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On Limit cycles for Bernoulli and Riccati differential equations

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Dedication

In the name of Allah, the Most Gracious, the Most Merciful

Praise be to Allah, by whose grace good deeds are completed, and through whose favor goals are reached. With His guidance, success is attained after long struggle and perseverance.
Since my early childhood, I have loved learning and was deeply passionate about mathematics. I faced many hardships and overcame obstacles, with no one beside me except my generous Lord, who never abandoned me and was with me at every step.
It was my childhood dream to become a mathematics teacher, to carry the torch of knowledge and inspire future generations just as my own teachers once inspired me.
And today, by the grace of Allah first and foremost, and then thanks to everyone who played a role in my life, that dream has come true.
*To my beloved father, **Mustapha Dahou**, The pillar of my life and my constant support, Who taught me that true strength lies in faith and determination, And showed me how to see challenges as opportunities, You have been both my greatest teacher and dearest father. May Allah reward you abundantly.*
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Abstract

This thesis presents a comprehensive study of planar polynomial differential systems, which are fundamental in the qualitative analysis of differential equations. Among the dynamic behaviors of interest are limit cycles closed periodic solutions that characterize long term system behavior and stability. The central problem lies in proving the existence, number, and stability of such cycles, especially in specific cases such as Bernoulli and Riccati equations .Within this framework, the thesis reviews key preliminary notions such as vector fields, equilibrium points, invariant curves, and Darboux integrability, alongside analytical tools like the Poincaré map and the Hartman Grobman theorem for classifying behavior near critical points.

The core contribution consists in studying and interpreting the results from Clàudia Valls' article [44], where I reformulated and simplified the theoretical proofs concerning rational limit cycles, and enriched them with illustrative examples and diagrams aimed at enhancing understanding.

Keywords: Planar polynomial differential systems, vector field, periodic solutions, phase portrait, equilibrium points, integrability, rational limit cycle, Bernoulli equation, Riccati equation.

Résumé

Ce mémoire présente une étude approfondie des systèmes différentiels polynomiaux plans, qui jouent un rôle essentiel dans l'analyse qualitative des équations différentielles. Parmi les comportements dynamiques les plus significatifs figurent les cycles limites, qui sont des solutions périodiques fermées représentant la stabilité et l'évolution à long terme du système. Le principal enjeu réside dans la démonstration de l'existence, du nombre et de la stabilité de ces cycles, notamment dans des cas particuliers comme les équations de Bernoulli et de Riccati. Dans ce cadre, le mémoire passe en revue un ensemble de notions fondamentales telles que les champs de vecteurs, les points d'équilibre, les courbes invariantes et l'intégrabilité au sens de Darboux, ainsi que des outils analytiques comme l'application de Poincaré et le théorème de Hartman Grobman, utiles pour la classification du comportement local près des points critiques.

La contribution principale de ce travail consiste en l'étude et l'interprétation des résultats de l'article [44] de Clàudia Valls, dont les démonstrations théoriques sur l'existence de cycles limites rationnels ont été reformulées, simplifiées et enrichies par des exemples illustratifs et des schémas explicatifs facilitant la compréhension.

Mots clés : Systèmes différentiels polynomiaux plans, champ de vecteurs, solutions périodiques, portrait de phase, points d'équilibre, intégrabilité, cycles limites rationnels, équation de Bernoulli, équation de Riccati.

الملخص

أعطت هذه المذكرة دراسة شاملة للأنظمة التفاضلية متعددة الحدود في المستوى، باعتبارها من المواضيع الأساسية في التحليل النوعي للمعادلات التفاضلية. ومن بين السلوكيات الديناميكية التي تبرز في هذا السياق ما يُعرف بالدورات الحدية، وهي حلول دورية مغلقة تعكس استقرار النظام وسلوكه طويل الأمد. وتكمن الإشكالية في إثبات وجود مثل هذه الدورات وتحديد عددها واستقراريتها، لا سيما في حالات خاصة كمعادلات برنولي وريكاتي. في هذا الإطار، تستعرض المذكرة جملة من المفاهيم التمهيدية مثل الحقول الاتجاهية، نقاط التوازن، المنحنيات الثابتة، والتكامل حسب داربو، إلى جانب أدوات تحليلية مثل خريطة بوانكاريه ومبرهنة هارتمان-غروبمان، التي تساعد على تصنيف السلوك المجاور للنقاط الحرجة .

