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كلية العلوم والتكنولوجيا

Faculty of Science and Technology

قسم الرياضيات والإعلام الآلي

Department of Mathematics and Informatics

Thesis of end study, with a view to obtaining the diploma

Master

Domaine: Mathematics and Informatic **. Field:** Mathematics

Specialty: Functional Analysis and Applications.

Theme

Family of differential systems with explicit algebraic and non-algebraic limit cycles

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الحمد لله الذي يسر البدايات وأكمل النهايات وبلغنا الغايات.

استاذي بن كينة عبد الكريم الذي كان رمز العطاء وانار دربي بالعلم والمعرفة فشكرا لك ما قدمت وما تقدم و شكرا لك بألف لغة.

أوجه لنفسي كل التقدير والحب بسبب عدم استسلامي للعديد من الصعاب التي مرت عليا فكان امامي خيارين إما أن أستسلم وأقول لا أستطيع فعل ذلك أو أن أقاوم نفسي وأكمل الطريق الذي بدأت فيه فالحمد لله الذي مدني بعزيمة وإرادة .

من قال أنا لها نالها.

يقال ان خلف كل رجل عظيم إمرأة عظيمة إنما يا سيد قلبي أنت من صنعت هذه المرأة فأنت سندي وفخري لك كل الشكر.

أولادي أجمل ما في حياتي وبهم تحلو حياتي حفظكم الله ورعاكم.

رسالة شكر وامتنان لاهلي لوقوفكم بجانبي دوماً، فكنتم نعم العون والناصح.

صديقتي بشيري بشرى مهما استخدمت من كلمات الشكر وعبارات التقدير لا تفييك قدرك فحروف اللغة قليلة وسأظل عاجزة عن التعبير.

لولا ان وفقني الله ويسر لي أسباب النجاح ما كنت لأنجح .

فالحمد لله الذي بنعمته تتم الصالحات.

Abstract:

In this memory, we present some fundamental notions of the qualitative theory of differential equations, precisely on nonlinear polynomial planar differential equation systems which aim to find the properties of their solutions (such as equilibrium points, invariant curves, and periodic solutions).We give details about the research of A.Gasull, H.Gizcomini, J.Torregrosa. In the article entitled: "Explicit non-algebraic limit cycles for polynomial systems" published by: "Elsevier" on January 3, 2006.

Keywords : Sixteenth Hilbert problem, planar polynomial differential system, curve invariant, hyperbolic limit cycle.

Résumé:

Dans ce mémoire, nous présentons quelques notions fondamentales de la théorie qualitative d'équations différentielles, précisément sur différentiel plan polynomial non linéaire systèmes d'équations qui visent à trouver les propriétés de leurs solutions (telles que points d'équilibre, courbes invariantes et solutions périodiques).Nous donnons des détails à propos des recherches de A. Gasull, H. Gizcomini, J. Torregrosa. Dans l'article intitulé:" Cycles limites explicites non algébriques pour les systèmes polynomiaux " publié par: "Elsevier" le 3 janvier 2006**.**

Mots clés : Seizième problème de Hilbert, système différentiel polynomiale planaire, courbe invariante, cycle limite hyperbolique.

الملخص :

في هذه المذكرة ، نقدم بعض المفاهيم الاساسية حول النظرية النوعية للمعادلات التفاضلية , على وجه التحديد على جملة معادلات التفاضلية المستوية غير الخطية متعددة الحدود والتي تهدف إلى إيجاد خصائص حلولها (مثل نقاط توازن, منحنيات ثابتة وحلول دورية),نقدم تفاصيل حول بحث أ. جاسول ،هـ. جيزكوميني، ج. توريجروسا. في مقال بعنوان: "دورات حدية و صريحة غير جبرية لجمل تفاضلية "نشرته :"إلسفير " في 3 يناير 2006

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GENERAL INTRODUCTION

Differential equations have important applications and are powerful tools in the behaviorstudy of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. Direct resolution of a differential equation is usually difficult or impossible. However, another way out is possible. This is the qualitative study of differential equations. This study makes it possible to provide information on the of the solutions of a differential equation without the need to solve it explicitly, and it consists in examining the properties and the characteristics of the solutions of this equation, and to justify among these solutions, the existence or non-existence of an isolated closed curve form called a limit cycle.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of differential equations. Usually, we ask for the number of such limit cycles as orbits, and an even more difficult problem is to give an explicit expression of them. The limit cycles were introduced for the first time by Henri Poincaré in 1881 in his "Dissertation on the curves defined by a differential equation" [34]. Poincaré was interested in the qualitative study of the solutions of differential equations, i.e., points of equilibrium, limit cycles, and their stability. This makes it possible to have an overall idea of the other orbits of the studied systems.

The mathematician David Hilbert presented at the second international congress of mathematics [23] in 1900, 23 problems whose future awaits resolution through new methods that will be discovered in the century that begins. Problem number 16 is to know the maximum number and relative position of the limit cycles of a planar polynomial differential system of degree n. We denote H_n as this maximum number. Dulac [11] in 1923, offered a proof that H_n is finite. In recent years, several papers have studied the limit cycles of planar polynomial differential systems. The main reason for this study is Hilbert's 16th unsolved problem. Later on, Van der Pol [10] in 1926, Liénard [27] in 1928, and Andronov [2] in 1929 shown that the periodic solution of self-sustained oscillation of a circuit in a vacuum tube was a limit cycle.

This work is organized in the following way :

- **Chapter 1**: Contains some preliminary notions of differential systems, introductive and necessary for the understanding of all this work. Let's start with define differential systems, vector field, flow, phase portrait, point equilibrium, linearization of nonlinear differential systems in the vicinity of points of equilibria, periodic solutions and their stabilities.

- **Chapter 2:** In this chapter, we are more particularly interested in the study of cycles limits. We will start with the qualitative study of the limit cycles. We study the noexistence and existant of limit cycle .We study stability of limit cycles We study hyperbolic limit cycles

- **Chapter 3**: This chapter presents the work of A. Gasull, H.Gizcomini, J.Torregrosa . In the article entitled: "Explicit non-algebraic limit cycles for polynomial systems" published by: "Elsevier" on January 3, 2006. To prove that a planar polynomial vector field can have an explicit limit cycle that is not algebraic They prove that this system has at most one limit cycle and that when it exists, it can be explicitly found and given by quadratures. They provide valuable insights into the existence of hyperbolic systems and the existence of algebraic solutions. They provide that the limit cycle is not algebraic.

CHAPTER 1
PRELIMINARY CONCEPTS

1.1 Planar polynomial differential systems

Some basic concepts for the qualitative theory of polynomial differential systems are covered in this chapter .The concepts of vector field, flux, phase portrait, equilibrium point, and linearization of non-linear differential systems at equilibrium points will all be covered .We will begin with polynomial differential systems. Next, we study the equilibrium points' characteristics and stability .Then integrability and invariant curves. the basic theorems are reviewed, which include the Hartman-Grobman theorem, the existence and uniqueness theorem, and the theorems of the Lyapunov and Poincaré stability method of classification.

Definition : 1 *[33] A real planar polynomial differential system of degree m is a system*

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{1.1}$$

where P and Q are real polynomials in the variables x and y, the degree of the system

(1.1) is the maximum of the degrees of the polynomials P and Q .As usual the dot denotes derivative with respect to the independent variable t.

1.1.1 Vector field

Definition : 2 *[33] The vector field associated to the polynomial differential system (1.1) is an application*

$$\mathcal{X} : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$M(x,y) \longmapsto \frac{\overrightarrow{dM}}{dt} = \begin{pmatrix} p(x,y) \\ Q(x,y) \end{pmatrix},$$

Vector fields are essential for understanding the behavior of solutions to differential equations .They are used to define the local flow generated by a vector field .

In summary, vector fields provide a geometric interpretation of differential equations, allowing us to study the behavior of the solutions in a visual and intuitive way.

The graphical representation of a vector field on the plane consists in drawing a number of well chosen vectors $(x, \mathcal{X}(x))$ as in 1.1.

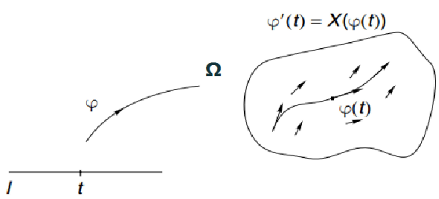


Figure 1.1: An integral curve.

Remark: 1 *[33] We note $\mathcal{X} = (P, Q)$ the vector field associated with the system (1.1) .*

It is also written in the form:

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Example: 1 *We give the following example*

$$\begin{cases} \dot{x} = 2y + x, \\ \dot{y} = -x + y. \end{cases} \tag{1.2}$$

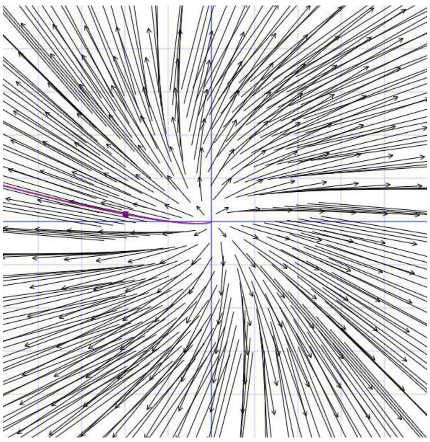


Figure 1.2: Vector field associated with the system (1.2)[36].

1.2 Solution and periodic solution

1.2.1 Solution of systems differential

Definition : 3 *[13] The differential system (1.1) solution is referred to as any differentiable function*

$$\varphi : \quad I \subset \mathbb{R} \longrightarrow \mathbb{R}^2,$$

$$t \longrightarrow \varphi(t) = (\varphi_1(t), \varphi_2(t)),$$

where I is $\mathbb{R}$ a interval which satisfies the following criteria:

1. *for* $t \in I, (t, \varphi(t)) \in \Omega$ *all an open of* $\mathbb{R}$.

2. *for all*

$$t \in I \begin{cases} \frac{d\varphi_1(t)}{dt} = P(\varphi_1(t), \varphi_2(t)), \\ \frac{d\varphi_2(t)}{dt} = Q(\varphi_1(t), \varphi_2(t)). \end{cases}$$

Definition : 4 *The orbit of the system (1.1) is called the representation of a solution* $X(t), t \in I$ *on the plane* $\mathbb{R}^2$.

Definition : 5 *[12] The graph of a solution* $\varphi : I \longrightarrow \mathbb{R}^2$ *is an integral curve or trajectory of the differential system (1.1).*

1. *The image* φ *of* $\mathbb{R}$ *of is an orbit of the differential system (1.1).*

2. *The space* $\mathbb{R}^2$ *or the solutions take their values is called the phase space.*

Definition : 6 *called periodic solution of system (1.1) ,all solution* $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ *for which there exists a real* $T > 0$ *such that:*

$$\forall t \in \mathbb{R}, x(t+T) = x(t) \ \textit{and} \ \ y(t+T) = y(t).$$

The smallest number $T > 0$ *is called the period of this solution.*

Remark: 2 *[12] If* $X(t)$ *has a period* T*, the solution also has a period* kT

Definition : 7 *[13] we call the phase plane associated with a system (1.1) the open* $\Omega \subseteqq \mathbb{R}^2$ *where their solutions have values.*

Example: 2 *The harmonic oscillator is governed by the differential equation*

$$\ddot{x} + w^2 x = 0,$$

which can be written in the form of the following systems:

$$\begin{cases} \dot{x} = y, \\ \\ \dot{y} = -w^2 x, \end{cases} \tag{1.3}$$

system integrates easily since $\frac{dy}{dx} = -w^2 \frac{x}{y}$ *this gives a set of solution* $y^2 + w^2 x^2 = c$*, which* $c \in \mathbb{R}$ *.In other words, this system has a continuous one-parameter family of periodic solutions represented in this phase plane by ellipses .*

1.2.2 Phase portrait

Trajectories that show the systems evolution across time are usually included in the phase portrait, along with crucial locations like equilibrium points, limit cycles, and other significant characteristics. The stability, periodicity, and general dynamics of the system described by the differential equations can all be understood by examining the phase portrait.

Definition : 8 *[29] A phase portrait is a phase space geometric representation of a dynamic system trajectories, where each set of initial conditions is represented by a curve or a point.*

Example: 3 *The associated phase portrait with the system (1.2)*

1.3 Singular point

Definition : 9 *[33] A point* (x^*, y^*)*is called a singular point of system (1.1) if it satisfies*

$$P(x^*, y^*) = Q(x^*, y^*) = 0.$$

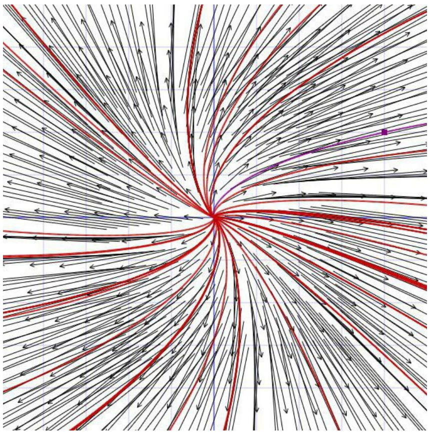


Figure 1.3: Phase portrait associated with the system (1.2)[36].

