^*(E)$ is given by:*

  $$V_f^*(E) = \inf\{V_f(Q) \mid Q \textit{ is open and } E \subset Q \subset G\}.$$

- *A set $E \subset G$ is measurable with respect to V_f if and only if there exist G_δ-sets K and H such that:*

$$E \subset K, \quad K \setminus E \subset H, \quad V_f(H) = 0.$$

- *In this case:*

$$V_f(E) = V_f(K).$$

- *The same regularity criterion applies to the measures ν_f^+ and ν_f^-.*

Application of caratheodory process

The measures V_f, ν_f^+, and ν_f^- are initially defined on the semiring of cells $\{P \mid \overline{P} \subset G\}$. These measures satisfy countable additivity and monotone continuity on the semiring.

Extension to σ-algebras

Using the Carathéodory extension process, these measures can be uniquely extended to σ-algebras $\mathfrak{A}_f$, $\mathfrak{A}_f^+$, and $\mathfrak{A}_f^-$, which include all Borel subsets of G. The extended measures preserve the notation V_f, ν_f^+, and ν_f^-.

Measurability of borel sets

Since open sets are measurable with respect to V_f, ν_f^+, and ν_f^-, all Borel subsets of G are also measurable. In particular, compact sets, G_δ-sets, and F_σ-sets are measurable.

Measurability of sets in $\mathfrak{A}_f$

Existence of G_δ-sets

For any $E \in \mathfrak{A}_f$, there exist G_δ-sets K and H such that:

$$E \subset K, \quad K \setminus E \subset H, \quad \text{and} \quad V_f(H) = 0.$$

Since all sets considered are subsets of G, we have $K \subset G$ and $H \subset G$.

Measurability of E

Assume $E \subset K \subset G$, $K \setminus E \subset H$, and $V_f(H) = 0$, where K and H are G_δ-sets.
Since the continuation of a measure by Carathéodory gives a complete measure, the set $K \setminus E$ is measurable.
Hence, the set $E = K \setminus (K \setminus E)$ is measurable.

Regularity of ν_f^+ and ν_f^-

The same arguments apply to ν_f^+ and ν_f^-, since they are defined similarly using $N_f^+(E, y)$ and $N_f^-(E, y)$.

Measurability of E

A set $E \subset G$ is measurable if there exist G_δ-sets K and H such that:

$$E \subset K, \quad K \setminus E \subset H, \quad V_f(H) = 0.$$

In this case:

$$V_f(E) = V_f(K).$$

The same regularity holds for ν_f^+ and ν_f^-.

Approximation by open sets

For any subset $E \subset G$, approximate E from above by open sets $Q \supset E$ such that $Q \subset G$. The outer measure $V_f^*(E)$ satisfies:

$$V_f^*(E) = \inf\{V_f(Q) \mid Q \text{ is open and } E \subset Q \subset G\}.$$

Approximation by compact sets

Similarly, approximate E from below by compact sets $K \subset E$. For any $\varepsilon > 0$, there exists a compact set $K \subset E$ such that:

$$V_f(E) \leq V_f(K) + \varepsilon.$$

Theorem 9.0.3. *Let $f : G \to \mathbb{R}^n$ be a mapping with locally-finite variation. Then the following statements hold:*

(I) If a set E is measurable with respect to the measure V_f, then the function $N_f(E)$ is Lebesgue measurable and

$$V_f(E) = \int_{\mathbb{R}^n} N_f(E)\, d\lambda_n,$$

and similar statements are valid for the measures ν_f^+ and ν_f^-.

(II) $\mathfrak{A}_f \subset \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-$.

(III) The inequality

$$\nu_f^+(E) + \nu_f^-(E) \geq V_f(E)$$

holds for any $E \in \mathfrak{A}_f$.

(IV) If E is a compact set such that $E \subset G$, then the values $V_f(E)$, $\nu_f^+(E)$, and $\nu_f^-(E)$ are finite.

(V) Let G' be an open subset of $\mathbb{R}^n$ such that $G' \subset G$ and let $f_1 = f|_{G'}$. Then

$$E \in \mathfrak{A}_{f_1}^+ \quad \Leftrightarrow \quad E \subset G' \text{ and } E \in \mathfrak{A}_f^+,$$

and the equality $\nu_{f_1}^+(E) = \nu_f^+(E)$ holds for such sets E. Similar statements are valid for sets $E \in \mathfrak{A}_{f_1}^-$ and $E \in \mathfrak{A}_{f_1}$.

Proof 9.0.4. Part (I): Integral representation

Step (I.1):Integration representation

If E is an open set in G, it can be written as a disjoint union of "cells" P_k (e.g., rectangles or other simple measurable sets).
By Lemma 9.0.1, the counting function $N_f(E)$ satisfies:

$$N_f(E) = \sum_{k=1}^{\infty} N_f(P_k).$$

By the definition of V_f, each term $V_f(P_k)$ can be expressed as:

$$V_f(P_k) = \int_{\mathbb{R}^n} N_f(P_k)\, d\lambda_n.$$

Since $N_f(P_k)$ is Lebesgue measurable by Theorem 7.2.1, $N_f(E)$ is also measurable. Using the additivity of V_f over disjoint sets, Summing over k, we get:

$$V_f(E) = \sum_{k=1}^{\infty} V_f(P_k) = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} N_f(P_k)\, d\lambda_n = \int_{\mathbb{R}^n} N_f(E)\, d\lambda_n.$$

For the measure V_f, ν_f^+, and ν_f^- , the proof is similar (in this cas , we refer to Theorem 9.0.1)

Step (I.2): G_δ-Sets.

- *If E is a G_δ-set (a countable intersection of open sets) with $V_f(E) < \infty$, it can be approximated by a decreasing sequence of open sets G_k such that $E = \bigcap_{k=1}^{\infty} G_k$, where the sets G_k are open.*

- *By Lemma 9.0.2, there exists an open set $G' \subset G$ such that:*

$$E \subset G' \quad \textit{and} \quad V_f(G') < V_f(E) + 1 < \infty.$$

- *Consider the open sets define:*

$$Q_k = \bigcap_{p=1}^{k} G_p \cap G' \quad \textit{Clearly}.$$

Then:

$$G \supset G' \supset Q_1 \supset Q_2 \supset \cdots \supset Q_k \supset \cdots, \quad \textit{and} \quad \bigcap_k Q_k = E.$$

- *Since $V_f(Q_1) \leq V_f(G') < \infty$, the monotonicity of V_f implies:*

$$V_f(E) = \lim_{k\to\infty} V_f(Q_k).$$

- *Since $V_f(Q_1) = \int_{\mathbb{R}^n} N_f(Q_1)\, d\lambda_n < \infty$, it follows that $N_f(Q_1, y) < \infty$ for almost all $y \in \mathbb{R}^n$.*
- *By Lemma 9.0.1, step (2) and (4), the equalities*

$$N_f(E, y) = \lim_{k\to\infty} N_f(Q_k, y) \quad \textit{and} \quad 0 \leq N_f(E, y) \leq N_f(Q_1, y)$$

hold for almost all $y \in \mathbb{R}^n$.

- *The function $N_f(E)$ is measurable (for almost all $y \in \mathbb{R}^n$, this function is the limit of the sequence $N_f(Q_k)$ of measurable functions).*

- *By the Lebesgue theorem on majorized convergence,*

$$\int_{\mathbb{R}^n} N_f(E)\, d\lambda_n = \lim_{k\to\infty} \int_{\mathbb{R}^n} N_f(Q_k)\, d\lambda_n = \lim_{k\to\infty} V_f(Q_k).$$

- *By the monotone continuity of the measure considered,*

$$\lim_{k\to\infty} V_f(Q_k) = V_f(E).$$

Step (I.3): Zero measure sets.