أما المشاركة الأساسية فتتمثل في دراسة وتفسير النتائج الواردة في المقال [56] لكلود فاديا ، حيث قمت بإعادة صياغة البراهين النظرية الخاصة بوجود الدورات الحدية الجذرية، مع تبسيطها وإثرائها بأمثلة تطبيقية ورسومات توضيحية تهدف إلى تعزيز الفهم..

الكلمات المفتاحية : أنظمة تفاضلية متعددة الحدود مستوية، حقل اتجاه، حلول دورية، مخطط الطور، نقاط التوازن، القابلية للتكامل، الدورات الحدية الكسرية، معادلة بيرنولي، معادلة ريكاتي.

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Introduction

Dynamical systems provide a fundamental mathematical framework for understanding and interpreting the evolution of various phenomena over time in fields such as biology, physics, engineering, and economics. Since the advent of differential equations, these systems have attracted significant interest from mathematicians aiming to study both their qualitative and quantitative behavior [13,28,42]. Given that most real world models are nonlinear, finding exact analytical solutions is often impossible. For this reason, numerical methods have become indispensable tools for approximating solutions. However, such methods usually yield information about the system's behavior over limited time intervals, without fully revealing its global dynamics.in the late 19th century, the French mathematician Henri Poincaré revolutionized this field with his seminal work Memoir on Curves Defined by a Differential Equation [40, 41], introducing a qualitative approach based on geometric and topological ideas. This allowed for the study of solution behavior without requiring explicit formulas. Foundational concepts such as phase portraits and return maps contributed to the establishment of modern qualitative theory in dynamical systems.

Among the central challenges in this field are the questions of integrability and the existence of periodic solutions particularly those known as limit cycles, which are closed and isolated trajectories. poincaré was the first to address this concept in his fundamental works [40]. Since then, many models in applied sciences have been formulated as planar differential systems that exhibit limit cycles [9, 12]. although some first-order differential equations, such as Bernoulli and Riccati equations, may appear structurally simple, the existence of relative limit cycles within these systems remains an unresolved issue which is known in the literature as an open problem in the context of qualitative analysis [44].

This thesis aims to study specific classes of planar differential equations and analyze their qualitative behavior, with a particular focus on the existence of limit cycles, whether algebraic, relative, or non-algebraic, various tools will be employed in this context, including first integrals, and algebraic curve analysis. the study is structured in three main chapters.

Chapter 1, introduces the fundamental concepts and tools used to investigate planar differential systems, such as direction fields, solutions, periodic orbits, phase portraits, and the classification of critical points, as well as important notions like invariant algebraic curves, first integrals, Darboux integrability, and integrating factors.

Chapter 2, focuses on the general theory of limit cycles, including their existence and stability, and presents the concept of Poincaré maps as a central analytical tool, with an emphasis on hyperstable limit cycles.

Chapter 3, constitutes the original contribution of this thesis, where we study two classes of first-order differential equations Bernoulli and Riccati. We prove the existence of an upper bound for the number of relative limit cycles and provide explicit examples that achieve this bound.

1 Preliminary concepts

In this chapter, we introduce the fundamental notions and tools used in the qualitative study of planar differential systems, these preliminary concepts include the structure and behavior of vector fields, the nature of solutions and phase portraits, as well as the classification and stability of equilibrium points, we also explore invariant curves and conditions for integrability, which are essential in analyzing the existence and behavior of periodic solutions and limit cycles.