Remark: 3 *[33] For the vector field, the concept of an equilibrium point is equivalent to that of a singular point.When examining the vector field in isolation, we prefer to refer to it as a equilibrium point, and when considering the trajectories, we speak of a point of equilibrium.*

Example: 4 *We give the system:*

$$\begin{cases} \dot{x} &= x(2 - x - y), \\ \dot{y} &= x - y, \end{cases}$$

then the singular points are : (0, 0) and (1,1).

1.3.1 Linearization and Jacobian matrix :

Most existing systems in nature are nonlinear. The most natural approach to studying the behavior of trajectories of a nonlinear autonomous differential system near a singular point is to reduce it to the study of the associated linear system We denote by $J_x(x^*, y^*)$ the Jacobian matrix associated with the vector field x in the neighborhood of a singular point (x^*, y^*) defined by

$$J_x(x^*, y^*) = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix}, \tag{1.4}$$

The linearization of the nonlinear system (1.1) is given by:

$$\begin{pmatrix} \acute{u} \\ \acute{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

1.3.2 Classification of singular points:

Definition : 10 *[12] Singular point stability is determined by the general stability theorems. Therefore, if the real eigenvalues (or real components of complex eigenvalues) are negative, the equilibrium point is asymptotically stable. Two examples of such equilibrium places are stable focus and stable node. If the real part of at least one eigenvalue is positive, the corresponding equilibrium point is unstable. It might be a saddle, for example.Lastly, we are dealing with the classical stability in the Lyapunov sense where the singular point is a center, or simply imaginary roots. Let's assume that the eigenvalues of the Jacobian matrix (1.4) are denoted by* λ_1 *and* λ_2*:*

1. *If* $0 < \lambda_1 < \lambda_2$ *then the singular point is unstable.*

   is said that the singular point is an "unstable node".

2. *If* $\lambda_1 < 0 < \lambda_2$ *then it is said that the singular point is a "saddle point".*

3. *If* $\lambda_1 < \lambda_2 < 0$ *then the singular point is asymptotically stable.*

   It is said that the singular point is a "stable node".

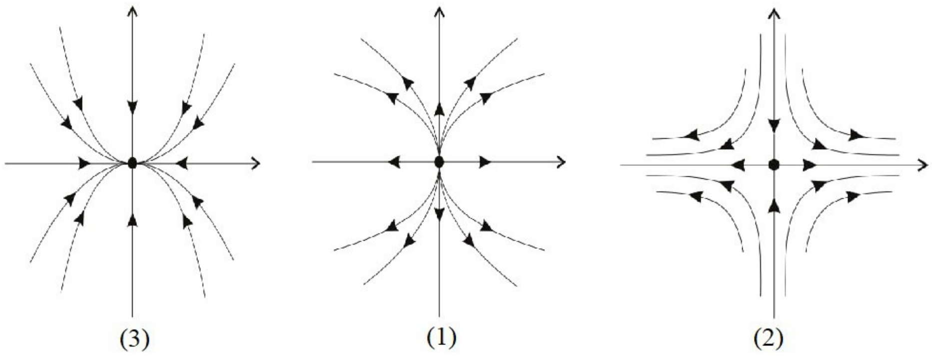


Figure 1.4: Real non-zero and distinct eigenvalues [29].

If λ_1 and λ_2 are distinct complex numbers (where their imaginary parts are not zero) (see figure 1.5):

1. If $Re(\lambda_1), Re(\lambda_2) < 0$ then the singular point is asymptotically stable.

   It is said that the point is a stable spiral sink "stable focus".

2. If $Re(\lambda_1), Re(\lambda_2) = 0$ then the singular point is a center.

3. If $Re(\lambda_1), Re(\lambda_2) > 0$ then the singular point is unstable.

   It is said that the point is an unstable spiral source "unstable focus".

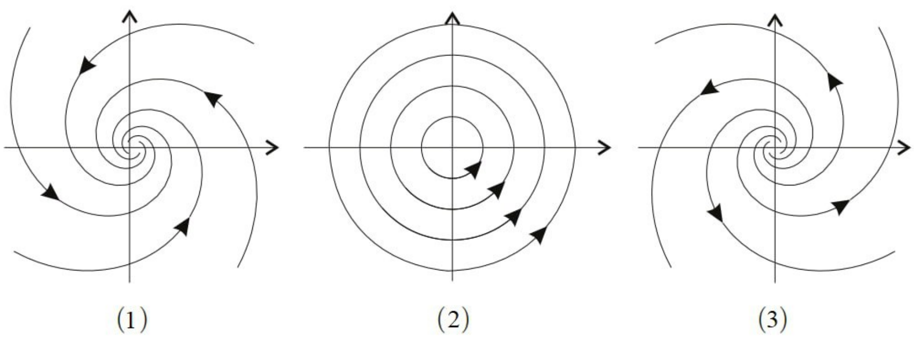


Figure 1.5: Complex eigenvalues[29].

If λ_1 and λ_2 are equal J_x is diagonalizable .

1. If $\lambda > 0$ then we say that the point is a "source in star".

2. If $\lambda < 0$ then we say that the point is a "sink in star".

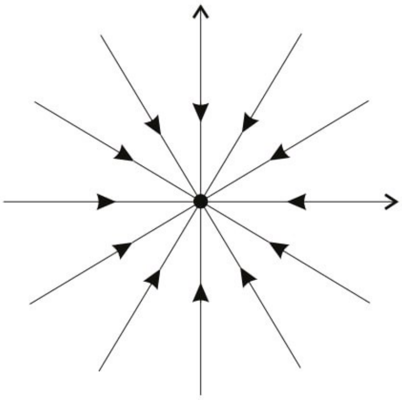


Figure 1.6: Eigenvalues are $\lambda_1 = \lambda_2$ [29].

A is not diagonalizable

1. If $\lambda > 0$ then the point is called a source.

2. If $\lambda < 0$ then the point is called a sink.

3. If $\lambda = 0$ then all points on the line kv are equilibrium points.

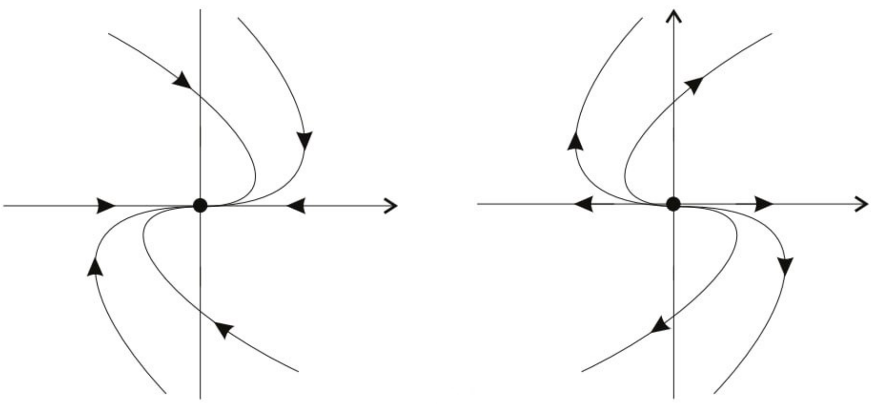


Figure 1.7: Real non-zero and distinct eigenvalues [29].

Remark: 4 *[12] We recall that a singular point p is hyperbolic if the eigenvalues of the linear part of the system at (x^*, y^*) have non zero real part.*

Theorem: 1 *[33] Let* (x^*, y^*) *be a hyperbolic singular point of system (1.7).Then in a neighborhood of, the trajectories of* (x^*, y^*) *the solutions of system (1.7) have the same shape as the trajectories of the solutions of its linearization.*

Example: 5 *We give the system*

$$\begin{cases} \dot{x} = -3x - y^2, \\ \dot{y} = 3x^4 - y, \end{cases} \tag{1.5}$$

The system (1.5) has a single singular point which is the origin (0, 0), and the system linearized at this point is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x}(0,0) & \frac{\partial P}{\partial y}(0,0) \\ \frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q}{\partial y}(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

. *this system has two negative real eigenvalues* $\lambda_1 = -3$ *and* $\lambda_2 = -1$ *Then the singular point (0, 0) is a stable node.*

Definition : 11 *[33] The singular point in the system (1.1) is called hyperbolic if the eigenvalues of the matrix* $J_x(x^*, y^*)$. *Have real part different from 0.Otherwise, the singular point is said to be non-hyperbolic.*

1.3.3 Topological equivalence:

Definition : 12 *[12] A homeomorphism of* $\mathbb{R}^2$ *is a bijective continuous map* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. *whose inverse bijection is continuous.*

Definition : 13 *[12] Two autonomous systems in the plane*

$$\begin{cases} \dot{x} = P_1(x(t), y(t)), \\ \\ \dot{y} = Q_1(x(t), y(t)), \end{cases} \tag{1.6}$$

and

$$\begin{cases} \dot{x} = P_2(x(t), y(t)), \\ \\ \dot{y} = Q_2(x(t), y(t)), \end{cases} \tag{1.7}$$

defined on two open sets U *and* V *in* $\mathbb{R}^2$ *respectively are topologically equivalent, if there exists a homeomorphism.*

$$h : U \longrightarrow V.$$

such that h transforms the orbits of (A) into orbits of (B) and preserves the orientation of the orbits. The following theorem allows us to reduce the study of a differential system in the neighborhood of a hyperbolic singular point to the study of a linear system topologically equivalent to (1.7) in the neighborhood of the origin.

1.3.4 Theorem of Hartman-Grobman

The Hartman-Grobman theorem, sometimes referred to as the linearization theorem, is an important finding in the mathematical study of dynamical systems.grasp the behavior of systems close to equilibrium points requires a grasp of this theorem.It asserts that a nonlinear system can be spatially topologically identical to its linearization around a hyperbolic equilibrium point under specific circumstances.

Theorem: 2 *(Hartman-Grobman,1967):[33]*

Let's assume that the Jacobian matrix at the singular point has two eigenvalues λ_1 *and* λ_2 *such that and, then the solutions of system (1.1) are approximately given by the solutions* *of the linearized system (1.1) in the vicinity of the singular point. In other words, the phase portrait of the linearized system constitutes, in the vicinity of this equilibrium point, a good approximation of that of the system (1.1).*

Remark: 5 *[12] If $Re(\lambda_{1,2}) = 0$, the singular point (x^*, y^*) is called a center for the linearized system. Determining its nature in the case of system (1.1) requires further investigations: this is the center problem.*

1.3.5 Stability of an singular point:

Definition : 14 *[12] Let (x^*, y^*) is singular point of the system (1.1).Note by $X^* = (P(x^*, y^*), Q(x^*, y^*))$ and $X(t) = (P(x(t), y(t), Q(x(t), y(t))))$.*

- *(x^*, y^*) is called stable if*

  $$\forall \varepsilon > 0, \exists \eta > 0, \|(x, y) - (x^*, y^*)\| < \eta \Rightarrow \forall t > 0, \|X(t) - X^*\| < \varepsilon.$$

- *(x^*, y^*) is called asymptotically stable if it is stable and :*

  $$\lim_{t \longrightarrow +\infty} \|X(t) - X^*\| = 0.$$

- *(x^*, y^*) is called exponentially stable if:*

  $\forall \varepsilon > 0, \exists \delta > 0, \alpha > 0$ and $\beta > 0$ such as:

  $\|X(t) - X^*\| < \varepsilon$, $\forall t > 0, \|X(t) - X^*\| < \alpha \|X(t) - X^*\| e^{-\beta t}$.

1.3.6 Stability in the sense of Lyapunov:

The concept of Lyapunov stability is rooted in the idea of finding Lyapunov functions that certify the stability or asymptotic stability of equilibrium points, with positive definite and negative definite functions playing key roles in stability analysis.

Definition : 15 *[29] Let* $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be a* C^1 *function* $\dot{X} = F(X)$*, a differential system,with* X_0 *singular points of system a Lyapunov* C^1 *function such as*

$$V(X_0) = 0 \quad \forall X \neq X_0 : V(X) > 0,$$
$$\forall X \neq X_0 : \overrightarrow{-gradV(X)}\dot{X} < 0.$$

Theorem: 3 *[29] Let be an* X_0 *equilibrium and is a Lyapunov function. Then*

- $\dot{V}(X) < 0$ *then* X_0 *is lyapunov stable .*
- $\dot{V}(X) \leq 0$ *then* X_0 *is asymptotically stable.*
- $\dot{V}(X) > 0$ *then* X_0 *is unstable .*

Example: 6 *[29] Consider the system:*

$$\begin{cases} \dot{x} = -x + 4y, \\ \dot{y} = -x - y^3, \end{cases}$$

let $V(x, y) = x^2 + 4y^2$*, then:*

$$\begin{cases} V(x, y) > 0, \qquad \forall (x, y) \neq (0, 0), \\ \overrightarrow{gradV(x, y)} \cdot F(x, y) = -2x^2 - 8y^4, \\ \overrightarrow{gradV(x, y)} \cdot F(x, y) < 0. \end{cases}$$

Thus, V is a Lyapunov function, and the system does not admit a closed orbit (all trajectories tend towards 0*).*

Remark: 6 *[29] T_0 distinguish between the first and second cases, the Lyapunov function is usually called strict Lyapunov function.*

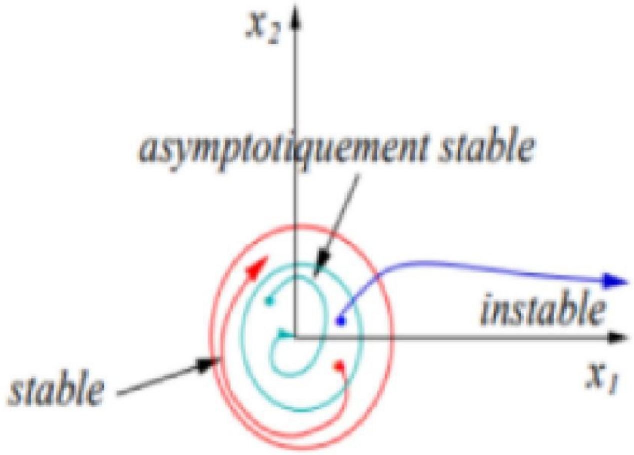


Figure 1.8: Different types of stability in the sense of Lyapunov [29].