- *If* $V_f(E) = 0$*,By Lemma 9.0.2 in (Step (4)), there exists a* G_δ*-set* $K \supset E$ *such that* $V_f(K) = 0$*.*
- *Since* $N_f(K, y) \geq 0$ *and* $\int_{\mathbb{R}^n} N_f(K)\, d\lambda^n = 0$*, it follows that* $N_f(K, y) = 0$ *almost everywhere in* $\mathbb{R}^n$*.*
- *By Lemma 9.0.2,* $0 \leq N_f(E, y) \leq N_f(K, y)$*; hence,* $N_f(E, y) = 0$ *for almost all* $y \in \mathbb{R}^n$*.*
- *In addition, the function* $N_f(E)$ *is measurable and*

$$\int_{\mathbb{R}^n} N_f(E)\, d\lambda_n = 0 = V_f(E).$$

Step (I.4): Integral representation.

- *Assume* $V_f(E) < \infty$*. By Lemma 9.0.2, there exists a* G_δ*-set* K *such that* $E \subset K$ *and* $V_f(K \setminus E) = 0$*.*
- *Since* $K = E \cup (K \setminus E)$ *and* $E \cap (K \setminus E) = \emptyset$*, Lemma 9.0.1 implies:*

$$N_f(K, y) = N_f(E, y) + N_f(K \setminus E, y).$$

- *From (I.2) and (I.3), we have* $N_f(K)$ *and* $N_f(K \setminus E)$ *are measurable functions. Hence,* $N_f(E)$ *is also measurable.*
- *Since* $V_f(K \setminus E) = 0$*, it follows from (I.3) that* $N_f(K \setminus E, y) = 0$ *almost everywhere. Thus,* $N_f(K, y) = N_f(E, y)$ *almost everywhere.*
- *Therefore:*

$$\int_{\mathbb{R}^n} N_f(E)\, d\lambda_n = \int_{\mathbb{R}^n} N_f(K)\, d\lambda_n = V_f(K) = V_f(E).$$

- *This completes the integral representation for sets* E *with* $V_f(E) < \infty$*.*
- *For the measures* ν_f^+ *and* ν_f^-*, the same reasoning applies, as they are derived from* V_f *via the Hahn decomposition.*

Part (II): Inclusion $\mathfrak{A}_f \subset \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-$

- *If* $E \in \mathfrak{A}_f$*, then by Lemma 9.0.2,* $E = K \setminus E_1$*, where* K *is a* G_δ*-set and* $V_f(E_1) = 0$*.*
- *By Lemma 9.0.2, there exists a* G_δ*-set* H *such that* $E_1 \subset H$ *and* $V_f(H) = 0$*.*
- *Since* $V_f(H) = \int_{\mathbb{R}^n} N_f(H)\, d\lambda_n = 0$*, it follows that* $N_f(H, y) = 0$ *almost everywhere.*
- *The remark following the definition of* N_f^+ *and* N_f^- *implies:*

$$N_f^+(H) + N_f^-(H) \leq N_f(H).$$

  Hence, $N_f^+(H)$ *and* $N_f^-(H)$ *vanish almost everywhere.*
- *Consequently:*

$$\nu_f^+(H) = \int_{\mathbb{R}^n} N_f^+(H)\, d\lambda_n = 0, \quad \nu_f^-(H) = \int_{\mathbb{R}^n} N_f^-(H)\, d\lambda_n = 0.$$

- *Since $E_1 \subset H$, the standard continuation argument shows that $E_1 \in \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-$. Therefore:*

$$E = K \setminus E_1 \in \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-.$$

- *Thus, $\mathfrak{A}_f \subset \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-$.*

Part (III): Variation inequality

- *If $E \in \mathfrak{A}_f$, then $E \in \mathfrak{A}_f^+ \cap \mathfrak{A}_f^-$. By Lemma 9.0.1, we have:*

$$N_f(E) = N_f^+(E) + N_f^-(E).$$

- *From (I), the functions $N_f(E), N_f^+(E)$, and $N_f^-(E)$ are measurable.*
- *Therefore:*

$$\nu_f^+(E) + \nu_f^-(E) = \int_{\mathbb{R}^n} N_f^+(E)\, d\lambda_n + \int_{\mathbb{R}^n} N_f^-(E)\, d\lambda_n.$$

- *Using the inequality $N_f^+(E) + N_f^-(E) \leq N_f(E)$, we conclude:*

$$\nu_f^+(E) + \nu_f^-(E) \leq \int_{\mathbb{R}^n} N_f(E)\, d\lambda_n = V_f(E).$$

- *This establishes the variation inequality.*

Part (IV): Finiteness for compact sets

- *Let E be a compact subset of G. Then the distance from E to $\mathbb{R}^n \setminus G$ is positive:*

$$d_0 = \inf\{\rho(x, z) \mid x \in E, z \in \mathbb{R}^n \setminus G\} > 0.$$

- *For each $x \in E$, let Δ_x be the open cube of diameter d_0 centered at x, and let Δ'_x be the corresponding closed cube.(let Δ'_x be the cell with the same faces).*

- *Clearly, if $x \in E$, then $\overline{\Delta'_x} \subset G$, i.e. :*

$$E \subset \bigcup_{x \in E} \Delta_x \subset \bigcup_{x \in E} \overline{\Delta'_x} \subset G.$$

- *Select a finite covering $\{\Delta_{x_k}\}_{k=1}^{p}$ from the covering $\{\Delta_x\}_{x \in E}$. .*
- *Since f is a mapping with locally-finite variation:*

$$V_f(\Delta'_{x_k}) < \infty \quad for \quad k = 1, \ldots, p.$$

- *By additivity of V_f:*

$$V_f(E) \leq \sum_{k=1}^{p} V_f(\Delta'_{x_k}) < \infty.$$

By (III), $\nu_f^+(E)$ and $\nu_f^-(E)$ are also finite.

Part (V): Restriction to subsets

Let $G' \subset G$ be an open subset, and let $f_1 = f|_{G'}$. For any set $E \subset G'$:

- *The multiplicity of a point x under f_1 is defined locally, so:*
$$k_{f_1}(x) = k_f(x) \quad \textit{for all } x \in G'.$$
- *This implies:*
$$f_1^{-1+}(y) = \{x \in f_1^{-1}(y) \mid k_{f_1}(x) > 0\} = \{x \in f^{-1}(y) \cap G' \mid k_f(x) > 0\} = f^{-1}(y) \cap G'.$$
and
$$\operatorname{card}(f_1^{-1+}(y) \cap E) = \operatorname{card}(f^{-1+}(y) \cap E) \Rightarrow N_{f_1}^+(E, y) = N_f^+(E, y).$$
- *For any $E \subset G'$, the counting functions satisfy:*
$$N_{f_1}^+(E, y) = N_f^+(E, y).$$

Step (V.1):

- *If $E \in \mathfrak{A}_{f_1}^+$, then $E \subset \operatorname{dom} f_1 = G'$.*
- *By Lemma 9.0.2 in (step (3)), there exist G_δ-sets K and H such that*
$$K, H \subset G', \quad E \subset K, \quad K \setminus E \subset H, \quad \textit{and} \quad \nu_{f_1}^+(H) = 0.$$
- *In this case,*
$$\nu_f^+(H) = \int_{\mathbb{R}^n} N_f^+(H)\, d\lambda_n = \int_{\mathbb{R}^n} N_{f_1}^+(H)\, d\lambda_n = \nu_{f_1}^+(H) = 0,$$
and Lemma 9.0.2 implies that $E \in \mathfrak{A}_f^+$.

Step (V.2):

- *Assume $E \in \mathfrak{A}_f^+$ and $E \subset G'$. By Lemma 9.0.2, there exist G_δ-sets K and H such that:*
$$K, H \subset G, \quad E \subset K, \quad K \setminus E \subset H, \quad \textit{and} \quad \nu_f^+(H) = 0.$$
- *Define $K_1 = K \cap G'$ and $H_1 = H \cap G'$. Clearly, K_1 and H_1 are G_δ-sets.*
- *Since $E \subset G'$, it follows that:*
$$E \subset K_1, \quad K_1 \setminus E \subset H_1, \quad \textit{and} \quad \nu_f^+(H_1) = 0.$$
- *By the same reasoning as in (V.1), we conclude that $\nu_{f_1}^+(H_1) = 0$.*
- *By Lemma 9.0.2, this implies $E \in \mathfrak{A}_{f_1}^+$.*
- *For such sets E, the equality of measures holds:*
$$\nu_f^+(E) = \int_{\mathbb{R}^n} N_f^+(E)\, d\lambda_n = \int_{\mathbb{R}^n} N_{f_1}^+(E)\, d\lambda_n = \nu_{f_1}^+(E).$$
- *The proof for $\mathfrak{A}_f^-$ is similar in the proof for $\mathfrak{A}_f$, we apply the obvious equality:*
$$f_1^{-1}(y) \cap E = f^{-1}(y) \cap E \quad \textit{for } E \subset G'.$$

This completes the proof of Theorem 9.0.3.