1.1 Planar polynomial differential systems

Definition 1.1. [32] A planar polynomial differential system is defined by two differential equations of the form

$$\begin{cases} \dot{x} = P(x(t), y(t)), \\ \dot{y} = Q(x(t), y(t)), \end{cases} \tag{1.1}$$

where $P(x(t), y(t))$ and $Q(x(t), y(t))$ are polynomial functions of the variables x and y. The system (1.1) is of degree n where $n = max(deg(P), deg(Q))$.
As usual the dot denotes derivative with respect to the independent variable t.

Definition 1.2. [42] A differential system is given by

$$\frac{dx}{dt} = f(t, x),$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. If the function f does not depend explicitly on the time variable t (that is, $f(t, x) = f(x)$), the system is called **autonomous** and can be written as

$$\dot{x} = f(x).$$

Otherwise, if f depends explicitly on t, the system is referred to as non-autonomous.

Definition 1.3. [39] A polynomial differential system in the plane is called $h$omogenous of degree n if it can be written in the form

$$\begin{cases} \dot{x} = P(x(t), y(t)) = \sum_{i+j=0}^{i+j=n} \alpha_{ij}\, x^i y^{n-j}, \\ \\ \dot{y} = Q(x(t), y(t)) = \sum_{i+j=0}^{i+j=n} \beta_{ij}\, x^i y^{n-j}. \end{cases}$$

Example 1.1. Consider the following planar differential system:

$$\begin{cases} \dot{x} = x^2 + y + 1, \\ \dot{y} = xy + 2. \end{cases}$$

This system is **non-homogeneous** because the right-hand sides contain terms of different degrees (for instance, x^2 is degree 2, y is degree 1, and 1 and 2 are constants of degree 0).

1.2 Vector field

Drawing the vector field before beginning a deep analysis of a differential system is quite practical and can give us important information about the many types of potential solutions. It is the vector that corresponds to each point in the space shown graphically. This vector will really be tangent to the differential system's trajectory as it passes through that location, as a result, we may get a reasonably accurate sense of the potential solutions and their asymptotic behavior from the vector field.

Definition 1.4. [15] Let Δ be an open set in $\mathbb{R}^2$ such that at every point $A \in \Delta$, there exists a vector defined as $\frac{d\vec{A}}{dt}$. There is a mapping

$$X : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad A(x, y) \mapsto \frac{d\vec{A}}{dt} = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix},$$

where $P(x, y)$ and $Q(x, y)$ are functions of class C^1 on Δ.

The vector field X associated with the system (1.1) is denoted by

$$X = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

This vector field can also be represented by the following first-order differential operator:

$$X \equiv P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

which acts on differentiable scalar function.

Remark 1.1.

1. We Assume That the functions P and Q are of class C^1. This assumption ensures that the Cauchy–Lipschitz conditions are satisfied for the system (1.1), so that for every initial condition (x_0, y_0), there exists a unique solution.

2. The plane formed by the variables x and y is called the phase plane.

3. On the curve $P(x, y) = 0$, known as the vertical isocline, the vector field is parallel to the y-axis, whereas on the curve $Q(x, y) = 0$, called the horizontal isocline, the vector field is parallel to the x-axis.

Example 1.2. Consider the system (1.2) of differential equations

$$\begin{cases} \dot{x} = -y + 2, \\ \dot{y} = x + 3y. \end{cases} \tag{1.2}$$

Rewrite the system in matrix A form as,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

,

find the eigenvalues of the coefficient matrix by solving,

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 1 = 0,$$

the eigenvalues are ,

$$\lambda = \frac{3 \pm \sqrt{5}}{2},$$

next, find a particular solution by solving,

$$A\mathbf{X}_p = -\begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

which yields,

$$\mathbf{X}_p = \begin{pmatrix} -6 \\ 2 \end{pmatrix},$$

the general solution is a linear combination of the homogeneous solutions plus the particular solution:

$$\mathbf{X}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \mathbf{X}_p.$$

Where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors corresponding to λ_1 and λ_2, respectively.