1.3.7 Stability in the Poincare sense:

Compared to Lyapunovs concept, stability in the Poincare sense is more precisely defined as the distance from a point M(x, y) to a periodic solution.

Definition : 16 *[41] Let $X(t) = x(x(t), y(t))$ a system solution (1.1) a periodic solution $\phi(t)$ of this system is stable in the poincare sense (or orbitally stable)if :*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|X(t_0) - \phi(t_0)\| < \delta \Longrightarrow d(t) = \inf_{t \in [0,t]} \|X(t) - \phi(t)\| < \varepsilon,$$

for $t \in [t_0, +\infty[$ $\phi(t)$ is asymptotically stable if it is stable and if in addition $\lim\limits_{t \longrightarrow +\infty} d(t) = 0.$

Remark: 7 *[41] Stability in the Lyapunov sense implies stability in the Poincare sense but the converse is not true.*

1.4 Invariant curve

The study of differential systems benefits greatly from using invariant curves, which can disclose crucial structural characteristics of the underlying dynamical system and shed light on the qualitative behavior of solutions.

Definition : 17 *[22] let* $U : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ *a function from class* $\mathbb{C}^1$ *in the open* Ω

$$C_U = \{(x, y) \in \Omega / U(x, y) = 0\},$$

is referred to as an invariant curve if a function K *of class* $\mathbb{C}^1$ *exists in* Ω *that satisfies the following relation, and that function is known as the cofactor:*

$$P(x, y)\frac{\partial U}{\partial x} + Q(x, y)\frac{\partial U}{\partial x} = K(x, y)U(x, y), \tag{1.8}$$

for all $(x, y) \in \Omega$.

Example: 7 *[33] The curve defined by the equation* $x^2 + y^2 = 1$ *is an invariant curve for the system :*

$$\begin{cases} \dot{x} &= x^2 + y^2 + y - 1, \\ \dot{y} &= x^2 + y^2 - x - 1, \end{cases}$$

$U(x, y) = x^2 + y^2 - 1,$

$$\begin{aligned} P(x, y)\tfrac{\partial U}{\partial x}(x, y) + Q(x, y)\tfrac{\partial U}{\partial y}(x, y) &= 2x(x^2 + y^2 + y - 1) + 2y(x^2 + y^2 - x - 1), \\ &= (2x + 2y)(x^2 + y^2 - 1), \end{aligned}$$

The cofactor is $K(x, y) = 2x + 2y$.

Remark: 8 *[22]*

- *When the cofactor* $K(x, y)$ *is a polynomial, is an invariant curve of polynomial cofactor.*
- *On the invariant curve, the gradient* $(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y})$ *of* U *is orthogonal to the vector field* $\mathcal{X}(P, Q)$, *so the vector field is tangent to the invariant curve at every point on this curve, hence this name is inspired by the fact that this curve is formed by solutions (or trajectories) of the vector field* $\mathcal{X}$.

Definition : 18 *[22] An invariant curve* $U(x, y) = 0$ *is said to be algebraic of degree* m *if* U *is a polynomial of degree* m*, otherwise we say that it is non-algebraic or transcendent. As you may recall the divergence of the system (1.1) is represented by the div notation.*

$$div(x, y) = \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}.$$

Definition : 19 *[22] An algebraic curve* $U(x, y) = 0$ *is irreducible, if* $U(x, y)$ *is a irreducible polynomial in the ring* $\mathbb{R}[x, y]$.

Theorem: 4 *[26] We consider the system (1.1) and* $\Gamma(t)$ *a periodic orbit of period* $T > 0$. *We suppos that* $U : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is an invariant curve*

$$\Gamma(t) = \{(x, y) \in \Omega : U(x, y) = 0\},$$

and $K(x, y) \in \mathbb{C}^1$ *is the cofactor specified in the equation (1.8) of the invariant curve* $U(x, y) = 0$. *We suppose that* $p \in \Omega$ *such as* $U(p) = 0$ *and* $\nabla U(p) \neq 0$ *so* P *is singular* *point of systeme (1.8) and*

$$\int_0^T div(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt. \tag{1.9}$$

1.5 Integrability and first integral

Definition : 20 *[5] The $\mathbb{R}$-polynomial system (1.1) is integrable on an open subset Ω of $\mathbb{R}^2$ if there exists a nonconstant analytic function $H : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ called a first integral of the system on Ω, which is constant on all solution curves $(x(t), y(t))$ of system (1.2) contained in Ω i.e. $H(X(t)) = cste \quad \forall t \in I$ onstant for all values of t for which the solution $(x(t), y(t))$ is defined and contained in Ω . Clearly H is a first integral of system (1.1) on U if and only if $XH = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} \equiv 0$ on Ω .*

1.5.1 Integrating factors

Definition : 21 *[3] Let Ω be an open subset of $\mathbb{R}^2$ and let $R : \Omega \longrightarrow \mathbb{R}$ be an analytic function which is not identically zero on Ω. The function R is an integrating factor of the $\mathbb{R}$-polynomial system (1.1) if onefactors of the following three equivalent conditions holds :*

1. $\frac{\partial (RP)}{\partial x} = -\frac{\partial (RQ)}{\partial y}$.
2. $div(RP, RQ) = 0$.
3. $XR = -Rdiv(P, Q)$.

Definition : 22 *[3] The first integral H associated to the integrating factor R is given by*

$$H(x, y) = \int R(x, y)P(x, y)dy + h(x). \tag{1.10}$$

where h is chosen such that $\frac{\partial H}{\partial x} = -RQ$ *Then*

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \dot{y} = RQ = -\frac{\partial H}{\partial x}. \tag{1.11}$$

In (1.10) we suppose that the domain of integration Ω *is well adapted to the specific expression. Conversely, given a first integral* H *of system (1.9) we always can find an integrating factor* R *for which (1.1) holds.*

Proposition: 1 *[12] If the* $\mathbb{R}$*-polynomial system (8.1) has two integrating factors* R_1 *and* R_2 *on the open subset* Ω *of* $\mathbb{R}^2$*, then in the open set* $\Omega \backslash \{R_2 = 0\}$ *the function* R_1/R_2 *is a first integral, provided* R_1/R_2 *is non-constant.*

1.5.2 Exponential factors

Definition : 23 *[1] Let* $g, h \in \mathbb{C}$ *be relatively prime polynomials. The function* $F = e^{g/h}$ *is an exponential factor of system (1.1) if* $XF/F = L \in \mathbb{C}[x, y]$*. In this case,* L *is called the cofactor of* F*. It has degree at most* $m - 1$*. The expression which defines* L *is often written as*

$$P\frac{\partial e^{g/h}}{\partial x} + Q\frac{\partial e^{g/h}}{\partial y} = Le^{g/h}. \tag{1.12}$$

Remark: 9 *[1] An exponential factor appears when an invariant algebraic curve has a geometric multiplicity greater than one. The exponential factors play the same role as invariant algebraic curves to obtain a first integral for the polynomial system. For more details on exponential factors than the ones given in this section .*

1.5.3 Inverse integrating factors

The inverse integrating factor is an important tool in the study of the existence and non-existence of limit cycles.

Definition : 24 *[26] A non-zero function* $V : \Omega \longrightarrow \mathbb{R}$ *is an inverse integrating factor of the system (1.1) on an open* $\Omega \subset \mathbb{R}^2$ *if* $V \in \mathbb{C}^1(\Omega), V \neq 0$ *on* Ω *which satisfies the following equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x}\right) V. \tag{1.13}$$

Theorem: 5 *[13] Let* V *an inverse integrating factor (1.1) on an open* $\Omega \in \mathbb{R}^2$

1. *the function* $\frac{1}{V}$ *on* $\Omega \backslash V = 0$ *is an inverse integrating factor of system (1.1) In addition, the function*

$$I(x,y) = -\int \frac{P(x,y)}{V(x,y)}dy + \int \left(\frac{Q(x,y)}{V(x,y)} + \frac{\partial}{\partial x}\int \frac{P(x,y)}{V(x,y)}dy\right) dx,$$

   is a first integral of the system (1.1) .

2. *if the system (1.1) has a first integral* I*, then the function*

$$V_1(x,y) = \frac{P}{-\frac{\partial I}{\partial y}} = \frac{Q}{-\frac{\partial I}{\partial y}},$$

   is an integrating factor of system (1.1).

1.5.4 Darboux first integral

The Darboux first integral is a significant class of first integrals exponential factors and the algebraic invariant curve can be used to define these functions.

Definition : 25 *[8]* $x = (P, Q)$ *of degree* n *is a polynomial planar complex vector field that admits* p *irreducible invariant algebraic curves* $f_i = 0$ *with cofactors.* K_i *for* $i = 1, ..., P$ *and* q *exponential factor* $\exp\left(\frac{g_i}{h_i}\right)$ *with cofactors* L_j *for* $j = 1, ..., q$ *so :*

1. *There exists* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ *if the*

function

$$f_1^{\lambda_1}...f_p^{\lambda_p}\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}...\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}. \tag{1.14}$$

is a first integral of the vector field $\mathcal{X}$.

2. *it exists* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p}\lambda_i K_i + \sum_{j=1}^{q}\mu_j L_j = -div(P,Q)$ *if the fanction (1.14) is a first integral of the vector field* $\mathcal{X}$.

Definition : 26 *[8] The Darboux function is the name given to the function (1.14). We say that the polynomial system (1.1) has a Darboux first integral if its first integral takes the form (1.14).*

Definition : 27 *[26] A Liouville function is a function that can be expressed by quadratures of elementary functions. The following theorem gives a relationship between the first integral and the inverse integrating factor.*

Theorem: 6 *[13] If a polynomial system has a Liouville first integral, then it has an inverse Darboux integrating factor.*

CHAPTER 2

ON THE THEORIE OF LIMIT CYCLES

2.1 Introduction

The existence of limit cycles was first detected by Poincaré en 1882, these cycles are characterized by their stability and are crucial for simulating the behavior of numerous oscillatory systems found in real life. Neighboring trajectories are drawn to a stable or attractive limit cycle as time approaches infinity. On the other hand, when time gets closer to negative infinity, an unstable limit cycle repels nearby trajectories. Semi-stable limit cycles behave in both repellent and appealing ways. Trajectories' stability and the existence of nearby trajectories that spiral toward or away from the limit cycle are among the properties of trajectories that are analyzed in the study of limit cycles. By forecasting the existence or absence of these periodic orbits, the Poincaré-Bendixson theorem is an important tool in determining the existence of limit cycles in two-dimensional nonlinear dynamical systems. In this chapter, we are particularly interested in the study of limit cycles. We will begin with the qualitative study of limit cycles. Then, we present the main results on existence and non-existence.

Definition : 28 *[13] A limit cycle of system (1.1) is an isolated periodic solution in the set of all the periodic solutions.*

Example: 8 *[33] Either the system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + x(1 - x^2 - y^2). \end{cases} \tag{2.1}$$

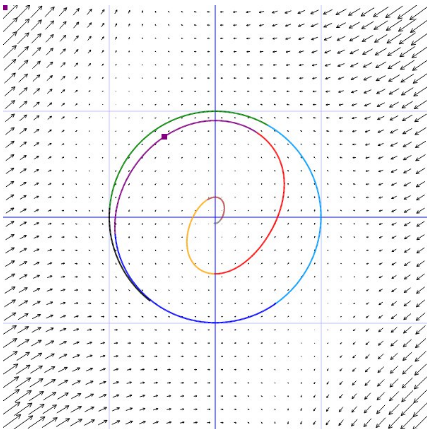


Figure 2.1: The phase portait of system (2.1)[36].

Example: 9 *[7] Either the system*

$$\begin{cases} \dot{x} = -2x - y - 2x(x^2 + y^2), \\ \dot{y} = x + 2y - 2y(x^2 + y^2), \end{cases} \tag{2.2}$$

In polar coordinates $x = r\cos(\theta)$ and $y = rsin(\theta)$, with $r > 0$, it becomes :

$$\begin{cases} \dot{r} = 2r(1 - r^2), \\ \dot{\theta} = 1, \end{cases} \tag{2.3}$$

Hence

$$\dot{r} = 0 \Longrightarrow r = \pm 1 \quad or \quad r = 0$$

Since $r > 0$, only the positive root $r = 1$ is accepted. So for $r = 1$ we have the periodic solution $(x(t), y(t)) = (\cos(t+\theta_0), \sin(t+\theta_0))$, with $\theta(0) = \theta_0$. In the phase plane there is a single limit cycle whose equation $x^2 + y^2 = 1$ and amplitude $r = 1$.