9.0.7 Definition Of orientation-preserving homeomorphism

On any topological n-manifold M, define an orientation *of M to be a function μ defined on M such that for each input $x \in M$, the output $\mu(x)$ is one of the two generators of the infinite cyclic group $H_n(M, M \setminus \{x\})$, and the following property holds:*

- *for each embedded open n-ball $B \subset M$, there exists a generator μ_B of the infinite cyclic group $H_n(M, M \setminus B)$ such that for each $x \in B$, the inclusion-induced homomorphism*

$$H_n(M, M \setminus B) \longrightarrow H_n(M, M \setminus \{x\})$$

  maps μ_B to $\mu(x)$.

- *Now one proves:*

  Theorem 9.0.4 (and definition)**.** *For every topological manifold M, exactly one of two possibilities holds:*

  1. *either M has exactly two orientations, in which case we say that M is* orientable*;*
  2. *or M has no orientations, in which case we say that M is* nonorientable.

- *This theorem is one of the preliminary steps to the proof of Poincaré Duality; see for example As explained in [**2**], orientation on a topological manifold involves choices of local generators of relative homology groups.*

- *In fact, proving that the relative homology groups $H_n(M, M \setminus B)$ and $H_n(M, M \setminus \{x\})$ are infinite cyclic is also one of the preliminary steps.*

- *Let M be a connected topological manifold.If M is nonorientable, then it makes no sense to ask whether a homeomorphism f of M preserves orientation, and the whole concept of "preserving orientation" is undefined for M.*

- *If on the other hand M is orientable, then to say that a homeomorphism $f : M \to M$ is* orientation-preserving *means that for either of the two orientations μ of M, and for any $x \in M$, the induced isomorphism*

$$f_* : H_n(M, M \setminus \{x\}) \longrightarrow H_n(M, M \setminus \{f(x)\})$$

  takes $\mu(x)$ to $\mu(f(x))$.

9.0.8 Definition Of reverse orientation homeomorphism

- *Let ϕ be the homomorphism and F an arbitrary lift. Then define*

$$\beta(x) = F(x) - x.$$

- *Note that this is continuous because ϕ is a homomorphism, and continuous itself (and so F, its lift, is too).*

- *See then that a fixed point of ϕ means*

$$\phi(x) = x \Rightarrow F(x) = x + k \Rightarrow \beta(x) \in \mathbb{Z}.$$

- *So it suffices to show that β is valued at least at two different integers.*
- *Then see, using that*
$$F(x+k) = F(x) + k$$
(easily proven as an exercise, by induction, making use of the fact that ϕ preserves orientation), that
$$\beta(1) = F(1) - 1 = F(0) + 1 - 1 = F(0) = \beta(0) + 2.$$
- *This means that β, a continuous function, increases by 2 between the inputs 0 and 1.*
- *This means it must take on two integer values in between, by the Intermediate Value Theorem (graph β against x if you're not convinced).*
- *And therefore, these two distinct values correspond to two distinct fixed points of our original function ϕ, as required.*

Theorem 9.0.5. *Let G_1 and G_2 be connected open subsets of $\mathbb{R}^n$ and let φ be a homeomorphism of G_1 onto G_2. If f is a continuous mapping of G_2 into $\mathbb{R}^n$, then the following statements hold:*

1. *f is a mapping with locally-finite variation $\Leftrightarrow$ $f \circ \varphi$ is a mapping with locally-finite variation;*
2. *$E \in \mathfrak{A}_{f\circ\varphi} \Leftrightarrow \varphi(E) \in \mathfrak{A}_f$; in this case,*
$$V_{f\circ\varphi}(E) = V_f(\varphi(E));$$
3. *if φ preserves orientation, then*
$$E \in \mathfrak{A}^+_{f\circ\varphi} \Leftrightarrow \varphi(E) \in \mathfrak{A}^+_f \quad \textit{and} \quad E \in \mathfrak{A}^-_{f\circ\varphi} \Leftrightarrow \varphi(E) \in \mathfrak{A}^-_f;$$
in this case,
$$\nu^+_{f\circ\varphi}(E) = \nu^+_f(\varphi(E)) \quad \textit{and} \quad \nu^-_{f\circ\varphi}(E) = \nu^-_f(\varphi(E));$$
4. *if φ reverses orientation, then*
$$E \in \mathfrak{A}^+_{f\circ\varphi} \Leftrightarrow \varphi(E) \in \mathfrak{A}^-_f \quad \textit{and} \quad E \in \mathfrak{A}^-_{f\circ\varphi} \Leftrightarrow \varphi(E) \in \mathfrak{A}^+_f;$$
in this case,
$$\nu^+_{f\circ\varphi}(E) = \nu^-_f(\varphi(E)) \quad \textit{and} \quad \nu^-_{f\circ\varphi}(E) = \nu^+_f(\varphi(E)).$$

Proof 9.0.5. Part (1): Locally finite variation

Step(1.1):

First, we prove that $N_{f\circ\varphi}(E) = N_f(\varphi(E))$ for any $E \subset G_1$.

- *Since φ is a bijection, the following implications hold:*
$$\varphi\left((f\circ\varphi)^{-1}(y) \cap E\right) = f^{-1}(y) \cap \varphi(E).$$
- *Taking the cardinality of both sides, we get:*
$$\operatorname{card}\left((f\circ\varphi)^{-1}(y) \cap E\right) = \operatorname{card}\left(f^{-1}(y) \cap \varphi(E)\right).$$
- *By definition of the counting function N, this implies:*
$$N_{f\circ\varphi}(E, y) = N_f(\varphi(E), y).$$

Step(1.2):

Let f be a mapping with locally finite variation, and let $\overline{P} \subset G_1$ be a cell. Then:

- *P is a G_δ-set, and since φ is a homeomorphism, $\varphi(P)$ is also a G_δ-set in G_2.*
- *By Lemma 9.0.2, $\varphi(P)$ the set is V_f-measurable.*
- *Since the set $\varphi(\overline{P})$ is compact, $V_f(\varphi(\overline{P})) < \infty$.*
- *Using (1.1) and Theorem 9.0.3, we have:*

$$V_{f\circ\varphi}(P) = \int_{\mathbb{R}^n} N_{f\circ\varphi}(P)\, d\lambda_n = \int_{\mathbb{R}^n} N_f(\varphi(P))\, d\lambda_n = V_f(\varphi(P)) < \infty.$$

$$V_f(\varphi(P)) < \infty$$

- *Thus, $f \circ \varphi$ is a mapping with locally finite variation.*
- *The converse statement follows similarly because φ^{-1} is also a homeomorphism.*

Part (2): Measurable sets and total variation

If $E \in \mathfrak{A}_{f\circ\varphi}$, then by Lemma 9.0.2, there exist G_δ-sets $K, H \subset G_1$ such that:

$$E = K \setminus E_1, \quad E_1 \subset H, \quad V_{f\circ\varphi}(H) = 0.$$

- *Since φ is a bijection, we have:*

$$\varphi(E) = \varphi(K) \setminus \varphi(E_1), \quad \varphi(E_1) \subset \varphi(H) \subset G_2, \quad \varphi(K) \subset G_2.$$

- *Since φ is a homeomorphism, $\varphi(K)$ and $\varphi(H)$ are G_δ-sets; hence, these sets are $\mathcal{V}_f$-measurable.*
- *Theorem 9.0.3and step (1.1) and step (1.2) imply that*

$$\mathcal{V}_f(\varphi(H)) = \int_{\mathbb{R}^n} N_f(\varphi(H))\, d\lambda_n = \int_{\mathbb{R}^n} N_{f\circ\varphi}(H)\, d\lambda_n = \mathcal{V}_{f\circ\varphi}(H) = 0.$$