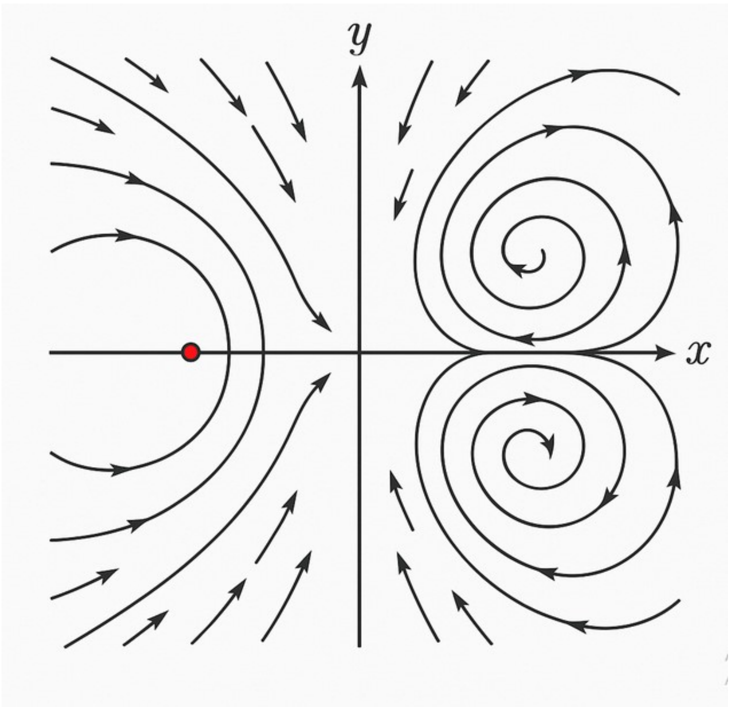


Figure 1.1: Vector field of system (1.2).

1.3 Solution and periodic solution

The basic terms solution and periodic solution in differential systems are clearly defined as follows: a solution is any function that satisfies the system's equations for given initial conditions, while a periodic solution is a function that repeats its values after a fixed period of time.

Definition 1.5. [39] A mapping $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\varphi(t) = (x(t), y(t))$ is called a *solution* of system (1.1) if

$$\dot{\varphi}(t) = \mathcal{X}(\varphi(t)), \quad \forall t \in I,$$

where $\mathcal{X} = (P, Q)$ is the associated vector field. if $\varphi_1(t) = (x_1(t), y_1(t))$ and $\varphi_2(t) = (x_2(t), y_2(t))$ are two solutions on I_1 and I_2 respectively, we say that $\varphi_2(t)$ is an extension of $\varphi_1(t)$ if $I_1 \subset I_2$ and $\varphi_1(t) = \varphi_2(t)$ for all $t \in I_1$. A solution is called maximal if it has no further extension.

Definition 1.6. [39] A solution $\varphi(t) = (x(t), y(t))$ of system (1.1) is called a *periodic solution* if there exists a real number $T > 0$ such that

$$\varphi(t + T) = \varphi(t), \quad \forall t \in \mathbb{R}.$$

The smallest such T is called the period of the solution.

1.4 Phase portrait

The solutions of a vector field $\mathcal{X}$ are represented as trajectories or orbits that illustrate how the system evolves over time. The collection of these trajectories is known as the *phase portrait*, which provides valuable insight into the qualitative behavior of the system, such as identifying equilibrium points and analyzing stability properties. The plane $\mathbb{R}^2$ is commonly referred to as the *phase plane*, where the behavior of dynamical systems is visually represented.

Definition 1.7. [39] Let $p \in \Delta$ be a point in the domain of the vector field $\mathcal{X} : \Delta \to \mathbb{R}^2$. The orbit of $\mathcal{X}$ through p, denoted by γ_p, is defined as the image of the maximal solution $\varphi_p : I_p \to \Delta$ that passes through p. In other words,

$$\gamma_p = \{\varphi_p(t) \mid t \in I_p\}.$$

Definition 1.8. [15] The phase portrait of a vector field $\mathcal{X}$ is the complete set of orbits that represent the solutions of the system in the (x, y)-plane. It provides a global view of the system's dynamics by displaying all trajectories (orbits) and equilibrium points.