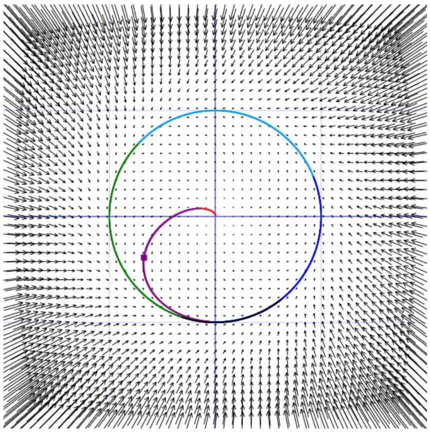


Figure 2.2: The phase portait of system (2.2)[36].

Definition : 29 *[25] An algebraic limit cycle is a limit cycle which is contained in the zeroes set of an invariant algebraic curve.*

Theorem: 7 *[30] Let $f \in \mathbb{R}[x, y]$ i.e. f is a polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ is an invariant algebraic curve of the polynomial system of differential equations (1.1) if for some polynomial $K \in \mathbb{R}[x, y]$, we have*

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf, \tag{2.4}$$

The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$.

Example: 10 *[35] Let the system be :*

$$\begin{cases} \dot{x} &= x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1), \\ \dot{y} &= y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1), \end{cases} \tag{2.5}$$

this system admits a single algebraic limit cycle of equation :

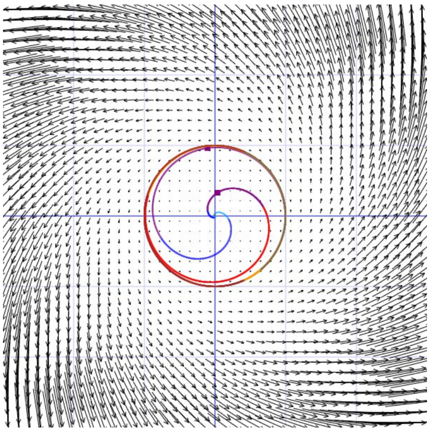


Figure 2.3: System limit cycle (2.5)[36].

Example: 11 *[6] The differential polynomial system of degree 3*

$$\begin{cases} \dot{x} = x + (y - x)(x^2 - yx + y^2), \\ \dot{y} = y - (y + x)(x^2 - yx + y^2). \end{cases} \tag{2.6}$$

has a unique non–algebraic limit cycle

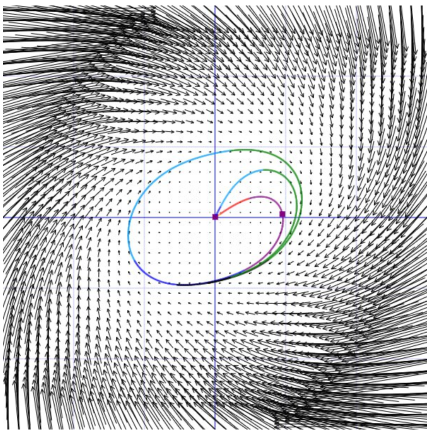


Figure 2.4: System limit cycle (2.2) [36].

2.2 The non existence and existence of limit cycles

2.2.1 The non existence of the limit cycle

Theorem: 8 *[46] (Bendixson). If the divergence $\partial P/\partial x + \partial Q/\partial y$ of system (1.1) has constant sign in a simply connected region U, and is not identically zero on any subregion of U, then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U .*

Proposition: 2 *[21] Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$ Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation (1.12). Let $\sum = (x, y) \in U : V(x, y) = 0$. If the divergence $\{\partial P/\partial x + \partial Q/\partial y\}$ of system (1.1) has constant sign in the simply connected domain of definition of $V \setminus \sum$ hen any limit cycle (in fact a closed trajectory) which lies entirely in the simply connected domain of definition of V must be contained in $\sum$.*

Proposition: 3 *[21] In the interior D of any closed trajectory of system (1.1) of a simply connected region we have*

$$\int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial x}\right) dxdy = 0. \tag{2.7}$$

Example: 12 *we give the following example :*

$$\begin{cases} \dot{x} = 4x^3y^2 - x^5y^4, \\ \dot{y} = x^4y^5 + 2x^2y^3, \end{cases} \tag{2.8}$$

we have

$$\begin{aligned} \tfrac{\partial P}{\partial x} + \tfrac{\partial Q}{\partial y} &= 12x^2y^2 - 5x^4y^4 + 5x^4y^4 + 6x^2y^2, \\ &= 18x^2y^2 > 0. \end{aligned} \tag{2.9}$$

then there can therefore be no limit cycle in $\mathbb{R}^2$

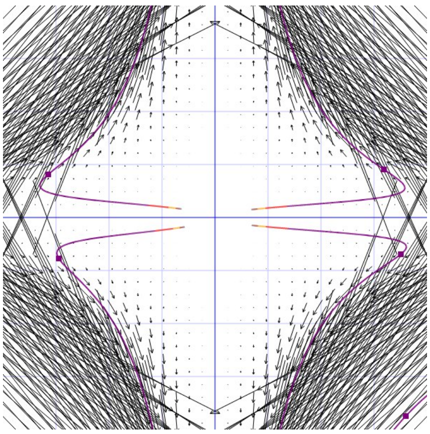


Figure 2.5: Phase portrait of the system 2.9[36].

Theorem: 9 *[21](Bendixon criterion) Let D be a related domain of* $\mathbb{R}^2$*. If* $div(\frac{dP}{dx} + \frac{dQ}{dy})$ *is non-zero and of constant sign on D, then the differential system (1.1) does not admit an entirely contained periodic solution in D.*

Example: 13 *[32] Consider the system*

$$\begin{cases} \dot{x} = x^3 + y, \\ \dot{y} = y - x. \end{cases} \tag{2.10}$$

$\frac{dP}{dx} + \frac{dQ}{dy} = 3x^2 + 1 > 0$ *at any point of* $\mathbb{R}^2$*, there can therefore be no limit cycle contained in* $\mathbb{R}^2$*.*

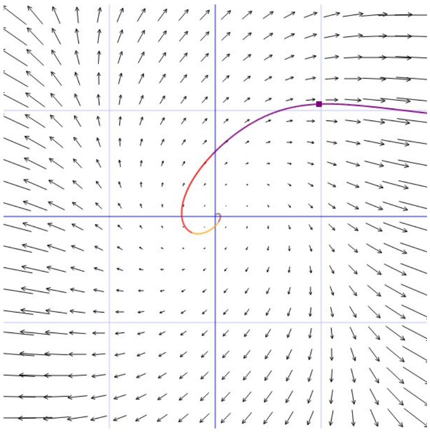


Figure 2.6: Phase portrait of the system 2.10[36].

Theorem: 10 *[46] (Bendixson-Dulac). If there exists a continuously differentiable function $B(x, y)$ in a simply connected region U such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion, then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U.*

Proposition: 4 *[21] Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$ Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation (1.12). Let $B(x, y)$ a continuously differentiable function in U such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign, then any limit cycle (in fact a closed trajectory) which lies entirely in the simply connected domain of definition of V must be contained in $\sum = \{(x, y) \in U : V(x, y) = 0\}$.*

Example: 14 *[32]Consider the system*

$$\begin{cases} \dot{x} & = x(3 - x - 3y), \\ \dot{y} & = y(2 - y - 3x). \end{cases} \tag{2.11}$$

Either the domain $D = \{(x, y) \in \mathbb{R}^2 / x > 0$ *and* $y > 0\}$

We have $\frac{dP}{dx} + \frac{dQ}{dy} = 5 - 5y - 5x$ *the quantity* $\frac{dP}{dx} + \frac{dQ}{dy}$ *cancels and changes sign in D. And the Bendixon criterion does not allow to conclude the non-existence of a closed orbit in D*

Let be the function $\psi(x, y) = \frac{1}{xy}$ *so* $\frac{d(\psi P)}{dx} + \frac{d(\psi Q)}{dy} = -\frac{1}{y} - \frac{1}{x}$

So, for everything $(x, y) \in D$ *the quantity* $\frac{d(\psi P)}{dx} + \frac{d(\psi Q)}{dy}$ *is negative and it can be concluded that the system does not admit a limiting cycle in the domain D.*

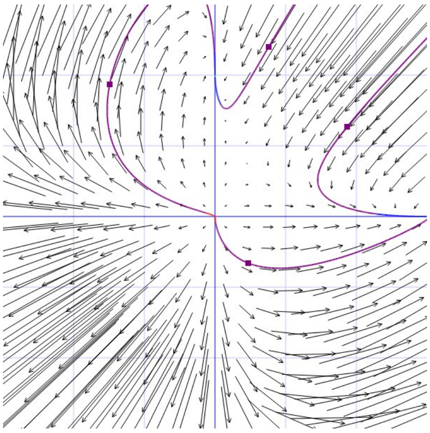


Figure 2.7: Phase portrait of the system 2.11[45].

Theorem: 11 *[21] (Cherkas). The vecteur field* $\mathcal{X}(x, y)$ *defined in the in the open subset U of* $\mathbb{R}^2$ *Suppose that in a simply connected domain* $U \subset \mathbb{R}^2$*, there exists a function* $\Psi(x, y)$ *f class* C^1 *and a number* $k > 0$ *such that*

$$k\Psi div\mathcal{X} + \mathcal{X}\Psi > 0,$$

then the domain U contains no limit cycles of system (1.1).

Corollaire: 1 *(Critical-point) A closed trajectory has a critical point in its interior. If we turn this statement around, we see that it is really a criterion for non-existence: it says that if a region* $\mathbb{R}$ *is simply-connected (i.e., without holes) and has no critical points, then it cannot contain any limit cycles. For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in* $\mathbb{R}$ *since* $\mathbb{R}$ *has no holes.*

2.2.2 The existences of limit cycles

Theorem: 12 *[19] (first criterion). Let* $(P\ ,\ Q)$ *be a* C^1 *vector field defined in the open subset* U *of* $\mathbb{R}^2$ *,* $(u(t), v(t)$ *a periodic solution of* $(P\ ,\ Q)$ *of period* $T, R : U \longrightarrow \mathbb{R}$ *a* C^1 *map such that* $\int_0^T R(u(t), v(t))dt \neq 0$ *and* $V = V(x, y)$ *a* C^1 *solution of the linear partial differential equation (1.12). Then the closed trajectory*

$$y = ((u(t), v(t)) \in U : t \in [0, T]),$$

is contained in

$$\sum = \{(x, y) \in U : V(x, y) = 0\}.$$

and γ *is not contained in a period annulus of* $(P\ ,\ Q)$*. Moreover, if the vector field* $(P\ ,\ Q)$ *is analytic, then* γ *is a limit cycle.*

Remark: 10 *[19] We note that theorem 12 shows that when we know explicitly the trajectories* $V(x, y) = 0$ *of a vector field* $(P\ ,\ Q)$ *through equation (1.12) for a given* $\mathbb{R}$*, we additionally have information on the periodic solutions of* $(P\ ,\ Q)$*, because if* γ *is a closed trajectory it must satisfy that either* γ *is contained in* $(x, y) \in U : V(x, y) = 0$*, or* $\int_\gamma R dt = 0$*.*

2.2.3 The Poincare-Bendixson Theorem

[39] Suppose $\mathbb{R}$ is the finite region of the plane lying between two simple closed curves D_1 and D_2, and Fis the velocity vector field for the system (1.1). If

1. at each point of D_1 and D_2, the field Fpoints toward the interior of $\mathbb{R}$.

2. R contains no critical points.

then the system (1.1) has a closed trajectory lying inside $\mathbb{R}$.

Example: 15 *[33] Consider the system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.12}$$

to solve this system, we pose :

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Then

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \end{cases}$$

Thus

$$\begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta = -r\sin\theta + r\cos\theta(1 - r^2), \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta = r\cos\theta + r\sin\theta(1 - r^2), \\ \dot{r} = -r\sin\theta + r\cos\theta(1 - r^2) + r\cos\theta + r\sin\theta(1 - r^2), \\ \dot{r} = r(1 - r^2). \end{cases} \tag{2.13}$$

And

$$\begin{cases} r(1-r^2)\sin\theta + r\dot{\theta}\cos\theta = r\cos\theta + rr(1-r^2)\sin\theta, \\ \dot{\theta}(r\cos\theta) = r\cos\theta + rr(1-r^2)\sin\theta - r(1-r^2)\sin\theta, \\ \dot{\theta} = \frac{r\cos\theta}{(r\cos\theta)}, \\ \dot{\theta} = 1. \end{cases} \tag{2.14}$$

had a limit cycle Γ *represented by* $\gamma(t) = (\cos t, \sin t)$. *The Poincaré map for* Γ *can be found by solving this system written in polar coordinates.*

$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = 1. \end{cases}$$

with $r(0) = r_0$ *and* $\theta(0) = \theta_0$. *The first equation can be solved either as a separable differential equation or as a Bernoulli equation. The solution is given by*

$$r(t, r_0) = \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.15}$$

and

$$\theta(t, \theta_0) = t + \theta_0,$$

If $\sum$ *is the ray* $\theta = \theta_0$ *through the origin, then* $\sum$ *is perpendicular to* Γ *and the trajectory through the point* $(r_0,\ \theta_0)\ \in \sum\ \cap \Gamma$*at* $t = 0$ *intersects the ray* $\theta = \theta_0$ *again at* $t = 2\pi$. *It follows that the Poincare map is given by*

$$P(r_0) = \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.16}$$

Clearly $P(1) = 1$ *corresponding to the cycle* Γ *and we see that*

$$P'(r_0) = e^{-4\pi} {r_0}^{-3} \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.17}$$

and that $P'(1) = e^{-4\pi} < 1$.