- *Since $\varphi(E_1) \subset \varphi(H)$ and the measure $\mathcal{V}_f$ is complete, we have $\varphi(E_1) \in \mathfrak{A}_f$ and $\varphi(E) \in \mathfrak{A}_f$ (noting that $\varphi(E) = \varphi(K) \setminus \varphi(E_1)$).*
- *We apply Theorem 9.0.3 and step (1.1) and step(1.2) once more to show that*

$$\mathcal{V}_f(\varphi(E)) = \int_{\mathbb{R}^n} N_f(\varphi(E))\, d\lambda_n = \int_{\mathbb{R}^n} N_{f\circ\varphi}(E)\, d\lambda_n = \mathcal{V}_{f\circ\varphi}(E).$$

- *Since φ^{-1} is a homeomorphism and $\varphi(E) \in \mathfrak{A}_f = \mathfrak{A}_f \circ \varphi \circ \varphi^{-1}$, it follows that*

$$E = \varphi^{-1}(\varphi(E)) \in \mathfrak{A}_{f\circ\varphi};$$

thus, statement (2) is proved.

Part (3): Multiplicity of points

- *We claim that the multiplicity of a point $x \in G_1$ under the mapping $f \circ \varphi$ is defined if and only if the multiplicity of the point $\varphi(x)$ under the mapping f is defined, and:*

$$k_{f\circ\varphi}(x) = \deg(\varphi) \cdot k_f(\varphi(x)).$$

- *Assume that the multiplicity $k_f(\varphi(x))$ is defined. By Lemma 8.3.2, there exists a neighborhood V of $\varphi(x)$ such that for any compact n-dimensional manifold K with boundary satisfying $K \subset V$, $\varphi(x) \in \mathrm{Int}(K)$, and the degree $\deg(f(K), f(\varphi(x)))$ equals $k_f(\varphi(x))$(i.e., the degree does not depend on K).*

- *Since φ is a homeomorphism, $\varphi^{-1}(V) = V_1$ is a neighborhood of x.*

- *If K_1 is a compact n-dimensional manifold with boundary such that $K_1 \subset V_1$ and $x \in \mathrm{Int}(K_1)$, then $K = \varphi(K_1)$ is also a compact n-dimensional manifold with boundary, $K \subset \varphi(V_1) = V$, and:*

$$\varphi(x) \in \varphi(\mathrm{Int}(K_1)) = \mathrm{Int}(\varphi(K_1)) = \mathrm{Int}(K).$$

- *Using the remark preceding Lemma 8.3.2, we have:*

$$\deg_{f\circ\varphi}(K_1, (f\circ\varphi)(x)) = \deg_{f\circ\varphi}(\mathrm{Int}\, K_1, (f\circ\varphi)(x))$$

  and

$$\deg_f(\varphi(K_1), f(\varphi(x))) = \deg_f(\mathrm{Int}\,\varphi(K_1), f(\varphi(x))).$$

- *From [Chapter VIII, §4, Corollary 4.6] [**13**], we have:*

$$\deg(f\circ\varphi(\mathrm{Int}(K_1)), (f\circ\varphi)(x)) = \deg(\varphi)\cdot\deg(f(\mathrm{Int}(K)), f(\varphi(x))).$$

- *Since* $\deg_{f\circ\varphi}(\mathrm{Int}\, K_1, (f\circ\varphi)(x)) = \deg\varphi \cdot \deg_f(\mathrm{Int}\,\varphi(K_1), f(\varphi(x)))$.

  And the above-mentioned properties of the set $K = \varphi(K_1)$ imply that the value

$$\deg_f(K, f(\varphi(x))) = k_f(\varphi(x))$$

  does not depend on K, we conclude that the value

$$\deg_{f\circ\varphi}(K_1, (f\circ\varphi)(x)) = \deg\varphi \cdot k_f(\varphi(x))$$

  does not depend on $K_1 \subset V_1$, i.e., the multiplicity

$$k_{f\circ\varphi}(x) = \deg\varphi \cdot k_f(\varphi(x))$$

  is defined.

- *Similarly, if the multiplicity $k_{f\circ\varphi}(x)$ is defined, then the multiplicity $k_f(\varphi(x))$ is defined as well (since φ^{-1} is a homeomorphism).*

Part (4): Positive and negative variations

Assume φ preserves orientation, i.e., $\deg(\varphi) = 1$,*and $N_f(\varphi(x)) = N_{f\circ\varphi}(x)$. Then:*

- *For any $y \in \mathbb{R}^n$ and $E \subset G_1$, the following equivalences hold:*

$$x \in (f\circ\varphi)_+^{-1}(y) \cap E \iff \begin{cases} x \in E, \\ x \in (f\circ\varphi)^{-1}(y), \\ k_{f\circ\varphi}(x) > 0, \end{cases}$$

- *which implies:*

$$\iff \begin{cases} \varphi(x) \in \varphi(E) \\ f(\varphi(x)) = y \\ k_f(\varphi(x)) > 0 \end{cases} \iff \varphi(x) \in f_+^{-1}(y) \cap \varphi(E),$$

- *Thus:*

$$f_+^{-1}(y) \cap \varphi(E) = \varphi((f\circ\varphi)_+^{-1}(y) \cap E).$$

- *Taking the cardinality of both sides, we get:*

$$\operatorname{card}(f_+^{-1}(y) \cap \varphi(E)) = \operatorname{card}((f\circ\varphi)_+^{-1}(y) \cap E),$$

- *which implies:*

$$N_f^+(\varphi(E), y) = N_{f\circ\varphi}^+(E, y).$$

- *A similar reasoning applies to the negative variation, proving:*

$$N_f^-(\varphi(E), y) = N_{f\circ\varphi}^-(E, y).$$

- *If φ is a homeomorphism that **reverses orientation**, i.e.,* $\deg(\varphi) = -1$.
  Then, for any measurable function f, the signed multiplicity function satisfies

$$k_f(\varphi(x)) = -k_{f\circ\varphi}(x).$$

- *In particular, for any $y \in \mathbb{R}^n$, we have the following identities:*

$$f_+^{-1}(y) \cap \varphi(E) = \varphi\left((f\circ\varphi)_-^{-1}(y) \cap E\right),$$

$$f_-^{-1}(y) \cap \varphi(E) = \varphi\left((f\circ\varphi)_+^{-1}(y) \cap E\right).$$

- *That is, the preimages of positive and negative multiplicities are interchanged under φ.*

- *Consequently, the multiplicities satisfy:*

$$N_f^+(\varphi(E), y) = N_{f\circ\varphi}^-(E, y) \quad , \quad N_f^-(\varphi(E), y) = N_{f\circ\varphi}^+(E, y).$$

Part (5): Positive and negative variations under homeomorphisms

- *Assume $E \in \mathfrak{A}^+_{f\circ\varphi}$. By Lemma 9.0.2, there exist G_δ-sets $K, H \subset G_1$ such that:*

$$E \subset K, \quad K \setminus E \subset H, \quad \nu^+_{f\circ\varphi}(H) = 0.$$

- *Since φ is a homeomorphism, $\varphi(K)$ and $\varphi(H)$ are G_δ-sets in G_2. Thus, these sets are measurable with respect to the measures ν^+_f and ν^-_f.*

- *If φ preserves orientation, then by Part (4) and Theorem 9.0.3, we have:*

$$\nu^+_f(\varphi(H)) = \int_{\mathbb{R}^n} N^+_f(\varphi(H))\, d\lambda_n = \int_{\mathbb{R}^n} N^+_{f\circ\varphi}(H)\, d\lambda_n = \nu^+_{f\circ\varphi}(H) = 0.$$

- *If φ reverses orientation, then:*

$$\nu^-_f(\varphi(H)) = \int_{\mathbb{R}^n} N^-_f(\varphi(H))\, d\lambda_n = \int_{\mathbb{R}^n} N^+_{f\circ\varphi}(H)\, d\lambda_n = \nu^+_{f\circ\varphi}(H) = 0.$$