Example 1.3 (Saddle Point)**.** Consider the system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y. \end{cases} \tag{1.3}$$

This is a linear system with a saddle at the origin.

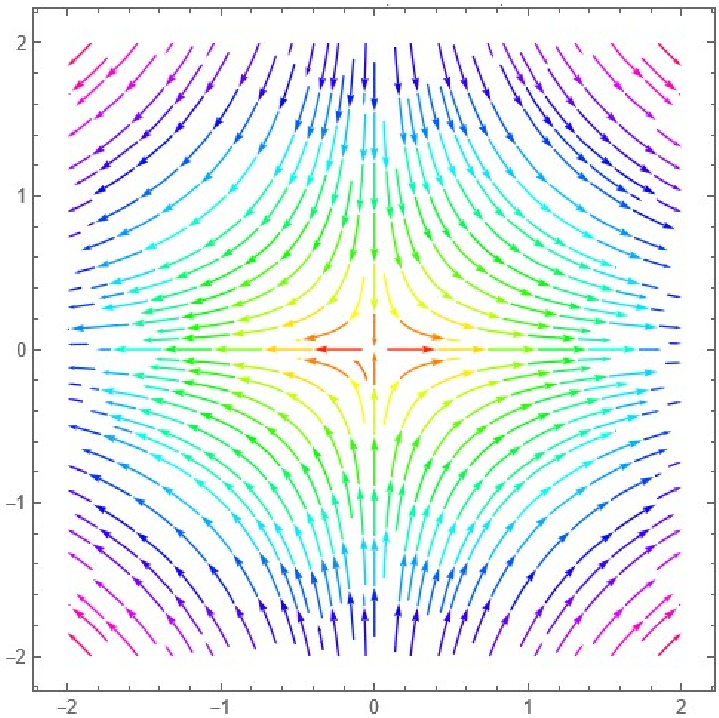


Figure 1.2: Phase portrait of a saddle point.

Example 1.4 (Stable Focus)**.** Consider the system

$$\begin{cases} \dot{x} = -x - y, \\ \dot{y} = x - y. \end{cases} \tag{1.4}$$

This system has a stable focus at the origin.

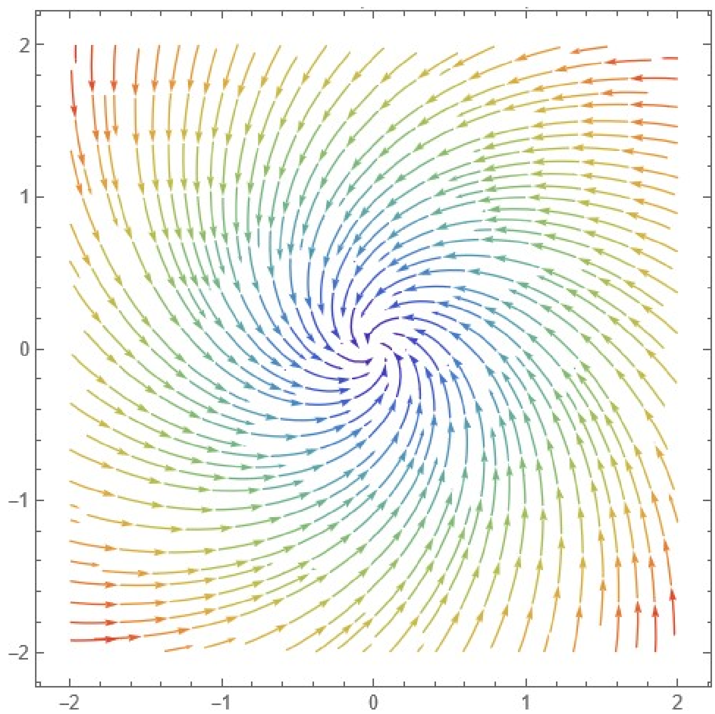


Figure 1.3: Phase portrait of a stable focus.