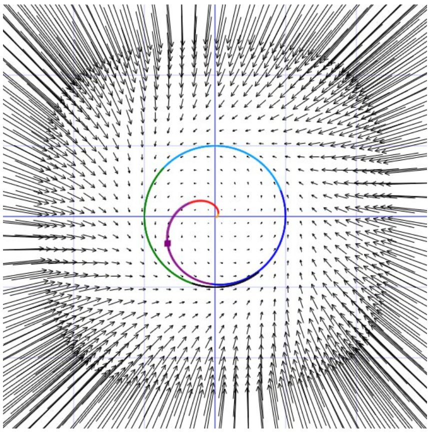


Figure 2.8: Phase portrait of the system 2.14[36].

Example: 16 *[32] Let be the following dynamic system :*

$$\begin{cases} \dot{x} & = y + (1 - x^2 - y^2)x, \\ \dot{y} & = -x + (1 - x^2 - y^2)y. \end{cases} \tag{2.18}$$

The function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ *is a Lyaponov function for this System.*

$$\begin{aligned} V & = x\dot{x} + y\dot{y}, \\ & = (x^2 + y^2)(1 - x^2 - y^2). \end{aligned}$$

If we consider the Domain D *defined by the circle of radius* $\frac{1}{2}$, *and the circle of radius 2,*

$$D = \left\{ (x, y) \backslash \frac{1}{4} \leq x^2 + y^2 \leq 4 \right\}, \tag{2.19}$$

we get an attractive domain for our system. The domain of therefore, we can conclude from the corollary of the Poincare-Bendixon theorem that there is at least one limit cycle completely contained in D. *We consider the domain* D_1 *defined by*

$$D_1 = \left\{ (x, y) \backslash x^2 + y^2 \leq 2 \right\}. \tag{2.20}$$

the domain D contains a single unstable equilibrium point (the origin), so we can conclude from the Poincare-Bendixon theorems that there exists at least one limit cycle.

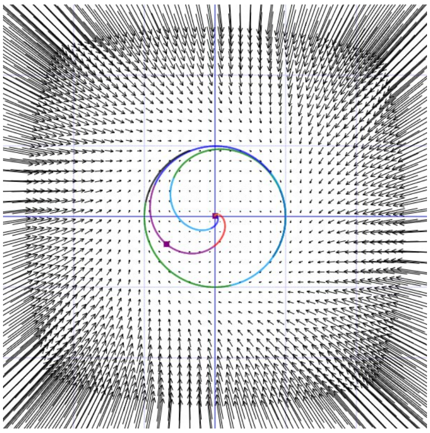


Figure 2.9: The phase portait of system 2.18[36] .

Theorem: 13 *[42] Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let V (x, y) be an inverse integrating factor If γ is a limit cycle of the vector field (P, Q) in the domain of definition of V, then γ is contained in $\sum = \{(x, y) \in U : V(x, y) = 0\}$.*

2.3 Stability of limit cycles

Theorem: 14 *[12] Let E be an open subset of $\mathbb{R}^2$ and suppose that $f \in C^1(E)$. Let $\gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincare map P(s) along a straight line $\sum$ normal to $\Gamma = x \in \mathbb{R}^2 | x = \gamma(t) - \gamma(0), 0 \leq t \leq T$ at $x = 0$ is given by*

$$P'(0) = exp \int_0^T \nabla.f(\gamma(t))dt,$$

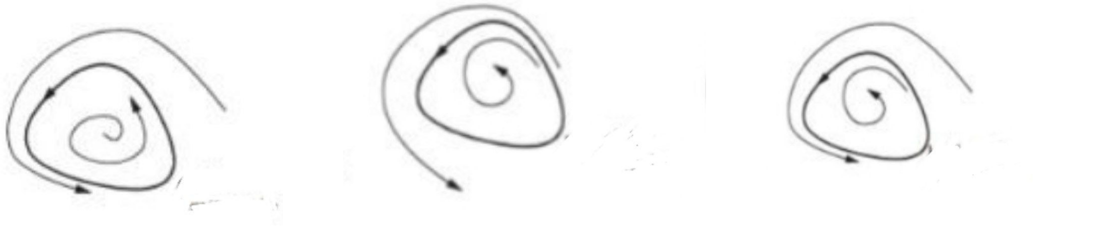
Corollaire: 2 *[24] Under the hypotheses of Theorem 14, the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla.f(\gamma(t))dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \nabla.f(\gamma(t))dt > 0.$$

It may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is zero.



Stable limite cycle. Unstable limite cycle. half stable limite cycle.

Figure 2.10: Classification of limit cycles [43].

Example: 17

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.21}$$

we have $\gamma(t) = (cost, sint)^T, \nabla.f(x, y) = 2 - 4x^2 - 4y^2$ *and*

$$\int_0^{2x} \nabla.f(\gamma(t))dt = \int_0^{2x} (2 - 4cos^2t - 4sin^2t)dt = -4\pi.$$

Thus, with $s = r - 1$*, it follows from Theorem 14 that*

$$p'(0) = e^{-4\pi},$$

which agrees with the result found in the system 2.21 by direct computation. Since $P'(0) < 1$*, the cycle* $\gamma(t)$ *is a stable limit cycle in this example.*

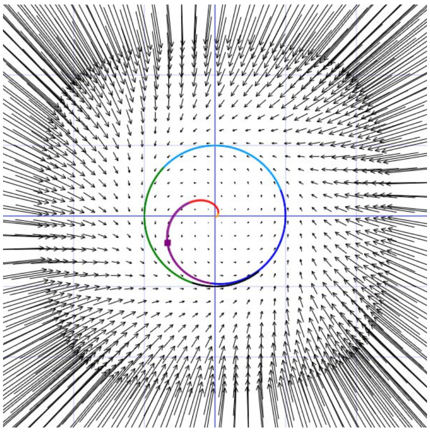


Figure 2.11: stable limit cycle [36] .

Definition : 30 *[24] Let $P(s)$ be the Poincare map for a cycle Γ of a planar analytic system (1.1)and let*

$$d(s) = P(s) - s,$$

be the displacement function. Then if

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0 \quad \text{and} \;\; d^{(k)}(0) \neq 0,$$

Γ is called a multiple limit cycle of multiplicity k. If $k = 1$ then Γ is called a simple limit cycle. We note that $\Gamma = \{x \in \mathbb{R}^2 | x = \gamma(t), 0 \leq t \leq T\}$ is a simple limit cycle of (1.1) if

$$\int_0^T \nabla . f(\gamma(t))dt \neq 0,$$

It can be shown that if k is even then Γ is a semi-stable limit cycle and if k is odd then Γ is a stable limit cycle if $d^{(k)}(0) < 0$ and Γ is an unstable limit cycle if $d^{(k)}(0) > 0$.

Example: 18 *[38]*

$$\begin{cases} \dot{x} = 3x - y - 3x(x^2 + y^2), \\ \dot{y} = x + 3y - 3y(x^2 + y^2), \end{cases}$$

the periodic solution is a stable limit cycle

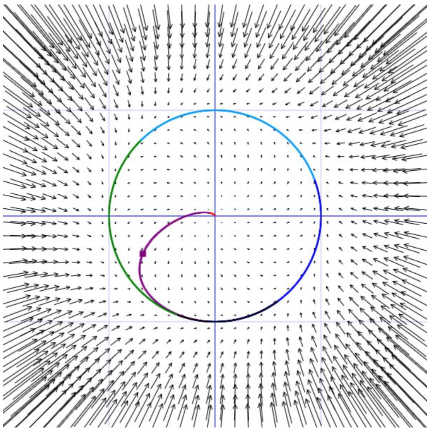


Figure 2.12: stable limit cycle [36] .

Example: 19

$$\begin{cases} \dot{x} = -5x - y + 5x(x^2 + y^2), \\ \dot{y} = x - 5y + 5y(x^2 + y^2), \end{cases}$$

the periodic solution is a unstable limit cycle

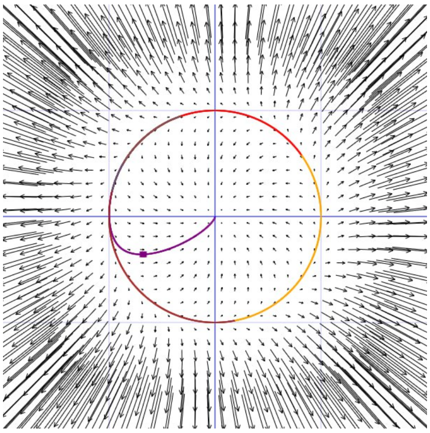


Figure 2.13: unstable limit cycle[36].

Example: 20 *[44] The system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.22}$$

has a limit cycle Γ *represented by*

$$\Gamma(\theta) = (\cos\theta, \sin\theta),$$

because we have

$$div(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \eta(2 - 4x^2 - 4y^2),$$

where $P = -y + \eta x(1 - x^2 - y^2)$ *and* $Q = x + \eta y(1 - x^2 - y^2)$, *and*

$$\int_0^T div(\Gamma(\theta))\,dt = \int_0^T div(\cos\theta, \sin\theta)\ dt,$$
$$= \eta \int_0^{2\pi} (4(\cos\theta)^2 + 4(\sin\theta)^2 - 2)\,d\theta,$$
$$= \eta \int_0^{2\pi} 2\,d\theta.$$

So the cycle $\Gamma(\theta) = (\cos(\theta), \sin(\theta))$ *is an unstable limit cycle if* $\eta > 0$ *and is a stable limit cycle if* $\eta > 0$*.*

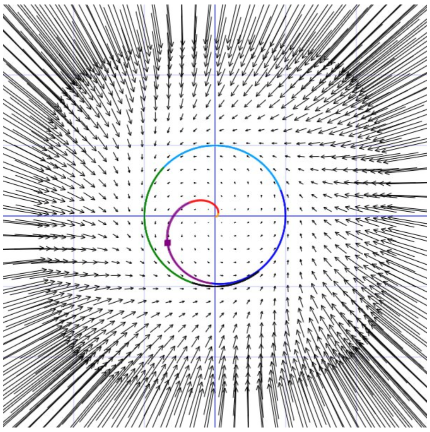


Figure 2.14: limit cycle of system (2.22) [36] .

2.4 The first return map

Probably the most basic tool for studying the stability of periodic orbits is the Poincare map or first return map, defined by Henri Poincare in 1881. The idea of the Poincare map is quite simple: If Γ is a periodic orbit of system (1.1), through the point (x_0, y_0) and Σ is a hyperplane perpendicular to Γ at (x_0, y_0), then for any point (x, y) in Σ sufficiently near (x_0, y_0), the solution of (1.1) through (x, y) at $t = 0$, $\Phi_t(x, y)$ will cross Σ again at a point $\Pi(x, y)$ near (x_0, y_0). The mapping $(x, y) \to \Pi(x, y)$ is called the Poincare map.

The next theorem establishes the existence and continuity of the Poincare map $\Pi(x, y)$ and of its first derivative $D\Pi(x, y)$.

Theorem: 15 *[24] Let E be an open subset of $\mathbb{R}^2$ and let $(P, Q) \in C^1(E)$. Suppose that $\Phi_t(x_0, y_0)$ is a periodic solution of (1.1) of period T and that the cycle.*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\},$$

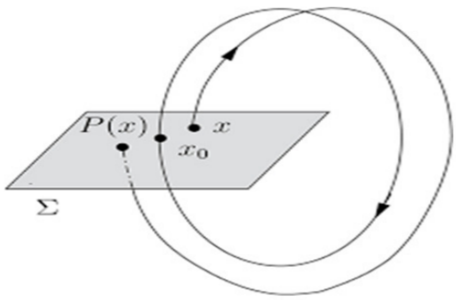


Figure 2.15: The poincare map [24].

is contained in E. Let Σ be the hyperplane orthogonal to Γ at (x_0, y_0), i.e.; let

$$\Sigma = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0, y - y_0) \cdot (P(x_0, y_0), Q(x_0, y_0)) = 0\}.$$

Then there is a $\delta > 0$ and a unique function $\tau(x, y)$, defined and continuously differentiable for $(x, y) \in N_\delta(x_0, y_0)$, such that

$$\tau(x_0, y_0) = T,$$

and

$$\Phi_{\tau(x,y)}(x, y) \in \Sigma,$$

for all $(x, y) \in N_\delta(x_0, y_0)$.

Definition : 31 *[33] Let Γ, Σ, δ, and $\tau(x, y)$ be defined as in Theorem (15). Then for*

$(x, y) \in N_\delta(x_0, y_0) \cap \Sigma$, *the function*

$$\Pi(x, y) = \Phi_{\tau(x,y)}(x, y),$$

is called the Poincare map for Γ *at* (x_0, y_0).

The following theorem gives the formula of $\Pi_0(0, 0)$.