- *Since $K \setminus E \subset H$,and φ is a bijection we have :*

$$\varphi(K \setminus E) = \varphi(K) \setminus \varphi(E) \subset \varphi(H).$$

- *The completeness of the measure ν^+_f implies that if φ preserves orientation, then:*

$$\varphi(K) \setminus \varphi(E) \in \mathfrak{A}^+_f \quad \Rightarrow \quad \varphi(E) = \varphi(K) \setminus (\varphi(K) \setminus \varphi(E)) \in \mathfrak{A}^+_f.$$

- *Similarly, if φ reverses orientation, the completeness of ν^-_f implies:*

$$\varphi(K) \setminus \varphi(E) \in \mathfrak{A}^-_f \quad \Rightarrow \quad \varphi(E) \in \mathfrak{A}^-_f.$$

- *Using step (4) and Theorem 9.0.3, we compute the measures:*

- *If φ preserves orientation, then:*

$$\nu^+_f(\varphi(E)) = \int_{\mathbb{R}^n} N^+_f(\varphi(E))\, d\lambda_n = \int_{\mathbb{R}^n} N^+_{f\circ\varphi}(E)\, d\lambda_n = \nu^+_{f\circ\varphi}(E).$$

- *If φ reverses orientation, then:*

$$\nu^-_f(\varphi(E)) = \int_{\mathbb{R}^n} N^-_f(\varphi(E))\, d\lambda_n = \int_{\mathbb{R}^n} N^+_{f\circ\varphi}(E)\, d\lambda_n = \nu^+_{f\circ\varphi}(E).$$

- *If $E \in \mathfrak{A}^-_{f\circ\varphi}$, the proof is analogous. In this case, the roles of ν^+_f and ν^-_f are swapped depending on whether φ preserves or reverses orientation.*

- *Finally, since φ^{-1} is also a homeomorphism, the implications:*

$$\varphi(E) \in \mathfrak{A}^+_f \Rightarrow E \in \mathfrak{A}^+_{f\circ\varphi}, \qquad \varphi(E) \in \mathfrak{A}^-_f \Rightarrow E \in \mathfrak{A}^-_{f\circ\varphi}$$

  hold. This completes the proof.

10

Manifolds with locally-finite variations

*The references for this chapter are [**43**]*

Definition 10.0.1. *Let G be an open subset of $\mathbb{R}^n$ and let $f_1, \ldots, f_m$ be the coordinate functions of a continuous mapping $f : G \to \mathbb{R}^m$ ($m \geq n$).*

- *For $\alpha = \{i_1, \ldots, i_n\}$, where $1 \leq i_1 < i_2 < \ldots < i_n \leq m$, we denote by f_α the mapping with coordinate functions $f_{i_1}, \ldots, f_{i_n}$:*

$$f_\alpha : \begin{cases} x_{i_1} = f_{i_1}(t_1, \ldots, t_n), \\ \ldots \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{i_n} = f_{i_n}(t_1, \ldots, t_n), \end{cases} \qquad (t_1, \ldots, t_n) \in G.$$

- *We say that f is a mapping with locally-finite variations if any of the mappings $f_\alpha : G \to \mathbb{R}^n$ is a mapping with locally-finite variation.*

Definition 10.0.2. *Let $M \subset \mathbb{R}^m$ be an n-dimensional manifold whose topology is induced from $\mathbb{R}^m$. We say that M is a manifold with locally-finite variations embedded into $\mathbb{R}^m$ if, for any point $x \in M$, there exists a neighborhood U of x in M and a homeomorphism f of $\mathbb{R}^n$ onto U with locally-finite variations (local parametrization of the neighborhood U).*

Remark 10.0.1. *If a local parametrization of a neighborhood U is a mapping with locally-finite variations, then any parametrization of this neighborhood is a mapping with locally-finite variations as well.*

- *Indeed, if $f : \mathbb{R}^n \to U$ and $g : \mathbb{R}^n \to U$ are homeomorphisms, then $\varphi = f^{-1} \circ g$ is a homeomorphism of the space $\mathbb{R}^n$ onto itself.*

- *In this case, $g = f \circ \varphi$, and the formula $g_\alpha = f_\alpha \circ \varphi$ is valid for any $\alpha = \{i_1, \ldots, i_n\}$; by Theorem 9.0.5 the mappings g_α and f_α are (or are not) mappings with locally-finite variations simultaneously.*

- *It is well known that if $U \subset \mathbb{R}^m$ and there exists a homeomorphism $f : \mathbb{R}^n \to U$, then it is possible to fix an orientation on U.*

- *For example, one may proceed as follows: let g be one more homeomorphism of $\mathbb{R}^n$ onto U and let $g = f \circ \varphi$.*

- *If $\deg \varphi = 1$, we include g into class I; if $\deg \varphi = -1$, we include g into class II. The orientation on U is fixed by a choice of one of the above classes as the class of ``positive" orientations.*

- *Let M be a manifold. Consider neighborhoods $U, V \subset M$ such that $U \cap V \neq \emptyset$; let $f : \mathbb{R}^n \to U$ and $g : \mathbb{R}^n \to V$ be positive parametrizations of the oriented neighborhoods U and V. In this case, $\varphi = f^{-1} \circ g$ is a homeomorphism of the open set $G_1 = g^{-1}(U \cap V) \subset \mathbb{R}^n$ onto $G_2 = f^{-1}(U \cap V) \subset \mathbb{R}^n$.*
  The neighborhoods U and V are oriented consistently if $\deg \varphi = 1$; the manifold M is orientable if it is possible to choose a consistent orientation for all of its parametrized neighborhoods.

Definition 10.0.3. *Let M be an oriented n-dimensional manifold with locally-finite variations embedded into $\mathbb{R}^m$ ($m \geq n$).*
Let $\alpha = \{i_1, \ldots, i_n\}$, where $i_1, \ldots, i_n \in \mathbb{N}$ and $1 \leq i_1 < \ldots < i_n \leq m$.
A subset $E \subset M$ is called small α^+-measurable if there exists a neighborhood U of a point of M and a positive parametrization $f : \mathbb{R}^n \to U$ of U such that $E \subset U$ and $f^{-1}(E) \in \mathfrak{A}^+_{f_\alpha}$.
For such a set E, we set:

$$\mu^+_\alpha(E) = \nu^+_{f_\alpha}(f^{-1}(E)) \quad \textit{by definition.}$$

Lemma 10.0.1. *The property of a subset E to be a small α^+-measurable set and the value $\mu^+_\alpha(E)$ do not depend both on the neighborhood $U \supset E$ and the positive parametrization f.*

Proof 10.0.1. *If $E = \emptyset$, the statement is obvious.*

- *Assume that $E \neq \emptyset$, $E \subset U \cap V$, f is a positive parametrization of the neighborhood U, and g is a positive parametrization of the neighborhood V.*

- *In this case, $\varphi = f^{-1} \circ g$ is a homeomorphism of the set $G_1 = g^{-1}(U \cap V)$ onto the set $G_2 = f^{-1}(U \cap V)$ and $\deg \varphi = 1$.*

- *Denote $\overline{f}_\alpha = f_\alpha|_{G_2}$ and $\overline{g}_\alpha = g_\alpha|_{G_1}$; in this case, $\overline{g}_\alpha = \overline{f}_\alpha \circ \varphi$.*

- *If $E_1 = g^{-1}(E) \in \mathfrak{A}^+_{g_\alpha}$, then the inclusion $E_1 \subset G_1$ and Theorem 9.0.3, Part (V), imply that $E_1 \in \mathfrak{A}^+_{\overline{g}_\alpha} = \mathfrak{A}^+_{\overline{f}_\alpha \circ \varphi}$.*
  Obviously, $E_2 = f^{-1}(E) = \varphi(E_1)$; since $\deg \varphi = 1$, we deduce from Theorem 9.0.5, step (3), that $\varphi(E_1) = E_2 \in \mathfrak{A}^+_{\overline{f}_\alpha}$. Theorem 9.0.3 implies now that $E_2 \in \mathfrak{A}^+_{f_\alpha}$.