Example 1.5 (Center)**.** Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases} \tag{1.5}$$

This is a linear center with closed orbits around the origin

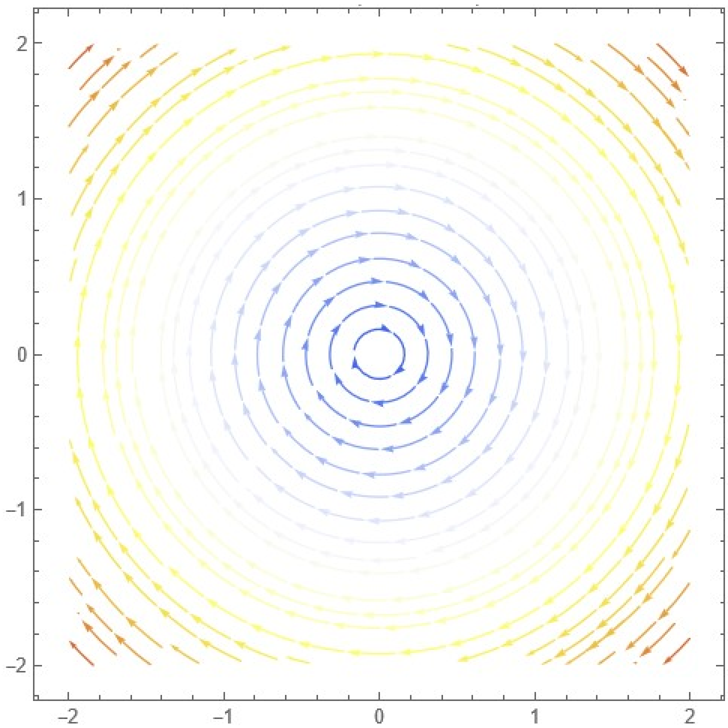


Figure 1.4: Phase portrait of a center equilibrium point.

Example 1.6 (Limit Cycle (Van der Pol Model))**.** Consider the Van der Pol oscillator

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + y(1 - x^2). \end{cases} \tag{1.6}$$

This system (1.6) exhibits a stable limit cycle.

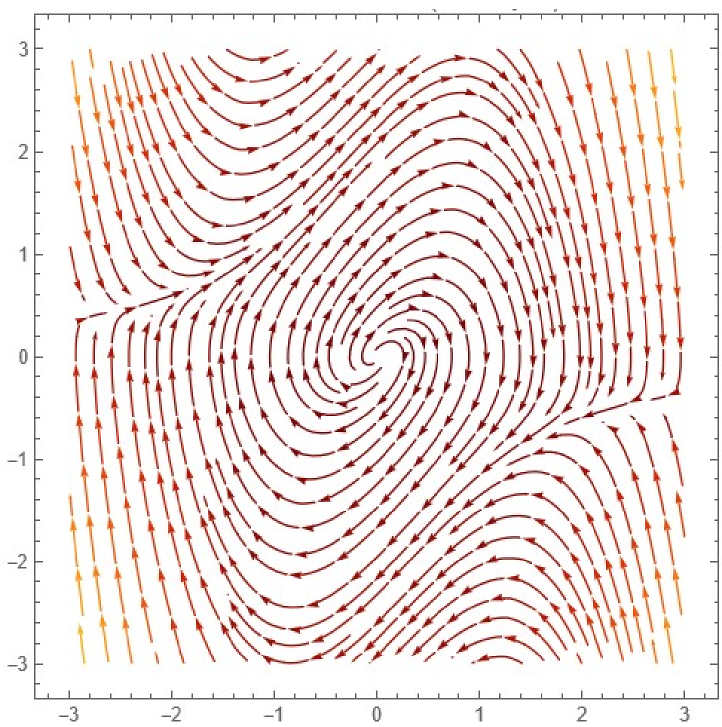


Figure 1.5: Phase portrait showing a limit cycle (Van der Pol oscillator).