Example: 21 *[24] Let the following system of equations:*

$$\begin{cases} \dot{x} = -y + x(4 - x^2 - y^2), \\ \dot{y} = x + y(4 - x^2 - y^2). \end{cases} \tag{1.12}$$

Note that $(0, 0)$ *is the only critical point of (1.12). In polar coordinates, the system (1.12) becomes:*

$$\begin{cases} \dot{r} = r(4 - r^2), \\ \dot{\theta} = 1. \end{cases} \tag{1.13}$$

The differential system (1.13) is equivalent to the following Bernoulli differential equation

$$\frac{dr}{d\theta} = 4r - r^3.$$

The general solution is

$$r(\theta) = \left[\frac{1}{4} + ce^{-8\theta}\right]^{-\frac{1}{2}}.$$

A solution of (1.13) satisfying the initial condition $r(0) = r_0$ *is given by*

$$r(\theta, r_0) = \left[\frac{1}{4} + \left[\frac{1}{r_0^2} - \frac{1}{4}\right] e^{-8\theta}\right]^{-\frac{1}{2}}.$$

For $\theta = 2\pi$, it follows that the Poincaré first return map is given by:

$$\Pi(r_0) = r(2\pi, r_0) = \left[\frac{1}{4} + \left[\frac{1}{r_0^2} - \frac{1}{4}\right] e^{-16\pi}\right]^{-\frac{1}{2}}.$$

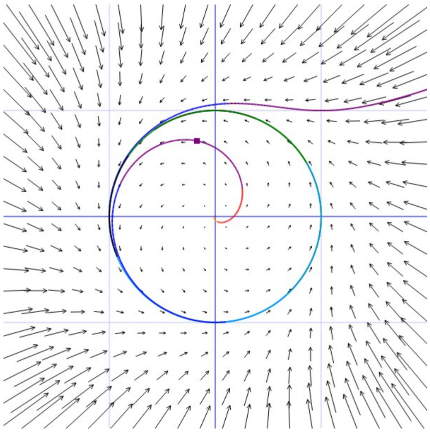


Figure 2.16: stable limit cycle [36] .

Definition : 32 *[24] A fixed point of the application Π is a point (x, y) such that $\Pi(x, y) = (x, y)$. It corresponds to a periodic orbit of the system (1.1).*

2.5 Stability of poincare map

Theorem: 16 *[33] Let $\Gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincare map $\Pi(s)$ along a straight line Σ normal to $\Gamma = \{(x, y) \in \mathbb{R}^2 : (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\}$ at $(x, y) = (0, 0)$ is given by*

$$\Pi'(0) = \exp \int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t))) \, dt.$$

Corollaire: 3 *[18] Under the hypotheses of (32), the periodic solution $\Gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t))) \, dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t)))\, dt > 0.$$

Example: 22 *[24] The system (1.12) has a limit cycle Γ_1 represented by*

$$\Gamma_1(\theta) = (2\cos\theta, 2\sin\theta),$$

we have $\nabla \cdot (P(x, y), Q(x, y)) = 8 - 4x^2 - 4y^2$ and

$$\begin{aligned} \textstyle\int_0^{2\pi} \nabla \cdot (P(\Gamma_1(\theta)), Q(\Gamma_1(\theta)))\, d\theta \ &= \textstyle\int_0^{2\pi} 8 - 4((2\cos(\theta))^2 + (2\sin(\theta))^2)\, d\theta, \\ &= -16\pi < 0. \end{aligned} \tag{2.23}$$

Then, the cycle Γ_1 is a stable limit cycle.

2.6 Hyperbolic limit cycle

Definition : 33 *[33] A limit cycle $\Gamma = (x(t), y(t)), t \in [0, T]$ is a T-periodic solution isolated with respect to all other possible periodic solutions of the system.*

A T-periodic solution Γ is a hyperbolic limit cycle if $\int_0^T div(\Gamma)dt$ is different from zero.

Example: 23 *[33] The system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.24}$$

coordinates the previous system becomes

$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = 1, \end{cases} \tag{2.25}$$

polar coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \tag{2.26}$$

so

$$\dot{r} = 0 \Leftrightarrow r = 0 \;\; or \;\; r = 1,$$

for

$$\gamma(t) = (\cos(t), \sin(t)),$$

$$\begin{aligned} \int_0^{2\pi} div(\gamma(t))dt \quad &= \int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)(\cos(t), \sin(t))\, dt, \\ &= \int_0^{2\pi} (1 - 3\cos^2(t) - \sin^2(t)) + (1 - \cos^2(t) - 3\sin^2(t))dt, \\ &= \int_0^{2\pi} (1 - 4\cos^2(t) - 4\sin^2(t))dt, \\ &= \int_0^{2\pi} -2dt = -4\pi \neq 0. \end{aligned} \tag{2.27}$$

So, system (1.12) has a hyperbolic limit cycle $\gamma(t) = (\cos(t), \sin(t))$*, which is stable.*

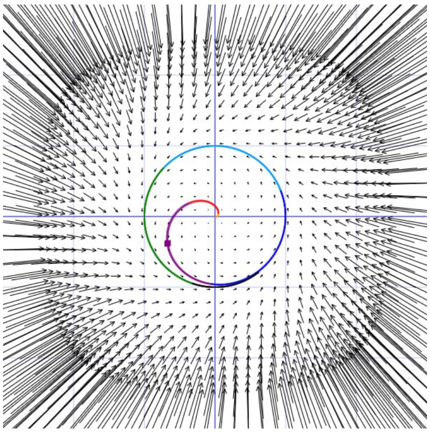


Figure 2.17: Hyperbolic limit cycle stable [36].

CHAPTER 3

ON THE EXPLICIT NON-ALGABRAIC LIMIT CYCLE OF CLASS OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEM

3.1 Introduction

In this chapter, we present studies conducted by A. Gasull and all in the article entitled "Explicit Non-Algebraic Limit Cycles for Polynomial Systems." published by :ELSEVIER 3 January 2006, They considered a system of the form:

$$\begin{cases} \dot{x} = P_n(x,y) + xR_m(x,y), \\ \dot{y} = Q_n(x,y) + yR_m(x,y), \end{cases} \tag{3.1}$$

where $P_n(x,y)$, $Q_n(x,y)$, and $R_m(x,y)$ are homogeneous polynomials of degrees n, n, and m respectively, with $n \leq m$.

They proved that this system has at most one limit cycle, and when it exists, it can be explicitly found and expressed in terms of quadratures. They studied a particular case where $n = 3$ and $m = 4$.

They demonstrate that this quintic polynomial system has an explicit limit cycle which is not algebraic. To their knowledge, there are no such examples in the existing literature.

The method they introduce to prove that this limit cycle is not algebraic can also be used to detect algebraic solutions for other families of polynomial vector fields or to demonstrate the absence of such solutions.

A function $F(x, y)$ is an algebraic solution of a real polynomial system $(\dot{x}, \dot{y}) = (X(x, y), Y(x, y))$ if:

$$\frac{\partial F(x,y)}{\partial x}X(x,y) + \frac{\partial F(x,y)}{\partial y}Y(x,y) = K(x,y)F(x,y). \tag{3.2}$$

Theorem: 17 *The planar differential system*

$$\begin{cases} \dot{x} = -(x-y)(x^2 - xy + y^2) + x(2x^4 + 2x^2y^2 + y^4), \\ \dot{y} = -(x+y)(2x^2 - xy + 2y^2) + y(2x^4 + 2x^2y^2 + y^4), \end{cases} \tag{3.3}$$

has exactly one limit cycle which is hyperbolic and non-algebraic. In polar coordinates, this limit cycle is

$$r = e^{(\frac{3}{2})\psi(\theta)-\theta}\left(a + \int_0^\theta \frac{\cos^4(s)+1}{cos^2(s)+1}e^{3\psi(s)-2s}ds\right)^{-\frac{1}{2}},$$

where $\psi(\theta) = 2\int_0^\theta \frac{1+\tan^2 s}{2+\tan^2 s}ds$ *and*

$$a = \frac{2e^{4\pi-6\sqrt{2}\pi}}{1-e^{4\pi-6\sqrt{2}\pi}}\int_0^{2\pi}\frac{\cos^4(s)+1}{\cos^2(s)+1}e^{3\psi(s)-2s}ds \approx 1.19903,$$

3.2 Systems with explicit limit cycles

Theorem: 18 *System (3.1) has at most one limit cycle, when it exists, it is hyperbolic and in polar coordinates it can be written as*

$$r = \left(\exp\left[\int_0^\theta \frac{f(s)}{g(s)}ds\right]\left[a + \int_0^\theta \frac{h(s)}{g(s)}\exp\left(-\int_0^s \frac{f(w)}{g(w)}dw\right)ds\right]\right)^{\frac{1}{n-m-1}}, \qquad (3.4)$$

where $a = \frac{AB}{1-A}$,

$$A = \exp\left(\int_0^{2\pi} \frac{f(s)}{g(s)}ds\right) \text{ and } B = \int_0^{2\pi} \frac{h(s)}{g(s)}\exp\left(-\int_0^s \frac{f(w)}{g(w)}dw\right)ds. \qquad (3.5)$$

Lemma: 1 *The system (3.1). In polar coordinates, it can be written as*

$$\begin{cases} \dot{r} = f(\theta)r^n + h(\theta)r^{m+1}, \\ \dot{\theta} = g(\theta)r^{n-1}, \end{cases} \qquad (3.6)$$

where

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$

$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta),$$

$$h(\theta) = R_m(\cos\theta, \sin\theta).$$

proof: 1

$$\begin{cases} r^2 = x^2 + y^2, \\ x = r\cos\theta, \\ y = r\sin\theta, \\ \tan\theta = \frac{x}{y}, \end{cases} \qquad (3.7)$$

$$\begin{cases} \dot{x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial t}, \\ \dot{y} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial t}. \end{cases} \qquad (3.8)$$

Lemma: 2 *System (3.1) has* $F(x,y) = yP_n(x,y) - xQ_n(x,y)$ *as an algebraic solution with cofactor* $(n+1)R_m + div(P_n, Q_n)$*. Notice that it is formed by a product of (complex or real) invariant straight lines through the origin.*

proof: 2 *By Using the homogeneity of* P_n *and* Q_n*, we know from Euler's formula that*

$$nP_n(x,y) = x\frac{\partial P_n}{\partial x} + y\frac{\partial P_n}{\partial y} \quad \textit{and} \quad nQ_n(x,y) = x\frac{\partial Q_n}{\partial x} + y\frac{\partial Q_n}{\partial y}.$$

Thus,

$$\begin{aligned}
&\left(y\tfrac{\partial P_n}{\partial x} - Q_n - x\tfrac{\partial Q_n}{\partial x}\right)(P_n + xR_m) + \left(P_n + y\tfrac{\partial P_n}{\partial y} - x\tfrac{\partial Q_n}{\partial y}\right)(Q_n + yR_m),\\
&= yP_n\tfrac{\partial P_n}{\partial x} + xyR_m\tfrac{\partial P_n}{\partial x} - Q_nP_n - xR_mQ_n - xP_n\tfrac{\partial Q_n}{\partial x} - x\tfrac{\partial Q_n}{\partial x}.xR_m + P_nQ_n\\
&+yR_mP_n + y\tfrac{\partial P_n}{\partial y}.Q_n + y\tfrac{\partial P_n}{\partial y}.yR_m - x\tfrac{\partial Q_n}{\partial y}Q_n - xy\tfrac{\partial Q_n}{\partial y}R_my,\\
&= R_m\left(xy\tfrac{\partial P_n}{\partial x} - xQ_n - x^2\tfrac{\partial Q_n}{\partial x} + yP_n + y^2\tfrac{\partial P_n}{\partial y} - xy\tfrac{\partial Q_n}{\partial y}\right)\\
&+yP_n\tfrac{\partial P_n}{\partial x} - Q_nP_n - x\tfrac{\partial Q_n}{\partial x}P_n + P_nQ_n + y\tfrac{\partial P_n}{\partial y}Q_n - x\tfrac{\partial Q_n}{\partial y}Q_n,\\
&= R_m\left(y(x\tfrac{\partial P_n}{\partial x} + y\tfrac{\partial P_n}{\partial y}) - xQ_n - x^2\tfrac{\partial Q_n}{\partial x} + yP_n + y^2\tfrac{\partial P_n}{\partial y} - xy\tfrac{\partial Q_n}{\partial y}\right)\\
&+\left(p_ny\tfrac{\partial P_n}{\partial x} - xP_n\tfrac{\partial Q_n}{\partial x} + y\tfrac{\partial P_n}{\partial y}Q_n - x\tfrac{\partial Q_n}{\partial y}Q_n\right),\\
&= R_m(ynP_n - xnQ_n - xQ_n + yP_n) + \tfrac{\partial P_n}{\partial x}(yp_n - xQ_n) + \tfrac{\partial Q_n}{\partial y}(yP_n + -xQ_n)\\
&+Q_n\left(x\tfrac{\partial P_n}{\partial x} + y\tfrac{\partial P_n}{\partial y}\right) - P_n\left(x\tfrac{\partial Q_n}{\partial x} + y\tfrac{\partial Q_n}{\partial y}\right),\\
&= (1+n)R_mF + \left(\tfrac{\partial P_n}{\partial x} + \tfrac{\partial Q_n}{\partial y}\right)F + nP_nQ_n - nP_nQ_n,\\
&= \left((1+n)R_m + \left(\tfrac{\partial P_n}{\partial x} + \tfrac{\partial Q_n}{\partial y}\right)\right)F.
\end{aligned} \tag{3.9}$$

Lemma: 3 *Let* $F(x,y)$ *be an algebraic solution of degree* ℓ *of the system*

$$\begin{cases} \dot{x} = P(x,y) + xR_m(x,y), \\ \dot{y} = Q(x,y) + yR_m(x,y). \end{cases} \tag{3.10}$$

Where $P(x,y)$ and $Q(x,y)$ are polynomials of degree less than or equal to n and $R_m(x,y)$ is a homogeneous polynomial of degree m, with $n \leq m$. Thus the homogeneous part of maximum degree of its cofactor is $dR_m(x,y)$.

proof: 3 *Since F is an algebraic solution of system (3.1), we know that*

$$\frac{\partial F}{\partial x}(P + xR_m) + \frac{\partial F}{\partial y}(Q + yR_m) = KF,$$

where K is the cofactor of F. Denote by $F_m(x,y)$ and by $K_m(x,y)$ the homogeneous parts of maximum degree of $F(x,y)$ and $K(x,y)$, respectively. By using the homogeneity of F_m, we know, from Euler's formula, that

$$dF_m = x\left(\frac{\partial F_m}{\partial x}\right) + y\left(\frac{\partial F_m}{\partial y}\right),$$

By equating the higher degree terms in the above equation, we obtain

$$\frac{\partial F_m}{\partial x}R_m = \frac{\partial F_m}{\partial x}xR_m + \frac{\partial F_m}{\partial y}yR_m = K_m F_m.$$

Thus, $K_m(x,y) = R_m(x,y)$ as we wanted to prove.