- *Similarly, $E_2 \in \mathfrak{A}^+_{f_\alpha} \Rightarrow E_1 \in \mathfrak{A}^+_{g_\alpha}$.*

- *In this case, the following equalities hold:*

$$\nu^+_{g_\alpha}(g^{-1}(E)) = \nu^+_{\overline{g}_\alpha}(E_1) \quad \textit{(Theorem 9.0.3)},$$

$$\nu^+_{\overline{g}_\alpha}(E_1) = \nu^+_{\overline{f}_\alpha \circ \varphi}(E_1) = \nu^+_{\overline{f}_\alpha}(\varphi(E_1)) \quad \textit{(Theorem 10.0.1)},$$

$$\nu^+_{\overline{f}_\alpha}(\varphi(E_1)) = \nu^+_{\overline{f}_\alpha}(E_2) = \nu^+_{f_\alpha}(f^{-1}(E)) \quad \textit{(Theorem 9.0.3)}.$$

- *Hence,*

$$\nu^+_{g_\alpha}(g^{-1}(E)) = \nu^+_{f_\alpha}(f^{-1}(E)).$$

Definition 10.0.4. *A subset $E \subset M$ is called α^+-measurable if, for any parametrized neighborhood $U \subset M$, the set $E \cap U$ is a small α^+-measurable set.*

Lemma 10.0.2. *The set of α^+-measurable subsets of an oriented n-dimensional manifold $M \subset \mathbb{R}^m$ with locally-finite variations forms a σ-algebra that contains all of Borel subsets of M.*

Proof 10.0.2. Step 1: Small α^+-measurable sets form a σ-algebra

- *Let $f : \mathbb{R}^n \to U$ be a positive parametrization of a neighborhood $U \subset M$. By definition of a small α^+-measurable set :*

$$E \subset U \text{ and } E \text{ is a small } \alpha^+\text{-measurable set} \iff f^{-1}(E) \in \mathfrak{A}^+_{f_\alpha}$$

- *where $\mathfrak{A}^+_{f_\alpha}$ is the σ-algebra of sets measurable with respect to the positive variation measure $\nu^+_{f_\alpha}$.*
- *Since $\mathfrak{A}^+_{f_\alpha}$ is a σ-algebra, we can apply operations (e.g., complements, countable unions, intersections) on preimages under f^{-1} to deduce that the collection of small α^+-measurable subsets of U also forms a σ-algebra.*
- *Thus, the family of small α^+-measurable subsets of U is closed under complementation, countable unions, and countable intersections.*

Step 2: Complementation in M

- *Let $E \subset M$ be an α^+-measurable set. For any parametrizable neighborhood $U \subset M$, the intersection $E \cap U$ is a small α^+-measurable set.*
- *By step (1), the complement of $E \cap U$ in U, denoted $(M \setminus E) \cap U = U \setminus (E \cap U)$, is also a small α^+-measurable set.*
- *Since U is arbitrary, the set $M \setminus E$ is α^+-measurable.*

  This shows that the family of α^+-measurable subsets of M is closed under complementation.

Step 3: Countable unions

- *Let $\{E_k\}$ be an at most countable family of α^+-measurable subsets of M, and let $E = \bigcup_k E_k$.*
- *For any parametrizable neighborhood $U \subset M$, the intersection $E_k \cap U$ is a small α^+-measurable set for each k.*
- *By item (1), the union $E \cap U = \bigcup_k (E_k \cap U)$ is a small α^+-measurable set.*
- *Since U is arbitrary, the set $E = \bigcup_k E_k$ is α^+-measurable.*
- *This shows that the family of α^+-measurable subsets of M is closed under countable unions.*

Step 4: Open sets are α^+-measurable

- *Assume $G \subset M$ is an open set. For any parametrizable neighborhood $U \subset M$, the intersection $G \cap U$ is open in U.*
- *Since $f : \mathbb{R}^n \to U$ is a positive parametrization, the preimage $f^{-1}(G \cap U)$ is open in $\mathbb{R}^n$.*
- *From the proof of Lemma 9.0.2, open subsets of the domain of definition of $f_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ are $\nu^+_{f_\alpha}$-measurable. Hence:*

$$f^{-1}(G \cap U) \in \mathfrak{A}^+_{f_\alpha}.$$

- *Therefore, $G \cap U$ is a small α^+-measurable set.*

- *Since U is arbitrary, the set G is α^+-measurable.*
- *By step (2) and (3), any Borel set (which can be expressed as a countable union, intersection, or complement of open sets) is also α^+-measurable.*

Conclusion

The proof demonstrates that the set of α^+-measurable subsets of M satisfies the following properties:

- *It is closed under complementation.*
- *It is closed under countable unions.*
- *It contains all open subsets of M, and hence all Borel subsets of M.*

Thus, the collection of α^+-measurable subsets of M forms a σ-algebra that contains all Borel subsets of M. Lemma 10.0.2 is fully proven.

Representation of α^+-measurable sets

Denote by $\mathfrak{A}_\alpha^+$ the σ-algebra of all α^+-measurable subsets of M. Since the topology of M has a countable basis of parametrizable neighborhoods, any set $E \in \mathfrak{A}_\alpha^+$ can be represented as a union of an at most countable family of small α^+-measurable sets:

$$E = \bigcup_k E_k, \quad E_k \subset U_k,$$

where U_k are parametrizable neighborhoods. If necessary, we can assume that the sets E_k are pairwise disjoint by considering their differences:

$$E_1' = E_1, \quad E_2' = E_2 \setminus E_1, \quad E_3' = E_3 \setminus (E_1 \cup E_2), \; \ldots$$

This ensures that the resulting family $\{E_k'\}$ consists of pairwise disjoint, small α^+-measurable sets.

Definition 10.0.5. *For $E = \bigsqcup_k E_k$ (a disjoint union of an at most countable family of small α^+-measurable subsets), we set*

$$\mu_\alpha^+(E) = \sum_k \mu_\alpha^+(E_k).$$

Lemma 10.0.3. *The function μ_α^+ is well-defined on the σ-algebra $\mathfrak{A}_\alpha^+$ and is a σ-finite complete measure. If $K \subset M$ is a compact set, then $\mu_\alpha^+(K) < \infty$.*

Proof 10.0.3. ***Step 1: Correctness of the definition***

- *By definition, for any measurable set $E \subset U = f(\mathbb{R}^n)$, where f is a positive parametrization of U:*

$$\mu_\alpha^+(E) = \nu_{f,\alpha}^+(f^{-1}(E)).$$

- *Since $\nu_{f,\alpha}^+$ is a measure (as established in earlier results), it follows that μ_α^+ satisfies the properties of a measure on subsets of U.*
- *By Lemma 10.0.1, the value $\mu_\alpha^+(E)$ does not depend on the choice of the parametrization f. This ensures that μ_α^+ is well-defined.*

- *Assume* $E = \bigcup_k E_k = \bigcup_p E'_p$*, where* $E_k \subset U_k$ *and* $E'_p \subset U'_p$ *are disjoint unions of small* α^+*-measurable sets.*
- *Define* $E_{kp} = E_k \cap E'_p$*. These sets* E_{kp} *are pairwise disjoint, small,* α^+*-measurable sets. Hence, the implication*

$$E_k = \bigsqcup_p E_{kp} \implies \mu_\alpha^+(E_k) = \sum_p \mu_\alpha^+(E_{kp})$$

holds (we refer to the countable additivity of the measure μ_α^+ *on subsets of* U_k*).*
- *Similarly:*

$$\mu_\alpha^+(E'_p) = \sum_k \mu_\alpha^+(E_{kp}),$$

and:

$$\sum_k \mu_\alpha^+(E_k) = \sum_k \sum_p \mu_\alpha^+(E_{kp}) = \sum_p \sum_k \mu_\alpha^+(E_{kp}) = \sum_p \mu_\alpha^+(E'_p).$$

- *Thus, the value* $\mu_\alpha^+(E)$ *does not depend on the representation of* E *as a disjoint union of small* α^+*-measurable sets. This establishes the correctness of the definition of* μ_α^+*(by properties of sums of nonnegative families, we can take sums of subfamilies and change the order of summation).*