The next lemma, which has an elementary proof, collects some easy remarks on the structure of the cofactors.

Lemma: 4 *Consider system (3.1) and define*

$$V(x,y) = \left(r^{m-n+1}G(\theta,a) - 1\right)(yP_n(x,y) - xQ_n(x,y)), \tag{3.11}$$

where $G(\theta,\rho_0)$ is the function given in (3.5) and $\rho_0 = a$ is the value for which this function is 2π-periodic. Then, whenever it is defined, $1/V(x,y)$ is an integrating factor *of the system and we call $V(x,y)$ an inverse integrating factor.*

proof: 4 *We use the following formula: let F_1 and F_2 be two solutions of $(\dot{x},\dot{y}) = (X(x,y), Y(x,y))$ with cofactors K_1 and K_2, respectively. Thus,*

$$div(X,Y)\frac{1}{F_1F_2} = \frac{1}{F_1F_2}(div(X,Y) - (K_1 + K_2)).$$

*We remark that the above formula, taking a denominator of the form $\Pi F_i^{\alpha_i}$, for some real or complex constants α_i, is indeed the key point of the Darboux theory of integrability, see [28]. Take $F_1(x,y) = yP_n(x,y) - xQ_n(x,y)$ and $F_2(x,y) = r^{m-n+1}G(\alpha,a) - 1$. By using Lemma 2 and Remark **??**, we know that their associated cofactors are*

$$K_1(x,y) = (n+1)R_m(x,y) + div(P_n(x,y), Q_n(x,y)),$$

and

$$K_2(x,y) = (m-n+1)R_m(x,y),$$

respectively. On the other hand, taking the vector field associated with system (3.1) we get

$$div(X,Y) = div(P_n,Q_n) + 2R_m + x\frac{\partial R_m}{\partial x} + y\frac{\partial R_m}{\partial y} = div(P_n,Q_n) + (2+m)R_m,$$

where we have used again Euler's formula. Collecting all the above results we get $div\left(\frac{(X,Y)}{F_1F_2}\right) \equiv 0$.

Remark: 11 *(i) When we apply the above lemma to systems (3.3) and (1) we get both non-algebraic and algebraic inverse integrating factors.*

(ii) In [20] it is proved that when $1/V(x,y)$ is an integrating factor of $\dot{x} = X(x,y)$ and

$\dot{y} = Y(x,y)$ and $V(x,y)$ is defined in the whole plane, all the limit cycles of the system are included in the curve $V(x,y) = 0$. This is the case of system (3.1), the limit cycle, whenever it exists, is given by the expression ,

$F_2(x,y) = r^{m-n+1}G(\theta, a) - 1 = 0.$

(iii) The equality div $\left(\frac{(X,Y)}{F_1F_2}\right) \equiv 0$ *also holds when instead of*

$F_2(x,y) = r^{m-n+1}G(\theta,\rho_0)|_{\rho_0=a} - 1$ *we take a different value of ρ_0, but in this case F_2 is indeed a multivalued function and the result of [20] cannot be applied.*

3.3 Algebraic limit cycles of subfamily differential system

The existence of limit cycles for a subfamily of system (3.1) has been studied in [16]. Here we prove that the limit cycle found there is algebraic .

Proposition: 5 *Consider the system*

$$\begin{cases} \dot{x} = -y + x(a + R_m(x,y)), \\ \dot{y} = x + y(a + R_m(x,y)), \end{cases} \tag{3.12}$$

where a is a real parameter and $R_m(x,y)$ is a homogeneous polynomial of degree m. Then, it has only two algebraic invariant curves $x^2 + y^2$ and $H(x,y) = G_m(x,y) - 1$, where $G(\theta) = G_m(\cos\theta, \sin\theta)$ satisfies $G' + maG + mR_m(\cos\theta,\sin\theta) = 0$. Furthermore, when the limit cycle exists, m is even and $H(x,y)$ contains a real oval which is the limit cycle of the system.

proof: 5 *From Lemma 2 we can see that $x^2 + y^2$ is an algebraic solution with cofactor $K(x,y) = 2a + 2R_m(x,y)$. Now, we study other possible algebraic solutions.*

Write the Fourier expansion of R_m:

$$R_m(\cos\theta, \sin\theta) = \sum_{k=-m}^{m} c_k e^{ki\theta}, \text{ where } c_k = c_{-k} \in \mathbb{C}, \text{ and } c_k = 0 \text{ when } k \not\equiv m \pmod 2,$$

Note that the solution of (3.12) starting at $r = r_0$ when $\theta = 0$ can be written as $r = r(\theta, r_0)$. Following the steps of the proof of Theorem 2.1 we obtain that

$$r^{-m} = r_0^{-m} + m \sum_{k=-m}^{m} \frac{c_k}{ki + ma} e^{-ma\theta} + G_m(\cos\theta, \sin\theta),$$

or

$$r_0^{-m} + m \sum_{k=-m}^{m} \frac{c_k}{ki + ma} r^m e^{-ma\theta} + G_m(r\cos\theta, r\sin\theta) = 1,$$

where $G_m(x, y)$ is the homogeneous polynomial of degree m defined by its Fourier expansion as

$$G_m(\cos\theta, \sin\theta) := -m \sum_{k=-m}^{m} \frac{c_k}{ki + ma} e^{ki\theta},$$

and $G(\theta) = G_m(\cos\theta, \sin\theta)$ satisfies $G' + maG + mR_m(\cos\theta, \sin\theta) = 0$.

By using the above expression we get that the only algebraic solution of system (3.12) is the one that satisfies

$$r_0^{-m} + m \sum_{k=-m}^{m} \frac{c_k}{ki + ma} = 0.$$

Moreover, it is easy to check that the cofactor of this algebraic solution, $H(x, y) = G_m(x, y) - 1$, is $K(x, y) = mR_m(x, y)$, see also the proof of Lemma 4. Notice that a necessary and sufficient condition for the existence of a real algebraic solution is that

$$\sum_{k=-m}^{m} \frac{c_k}{ki + ma} < 0.$$

Finally, the limit cycle exists when $G_m(\cos\theta, \sin\theta) > 0$ for $\theta \in [0, 2\pi]$. This can happen *only when m is even, see also [16].*

3.4 Applications

Example: 24 *Consider the system*

$$\begin{cases} x' = -(x-y)(x^2-xy+y^2)+x(2x^4+2x^2y^2+y^4), \\ y' = -(x+y)(2x^2-xy+2y^2)+y(2x^4+2x^2y^2+y^4). \end{cases} \tag{3.13}$$

and in polar coordinates, it can be written as:

$$\begin{cases} \dot{r}\cos\theta - \dot{\theta}r\sin\theta & = -(r\cos\theta - r\sin\theta)(r^2 - r^2\sin\theta), \\ & +r\cos\theta(2r^4\cos^4\theta + 2r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta), \\ \dot{r}\sin\theta + \dot{\theta}r\cos\theta & = -(r\cos\theta - r\sin\theta)(2r^2 - r^2\sin\theta), \\ & +r\cos\theta(2r^4\cos^4\theta + 2r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta), \end{cases} \tag{3.14}$$

$$\begin{aligned} \dot{r} \quad & = -(r\cos^2\theta - r\cos\theta\sin\theta)(r^2 - r^2\cos\theta\sin\theta), \\ & +r\cos^2\theta(2r^4\cos^4\theta + 2r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta), \\ & -(r\cos\theta\sin\theta + r\sin^2\theta)(2r^2 - r^2\cos\theta\sin\theta), \\ & +r\sin^2\theta(2r^4\cos^4\theta + 2r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta), \end{aligned} \tag{3.15}$$

$$\begin{aligned} \dot{r} \quad & = r^5(2\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta) + r^3(-\cos^2\theta - 2\sin^2\theta), \\ & = r^5\left[\cos^2\theta(2\cos^2\theta + \sin^2\theta) + sin^2\theta(\cos^2\theta + \sin^2\theta)\right] + r^3(-\cos^2\theta - 2\sin^2\theta), \\ & = r^5\left[\cos^2\theta(1+\cos^2\theta) + \sin^2\theta\right] + r^3\left[-\cos^2\theta - 2(1+\cos^2\theta)\right], \\ & = r^5(\cos^4\theta + 1) + r^3(\cos^2\theta - 2). \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
\dot{r}\cos\theta - \dot{\theta} r \sin\theta &= -(r\cos\theta - r\sin\theta)(r^2 - r^2\sin\theta\cos\theta),\\
&+r\cos\theta(2r^4\cos^4\theta + 2r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta),\\
&= r^5\cos\theta(\cos^4\theta+1) - r\cos\theta + r^3\cos^2\theta\sin\theta + r^3\sin\theta - r^3\cos\theta\sin^2\theta,\\
-\dot{\theta} r\sin\theta &= r^5\cos\theta(\cos^4\theta+1) - r^5\cos\theta(\cos^4\theta+1) - r^3\cos\theta(\cos^2\theta-2),\\
&-r^3\cos\theta + r^3\cos^2\theta\sin\theta + r^3\sin\theta - r^3\cos\theta\sin^2\theta,\\
&= r^3\left[(1-\cos\theta\sin\theta)(-\cos\theta+\sin\theta) - \cos\theta(-\sin^2\theta-1)\right],\\
&= r^2(-\cos\theta + \cos^2\theta\sin\theta + \sin\theta - \cos\theta\sin^2\theta + \cos\theta\sin^2\theta + \cos\theta),\\
\dot{\theta} &= -r^2(\cos^2\theta+1).
\end{aligned}
\tag{3.17}
$$

Eliminating the time betwen the variables r and θ we obtain the differential equation

$$\dot{r} = \tfrac{dr}{dt} \quad \dot{\theta} = \tfrac{d\theta}{dt},$$

$$
\begin{aligned}
&\tfrac{dr}{dt} = \tfrac{(\cos^4\theta+1)r^5}{-(\cos^2\theta+1)r^2} - \tfrac{(\cos^2\theta-2)r^3}{(\cos^2\theta+1)r^2},\\
&= -\tfrac{(\cos^4\theta+1)}{(\cos^2\theta+1)}r^3 - \tfrac{(\cos^2\theta-2)}{(\cos^2\theta+1)}r.
\end{aligned}
\tag{3.18}
$$

we introduce the change of variable $r = \frac{1}{\sqrt{\rho}}$ obtainis .

$$\rho = \frac{1}{r^2} \quad , \quad \frac{d\rho}{dr} = \frac{-2}{r^3},$$

$$\tfrac{d\rho}{d\theta} = \tfrac{d\rho}{dr}\tfrac{dr}{d\theta} \quad = -\tfrac{(\cos^4\theta+1)}{(\cos^2\theta+1)}r^3\tfrac{-2}{r^3} - \tfrac{(\cos^2\theta-2)}{(\cos^2\theta+1)}r\tfrac{-2}{r^3}, \tag{3.19}$$

we obtain the linear equation

$$\frac{d\rho}{d\theta} = 2\frac{(\cos^4\theta+1)}{(\cos^2\theta+1)} + 2\frac{(\cos^2\theta-2)}{(\cos^2\theta+1)} \times \rho, \tag{3.20}$$

Notice that system (3.13) has a periodic orbit if and only if Eq. (3.20) has a strictly positive 2π periodic solution. The solution satisfying that $\rho = \rho_0 > 0$ when $\theta = 0$ is:

$$\rho(\theta;\rho_0) = e^{-3\Phi(\theta)+2\theta}\left(\rho_0 + 2\int_0^\theta \frac{\cos^4(s)+1}{\cos^2(s)+1} e^{3\Phi(s)-2s} ds\right) > 0, \; = \left[e^{\psi(\theta)}\right]\left[\rho_0 - \int_0^\theta \varphi(s) e^{-\varphi(s)} ds\right]. \tag{3.21}$$

Hence

$$\varphi(s) = \frac{1+\cos^4 s}{1+\cos^2 s}, \tag{3.22}$$

$$\varphi(s) = -2\int_0^s \frac{-\cos^2 z}{1+\cos^2 z} dz, \tag{3.23}$$

The initial condition of the limit cycle is given by the equation $\rho(2\pi) = \rho(0) = \rho_0^*$. Hence,

$$\rho_0^* = \frac{e^{2(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)}}{1-e^{2(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)}},$$

$$\int_0^{2\pi} \frac{\cos^4 s+1}{\cos^2 \theta+1} e^{-(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)},$$

$$2\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds < 2\int_0^{2\pi} \frac{-1}{\cos^2 s+1} ds < 0,$$

Thus

$$0 < e^{2(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)} < 1,$$

$$\rho_0 = \frac{e^{2(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)}}{1-e^{2(\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)}} > 0,$$

on the other side

$$\int_0^{2\pi} \frac{\cos^4 s+1}{\cos^2 \theta+1} e^{(-2\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)} = 2\int_0^{2\pi} \varphi(s) e^{-\psi(s)} ds > 0,$$

$$\frac{d\pi}{d\rho_0}(\rho_0) = e^{(2\int_0^{2\pi} \frac{\cos^2 s-2}{\cos^2 s+1} ds)} < 1,$$

$$\rho_0^* = \frac{2e^{4\pi-6\sqrt{2}\pi}}{1-e^{4\pi-6\sqrt{2}\pi}} \int_0^{2\pi} \frac{\cos^4(s)+1}{\cos^2(s)+1} e^{3\Phi(s)-2s} ds > 0. \tag{3.24}$$

This value can be computed numerically, giving $\rho_0^* \approx 1.1990$. The intersection of the limit cycle with the OX^+ axis is the point having $r_0^* = \frac{1}{\sqrt{\rho_0^*}} \approx 0.9132$. Since the Poincaré return map is $\Pi(\rho_0) = \rho(2\pi; \rho_0)$, we have $\Pi'(\rho_0) = e^{(4-6\sqrt{2})\pi} < 1$ for all ρ_0 and $\dot{\theta} < 0$. Thus, we conclude that the limit cycle of system (3.13) is hyperbolic and unstable.

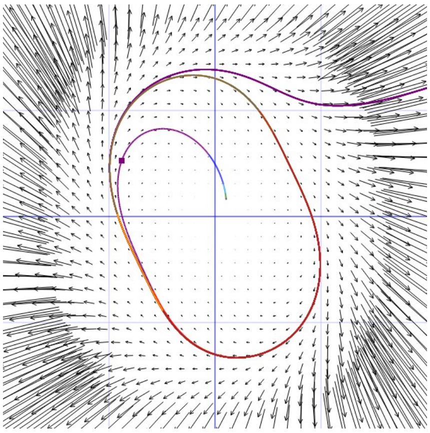


Figure 3.1: The phase portait of system 3.13[36].

3.5 A method for studying the existence of algebraic solutions

Let $F(x,y)$, $K(x,y)$, $X(x,y)$, and $Y(x,y)$ be real analytic functions such that

$$\frac{\partial F(x,y)}{\partial x} X(x,y) + \frac{\partial F(x,y)}{\partial y} Y(x,y) = K(x,y)F(x,y). \tag{3.25}$$

Thus, it is clear that the set $\{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$ is formed by solutions of the system

$$\begin{cases} \dot{x} = X(x,y), \\ \dot{y} = Y(x,y). \end{cases} \tag{3.26}$$

Fixing an analytic solution of (3.26) of the form $y = \psi(x)$, we can consider the following Taylor expansions in z:

$$F(x, z + \alpha(x)) = F_0(x) + zF_1(x) + z^2F_2(x) + \cdots,$$

$$K(x, z + \alpha(x)) = K_0(x) + zK_1(x) + z^2K_2(x) + \cdots,$$

$$X(x, z + \alpha(x)) = X_0(x) + zX_1(x) + z^2X_2(x) + \cdots,$$

$$Y(x, z + \alpha(x)) = Y_0(x) + zY_1(x) + z^2Y_2(x) + \cdots.$$

Notice that $\alpha'(x) = \frac{Y_0(x)}{X_0(x)}$. Then, Eq. (3.25) can be written as

$$\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\left(X^{k-i}(x)F_i'(x) + \left(iY^{k-i+1} - i\alpha'(x)X^{k-i+1} - K^{k-i}\right)F_i(x)\right)\right)z^k = 0.$$

The functions $F_k(x)$ can be obtained recurrently from the above relation by solving the linear differential equations in $F_k(x)$, obtained by vanishing each coefficient in z^k. In particular, for $k = 0$ and 1, we get

$$X_0(x)F_0'(x) - K_0(x)F_0(x) = 0,$$

$$X_0(x)F_1'(x) + (Y_1(x) - \alpha'(x)X_1(x) - K_0(x))\,F_1(x) + X_1(x)F_0'(x) - K_1(x)F_0(x) = 0.$$

We obtain $F_0(x) = C_0 \exp\left(\int_0^x \frac{K_0(s)}{X_0(s)}ds\right)$, where C_0 is an arbitrary constant, and similarly we could get $F_1(x)$.

When $\alpha(x)$ is a polynomial (resp. rational) function and $F(x,y)$, $K(x,y)$, $X(x,y)$, and $Y(x,y)$ are polynomials with real or complex coefficients, the linear differential equations for each $F_k(x)$ described in the above algorithm give us a collection of necessary conditions for the existence of an algebraic solution $F(x,y)$. The conditions are that, for each k, the functions $F_k(x)$ must be polynomials (resp. rational functions). For instance, for $k = 0$, the first necessary condition is that the primitive of the rational function

$$\frac{K_0(x)}{X_0(x)} = \frac{K(x, \alpha(x))}{X(x, \alpha(x))},$$

must be a linear combination of logarithms of polynomials. Furthermore, the coefficients of the logarithms have to be natural (resp. integer) numbers.

The necessary conditions obtained for the existence of algebraic solutions restrict the possible cofactors of F. These restrictions give the key for searching the possible algebraic solutions of system (3.26), see Remark 13. As we will see, in our case we only need to apply the described method for $k = 0$, but we remark that in other situations, by using it for bigger k, it can give more information about the existence or nonexistence of algebraic solutions.

Remark: 12 *Notice that the above method can only be applied when the candidate $F(x, y)$ to be an algebraic solution of system (3.26) does not contain the factor $y - \alpha(x)$.*

Remark: 13 *Assume that system (3.26) is fixed and it is polynomial. Notice that Eq. (3.25) that gives the possible set of algebraic solutions of system (3.26) is equivalent to a set of quadratic equations where the unknowns are the coefficients of F and the coefficients of K. In general, it is very hard to solve this system of equations, even by using algebraic manipulators. On the other hand, the method developed in this section imposes restrictions on the cofactor K for the existence of F. Ideally, if K is totally known, the system to be solved will be linear and so the problem of knowing the existence or not of algebraic solutions of a given degree would be a much easier task. In any case, any information on K makes the problem simpler. Another method to impose conditions on K is developed in [9].*

3.6 A non-algebraic limit cycle

This section is devoted to proving that the limit cycle of system (3.3) is not algebraic. We will prove that the only algebraic solutions of the system are the ones given in Lemma 2. The algebraic solution given by this lemma is $yP_n - xQ_n = (2x^2+y^2)(x^2+y^2)$. Concretely, the curves $x^2+y^2=0$ and $2x^2+y^2=0$ have the cofactors $2(2x^4+2x^2y^2+y^4-x^2-2y^2)$ and $2(2x^4+2x^2y^2+y^4-x^2+xy-2y^2)$, respectively. These two curves coincide with the four complex lines $y=\pm ix$ and $y=\pm\sqrt{2}ix$.

As we will see, the first step ($k=0$) applied to each one of the four complex lines, $y=\pm ix$ and $y=\pm\sqrt{2}ix$, will give enough restrictions to prove that the only algebraic solutions of system (3.3) are the ones described above.

Assume that the differential system has a real or complex algebraic solution F and that it does not contain any of the given four lines as a factor. By using Lemma 4, it is not restrictive to assume that F is real and that its cofactor is an even function, i.e., $K(-x,-y)=K(x,y)$. Since the degree of the vector field (4) is 5, we know that the degree of $K(x,y)$ is at most 4. By the above restrictions on $K(x,y)$ and by using also Lemma 3, we can write it as the real polynomial ...

$$K(x,y) = a_{00} + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ell(2x^4+2x^2y^2+y^4),$$

where ℓ is the degree of the corresponding algebraic curve $F(x,y)=0$.

We apply the first step of our method, i.e., we take $k=0$. By considering the cases $\eta(x)=\pm ix$ we obtain .

$$
\begin{aligned}
\int \frac{K(x,\alpha(x))}{X(x,\alpha(x))}dx &= \int \frac{K(x,\pm ix)}{X(x,\pm ix)}dx,\\
&= \int \frac{K_0(x)}{X_0(x)}dx,\\
&= \frac{a_{00}(-1\pm i)}{4x^2+1}\\
&\quad + \frac{1}{2}(a_{20}+a_{11}-a_{02}\pm i(-a_{20}+a_{11}+a_{02}+a_{00}))\log(x),\\
&\quad + \frac{1}{8}(-a_{20}-a_{11}+a_{02}+2\phi\pm i(a_{20}-a_{11}-a_{02}-a_{00}))\log(2+2x^2+x^4),\\
&\quad + \frac{1}{4}(a_{20}-a_{11}-a_{02}-a_{00}\pm i(a_{20}+a_{11}-a_{02}-2\phi))\arctan(x^2+1).
\end{aligned}
$$

By forcing $F(x,\eta(x)) = F(x,\pm ix) = F_0(x) = C_0\exp\left(\int \frac{K_0(x,\pm ix)}{X_0(x,\pm ix)}dx\right)$,to be a polynomial, with C_0 an arbitrary constant, we obtain a first set of necessary conditions:

$$
\begin{aligned}
a_{20}-a_{11}-a_{02}-a_{00} &= 0,\\
a_{20}+a_{11}-a_{02}-2\phi &= 0,\\
a_{00} &= 0.
\end{aligned}
$$

The same computations can be done for the other pair of algebraic solutions, $y = \pm i\sqrt{2}x$, that is,

$$
\begin{aligned}
\int \frac{K(x,\eta(x))}{X(x,\eta(x))}dx &= \int \frac{K(x,\pm i\sqrt{2}x)}{X(x,\pm i\sqrt{2}x)}dx,\\
&= \int \frac{K_0(x)}{X_0(x)}dx,\\
&= -\frac{a_{00}}{6x^2+1},\\
&\quad + \frac{1}{9}\left(3a_{20}-6a_{02}-2a_{00}\pm 3\sqrt{2}ia_{11}\right)\log(x),\\
&\quad + \frac{1}{18}\left(-3a_{20}+6a_{02}+9\phi+2a_{00}\mp 3\sqrt{2}ia_{11}\right)\log(3+2x^2).
\end{aligned}
$$

As in the previous case, we obtain a second set of necessary conditions:

$$\{a_{11} = 0, \quad a_{00} = 0\}.$$

Collecting all the obtained equations, we get that the degree of the invariant algebraic curve is $\phi = 0$, or in other words, such a curve does not exist.

Remark: 14 *if a planar system has an explicit non-algebraic solution which is in the zero level set of a Liouvillian function then it has a Darboux integrating factor and therefore the whole system is integrable by quadratures. Notice that this is the case for system it has a non-algebraic Liouvillian limit cycle and it can be transformed into a Bernoulli equation. Consequently, if we would like to have an explicit non-algebraic limit cycle for a planar system, which is not integrable by quadratures, we should look for a limit cycle given by a non-Liouvillian function.*

CONCLUSION

In this memory, we presented some results concerning the qualitative study of certain classes of nonlinear planar polynomial differential systems .

A differential system needs to know whether it admits a periodic solution or not; moreover, periodic solution, and limit cycle. The results obtained in this work are articulated ,around these questions .

First of all, in Chapter 1, we recalled some basic notions ,concerning the qualitative theory of differential systems.

In Chapter 2, we are more particularly interested in the study of cycles limits.

We will start with the qualitative study the theorie of the limit cycles. We study the noexistence and existant of limit cycle .We study stability of limit cycles We study hyperbolic limit cycles .

This chapter presents the works of A. Gasol, and all. In an article entitled "Explicit non-algebraic limit cycles of polynomial systems,"Published by "Eelsevier" on January 3, 2006. To prove that the polynomial is planarA vector field can have an explicit limit cycle that is not algebraic. Prove that this system has at most one limit cycle and that, when it exists, it can be explicit.They are found and given by squaring. They provide valuable insights into the presence of hyperbolic systems and the existence of algebraic solutions.

They provide that.The limit cycle is not algebraic. î

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كلية العلوم والتكنولوجيا

قسم الرياضيات والإعلام الآلي

جامعة غرداية

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شهادة الترخيص بالإيداع

أنا الأستاذ: كينه عبد الكريم

بصفتي -مشرف- رئيس – ممتحنا [1]- والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـــــ

Family of differential systems with explicit algebraic and non-algebraic limit cycles...

من انجاز الطالب(ة): شنيني هجيرة

الكلية: العلوم والتكنولوجيا.

القسم: الرياضيات والاعلام الالي.

الشعبة: رياضيات.

التخصص: تحليل دالي وتطبيقات.

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أشهد ان الطالب (الطلبة) قد قام (قاموا) بالتعديلات والتصحيحات المطلوبة من طرف لجنة المناقشة وان المطابقة بين النسخة الورقية والالكترونية استوفت جميع شروطها.

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