Step 2: Countable additivity

- *Let* $E = \bigcup_k E_k$ *be a disjoint at most countable family, where each* $E_k \in \mathfrak{A}_\alpha^+$*.*
- *Each* E_k *can be expressed as a disjoint union of small* α^+*-measurable sets:*

$$E_k = \bigcup_p E_{kp}.$$

Then:

$$E = \bigcup_k E_k = \bigcup_k \bigcup_p E_{kp},$$

- *where all the sets* E_{kp} *are disjoint. By definition:*

$$\mu_\alpha^+(E) = \sum_{k,p} \mu_\alpha^+(E_{kp}).$$

- *Using the property of sums of nonnegative families:*

$$\mu_\alpha^+(E) = \sum_{k,p} \mu_\alpha^+(E_{kp}) = \sum_k \left(\sum_p \mu_\alpha^+(E_{kp}) \right) = \sum_k \mu_\alpha^+(E_k).$$

- *Thus,* μ_α^+ *is countably additive.*

Step 3: Completeness and σ-finiteness

- *The completeness and* σ*-finiteness of* μ_α^+ *on subsets of a parametrizable neighborhood* U *follow from the corresponding properties of* $\nu_{f,\alpha}^+$ *(see Lemma 9.0.2).*
- *For any* $E \in \mathfrak{A}_\alpha^+$*,* E *can be represented as a union of an at most countable family of disjoint, small* α^+*-measurable sets:*

$$E = \bigcup_k E_k.$$

- *Since* μ_α^+ *is countably additive and each* $\mu_\alpha^+(E_k)$ *is finite,* μ_α^+ *is* σ*-finite on* $\mathfrak{A}_\alpha^+$*.*

Step 4: Finiteness on compact sets

- *Let $K \subset M$ be a compact set.*

  For each $x \in K$, there exists a neighborhood $U_x = f(\mathbb{R}^n)$ such that f is a positive parametrization of U_x. Let $P_x \subset \mathbb{R}^n$ be a cell such that $f^{-1}(x) \in \operatorname{Int}(P_x)$.

- *Define:*

$$G_x = f(\operatorname{Int}(P_x)).$$

- *Since f is a homeomorphism, G_x is open and $G_x \subset U_x$. By definition:*

$$\mu_\alpha^+(G_x) = \nu_{f,\alpha}^+(f^{-1}(G_x)) = \nu_{f,\alpha}^+(\operatorname{Int}(P_x)) \leq \nu_{f,\alpha}^+(\overline{P_x}) < \infty,$$

  because the closure $\overline{P_x}$ is compact and f_α has locally finite variation (Theorem 9.0.3).

- *For any $x \in K$, we have $x \in G_x$, i.e.,*

$$K \subset \bigcup_{x \in K} G_x.$$

- *Since $K \subset \bigcup_{x \in K} G_x$, we can extract a finite subcover:*

$$K \subset \bigcup_{k=1}^{p} G_{x_k}.$$

- *Thus, this implies:*

$$\mu_\alpha^+(K) \leq \sum_{k=1}^{p} \mu_\alpha^+(G_{x_k}) < \infty.$$

- *This proves that $\mu_\alpha^+(K) < \infty$ for any compact set $K \subset M$.*

Conclusion

The proof demonstrates that μ_α^+ is well-defined, countably additive, σ-finite, and complete on $\mathfrak{A}_\alpha^+$. Additionally, $\mu_\alpha^+(K) < \infty$ for any compact set $K \subset M$.

Extension to μ_α^- and μ_α

In a similar way, one defines μ_α^- and shows that it satisfies analogous properties. The oriented measure $\mu_\alpha(E)$ is defined for sets $E \in \mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-$ with finite values of $\mu_\alpha^+(E)$ and $\mu_\alpha^-(E)$ as:

$$\mu_\alpha(E) = \mu_\alpha^+(E) - \mu_\alpha^-(E).$$

(the oriented measure of the α-projection of the set E). The collection $\mathfrak{R}_M$ of all such sets forms a δ-ring of subsets of M.

10.0.1 The σ-algebra $\mathfrak{A}_\alpha^-$

Similar to $\mathfrak{A}_\alpha^+$, the collection of subsets of M that are measurable with respect to the negative variation $\nu_{f,\alpha}^-$ forms a σ-algebra denoted $\mathfrak{A}_\alpha^-$.
This σ-algebra contains all Borel subsets of M, as shown in Lemma 9.0.1.

1. *By analogy with $\mathfrak{A}_\alpha^+$, the σ-algebra $\mathfrak{A}_\alpha^-$ consists of subsets of M that are measurable with respect to the negative variation $\nu_{f,\alpha}^-$. Closure under standard operations follows as in Lemma 10.0.2.*

2. *Since $\nu_{f,\alpha}^-$ is defined via positive parametrizations and open sets are measurable, all Borel subsets of M belong to $\mathfrak{A}_\alpha^-$.*

The measure μ_α^-

On $\mathfrak{A}_\alpha^-$, a complete σ-finite measure μ_α^- is defined. For any compact set $K \subset M$, we have $\mu_\alpha^-(K) < \infty$.

1. *The measure is defined by:*
$$\mu_\alpha^-(E) = \nu_{f,\alpha}^-(f^{-1}(E)),$$
where f is a positive parametrization of a neighborhood $E \subset U$.

2. *For any compact $K \subset M$, it can be covered by finitely many such U_k with $\mu_\alpha^-(U_k) < \infty$, so $\mu_\alpha^-(K) < \infty$.*

Oriented measure μ_α

1. *For $E \in \mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-$ with finite $\mu_\alpha^+(E)$ and $\mu_\alpha^-(E)$, the oriented measure is defined as:*
$$\mu_\alpha(E) = \mu_\alpha^+(E) - \mu_\alpha^-(E).$$
This reflects the net contribution under the projection associated with α.

2. *μ_α is well-defined and inherits additivity and finiteness from $\mu_\alpha^\pm$.*

The collection $\mathfrak{R}_M$

Define:
$$\mathfrak{R}_M = \{E \subset M \mid E \in \bigcap_\alpha (\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-) \text{ and } \mu_\alpha^+(E), \mu_\alpha^-(E) < \infty \text{ for all } \alpha\}.$$
Then $\mathfrak{R}_M$ is a δ-ring of subsets of M, meaning it is closed under differences, finite unions, and countable intersections.

1. *Properties:*
   - *Closed under differences, since $\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-$ is a σ-algebra.*
   - *Closed under finite unions.*
   - *Closed under countable intersections.*

Simple n-vectors in $\mathbb{R}^m$

Let e_i denote the standard basis vectors in $\mathfrak{R}^m$. For $n \leq m$ and indices $1 \leq i_1 < \cdots < i_n \leq m$, the simple n-vectors $e_\alpha = e_{i_1} \wedge \cdots \wedge e_{i_n}$ form a basis for the space of n-vectors in $\mathfrak{R}^m$.

1. *For indices $1 \leq i_1 < \cdots < i_n \leq m$, the wedge product $e_\alpha = e_{i_1} \wedge \cdots \wedge e_{i_n}$ defines a simple n-vector.*
2. *Each α defines a projection onto the subspace spanned by the e_{i_k}, and the measures $\mu_\alpha^\pm$ and μ_α are constructed with respect to these projections.*

Conclusion

- *This explanation clarifies the construction of the σ-algebras $\mathfrak{A}_\alpha^-$, the measures μ_α^-, and the oriented measure μ_α.*
- *Additionally, it defines the δ-ring $\mathfrak{R}_M$ and the role of simple n-vect*

Definition 10.0.6. *The mapping from $\mathfrak{R}_M$ into the space of n-vectors defined by*

$$\mu_M(E) = \sum_\alpha \mu_\alpha(E) e_\alpha, \quad E \in \mathfrak{R}_M,$$

is called the standard vector measure on an n-dimensional orientable manifold M with locally-finite variations that is embedded into $\mathbb{R}^m$ (with $m \geq n$).

Theorem 10.0.1. *Let M be an orientable n-dimensional manifold with locally-finite variations that is embedded into $\mathbb{R}^m$ ($m \geq n$).*
Then the following statements hold:

1. *Compact subsets of the manifold M belong to $\mathfrak{R}_M$;*
2. *The function μ_M is countably additive on $\mathfrak{R}_M$;*
3. *If M' is a manifold with the same support and inverse orientation, then $\mathfrak{R}_{M'} = \mathfrak{R}_M$ and $\mu_{M'} = -\mu_M$.*

Proof 10.0.4. Step 1: Compact subsets belong to $\mathfrak{R}_M$

By Lemmas 10.0.2 and 10.0.3, any compact set $K \subset M$ satisfies:

1. *K is α^+-measurable and α^--measurable for any $\alpha = \{i_1, i_2, \ldots, i_n\}$,*
2. *The values $\mu_\alpha^+(K)$ and $\mu_\alpha^-(K)$ are finite.*

- *Thus, $K \in \mathfrak{R}_M$ by definition, since $\mathfrak{R}_M$ consists of sets that are measurable with respect to all projections α and have finite measures μ_α^+ and μ_α^-.*
- *This proves that compact subsets of M belong to $\mathfrak{R}_M$.*

Step 2: Countable additivity of μ_M

- *Recall that the space of n-vectors in $\mathbb{R}^m$ is finite-dimensional. Convergence of series in this space is equivalent to coordinate-wise convergence in any basis.*
- *For each projection α, the measures μ_α^+ and μ_α^- are countably additive (Lemma 10.0.3).*
- *Since μ_M is defined as a vector-valued function whose components correspond to μ_α^+ and μ_α^-, the countable additivity of μ_M follows from the countable additivity of its components.*
- *Thus, μ_M is countably additive on $\mathfrak{R}_M$.*

Step 3: Behavior under orientation reversal

- *Let M' be the manifold with the same support as M but with the opposite orientation. Assume $f : \mathbb{R}^n \to U \subset M$ is a positive parametrization of a subset $U \subset M$.*
- *Define $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ as a homeomorphism that reverses orientation, such as:*

$$\text{for example} \quad \varphi(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

- *Then $g = f \circ \varphi$ is a positive parametrization of U in M', because reversing the orientation of φ compensates for the reversed orientation of M'.*
- *Let $E \subset U$ be a small α^+-measurable set in M. By definition:*

$$f^{-1}(E) \in \mathfrak{A}^+_{f,\alpha} \quad , \quad \mu_{\alpha^+,M}(E) = \nu^+_{f,\alpha}(f^{-1}(E)).$$

- *Using Theorem 9.0.5, we know that φ reverses orientation, so:*

$$g^{-1}(E) = \varphi^{-1}(f^{-1}(E)) \in \mathfrak{A}^-_{f\circ\varphi,\alpha} = \mathfrak{A}^-_{g,\alpha},$$

and:

$$\mu_{\alpha^+,M}(E) = \nu^+_{f,\alpha}(f^{-1}(E)) = \nu^-_{f\circ\varphi,\alpha}(\varphi^{-1}(f^{-1}(E))) = \nu^-_{g,\alpha}(g^{-1}(E)) = \mu_{\alpha^-,M'}(E). \quad (10.1)$$

- *Thus, E is a small α^--measurable set in M', and:*

$$\mu_{\alpha^+,M}(E) = \mu_{\alpha^-,M'}(E).$$

- *Similarly, applying φ^{-1} shows that the classes of small α^+-measurable sets in M and small α^--measurable sets in M' coincide. Hence:*

$$\mathfrak{A}_{\alpha^+,M} = \mathfrak{A}_{\alpha^-,M'}, \quad \mathfrak{A}_{\alpha^-,M} = \mathfrak{A}_{\alpha^+,M'}.$$

- *Decompose an arbitrary set $E \in \mathfrak{A}_{\alpha^+,M}$ into a disjoint union of small measurable sets. Using the above equalities eq 10.1, it follows that:*

$$\mu_{\alpha^+,M}(E) = \mu_{\alpha^-,M'}(E) \quad , \quad \mu_{\alpha^-,M}(E) = \mu_{\alpha^+,M'}(E).$$

- *If $E \in \mathfrak{R}_M$, then:*

$$E \in \bigcap_\alpha \left(\mathfrak{A}_{\alpha^+,M} \cap \mathfrak{A}_{\alpha^-,M}\right) = \bigcap_\alpha \left(\mathfrak{A}_{\alpha^-,M'} \cap \mathfrak{A}_{\alpha^+,M'}\right).$$

- *All the numbers $\mu_{\alpha^+,M}(E) = \mu_{\alpha^-,M'}(E)$ and $\mu_{\alpha^-,M}(E) = \mu_{\alpha^+,M'}(E)$ are finite. This statement is equivalent to the following statements:*

$$E \in \mathfrak{R}_M$$

- *Thus:*

$$\mu_{\alpha,M'}(E) = \mu_{\alpha^+,M'}(E) - \mu_{\alpha^-,M'}(E) = \mu_{\alpha^-,M}(E) - \mu_{\alpha^+,M}(E) = -\mu_{\alpha,M}(E).$$

- *Taking the sum over all projections α, we conclude:*

$$\mu_{M'}(E) = -\mu_M(E).$$

Conclusion

The proof establishes the following:

- *Compact subsets of M belong to $\mathfrak{R}_M$.*
- *The function μ_M is countably additive on $\mathfrak{R}_M$.*
- *If M' has the same support as M but opposite orientation, then $\mathfrak{R}_{M'} = \mathfrak{R}_M$ and $\mu_{M'} = -\mu_M$.*

These results demonstrate the well-definedness and consistency of the oriented measure μ_M on an orientable manifold M.

Theorem 10.0.1 is fully proven.

Remark 10.0.2.

- *If a manifold M is orientable and disconnected, then this manifold admits more than two orientations (the orientation of M is determined by orientations of its components).*
- *Since any component admits exactly two orientations, we may apply Theorem 6 to components and show that $\mathfrak{R}_M = \mathfrak{R}_{M'}$ in the case of an arbitrary (not necessarily opposite) orientation of the manifold M'.*

Conclusion and open problems

In this work, we have introduced the class of n-dimensional manifolds embedded in $\mathbb{R}^m$ with locally-finite variations, together with a corresponding vector-valued measure that allows for the integration of differential forms in a manner analogous to the Lebesgue integral. This framework extends classical integration theory to a broader class of non-smooth geometric objects while preserving essential analytical properties such as countable additivity and convergence theorems.

Several important open problems and conjectures arise naturally from this study:

1. *The author conjectures that the definitions of a manifold with locally-finite variations and the associated standard n-vector-valued measure are independent of the choice of basis in the ambient space $\mathbb{R}^m$. It is further suggested that such manifolds may coincide with those possessing a locally-finite Favard measure or a locally-finite Hausdorff measure. In the second part of this work, it will be shown that smoothly embedded manifolds in $\mathbb{R}^n$ do indeed have locally-finite variations, and for these, basis independence of the integral is well known.*

2. *Another conjecture concerns the geometric examples of such manifolds. Specifically, the author proposes that the boundary of any n-dimensional convex body in $\mathbb{R}^n$ provides an example of an $(n-1)$-dimensional manifold with locally-finite variation, even when not necessarily smooth. For lower dimensions $k < n$, boundaries of k-dimensional convex sets may serve as similar examples.*

These open questions suggest promising directions for future research, particularly in clarifying the relationship between locally-finite variations and other measures commonly used in geometric measure theory, as well as in identifying new classes of manifolds admitting robust integration theories.

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جامـــعة غـــرداية
كليـــة العلوم والتكنولوجيـا

قسم : الرياضيات والإعلام الآلي
شعبة : الرياضيات
تخصص: التحليل الدالي وتطبيقاته

غرداية في : 03/07/2025

شهادة ترخيص بالتصحيح والايداع:

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بصفتي المشرف المسؤول عن تصحيح مذكرة تخرج (ليسانس/ماستر/دكتورا) المعنونة بـ:
Integration of differential forms on Manifolds With Locally - Finite Variations

من انجاز الطالب (الطلبة):
بحساني فاطمة الزهرة

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