Example 1.7 (Improper Node)**.** Consider the system

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = -x + 4y. \end{cases} \tag{1.7}$$

This linear system has an improper node at the origin

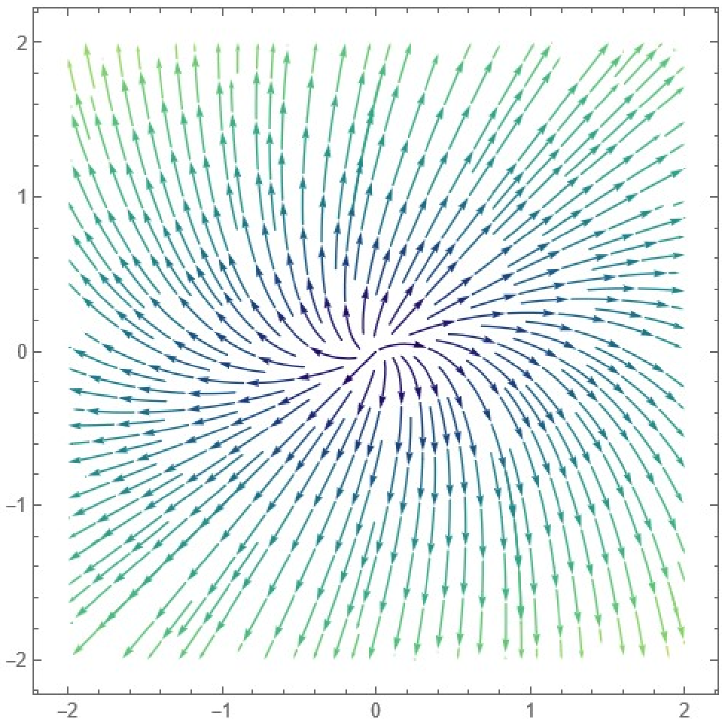


Figure 1.6: Phase portrait of an improper node.

1.5 Equilibrium points

Analyzing dynamical systems, equilibrium points are essential. when describing a dynamical system with multiple variables, Henri Poincaré $(1854 - 1912)$ demonstrated that it is sufficient to characterize the system without computing exact solutions. Determining the equilibrium points and evaluating their stability significantly simplifies the study of nonlinear systems near these points.

Definition 1.9. [39] A point (x_0, y_0) is called an *equilibrium point* (*singular point*) of the system (1.1) if

$$\begin{cases} P(x_0, y_0) = 0, \\ Q(x_0, y_0) = 0. \end{cases}$$

Remark 1.2. The concept of an equilibrium point is equivalent to that of a singular point in a vector field. We use the term *singular point* when referring to the vector field itself, while *equilibrium point* is used when focusing on the system's trajectories.

Remark 1.3. Equilibrium points occur at the intersections of the horizontal isocline (where $\dot{y} = 0$) and the vertical isocline (where $(\dot{x} = 0)$.

Proposition 1.1. *[39] Every nontrivial periodic orbit (i.e., limit cycle) of a planar differential system surrounds at least one equilibrium point.*

Example 1.8. Consider the following system:

$$\dot{x} = x(1 - y), \quad \dot{y} = y(x - 2).$$

To find the equilibrium points, we solve the system:

$$x(1 - y) = 0, \quad y(x - 2) = 0.$$

From the first equation, we get:

$$x = 0 \quad \text{or} \quad y = 1.$$

From the second equation, we get:

$$y = 0 \quad \text{or} \quad x = 2.$$

Now, considering the common solutions to both equations, we obtain:

- If $x = 0$ and $y = 0$, then this is an equilibrium point.
- If $x = 2$ and $y = 1$, then this is also an equilibrium point.

Other combinations, such as $x = 0, y = 1$ or $x = 2, y = 0$, do not satisfy both equations simultaneously.

Therefore, the equilibrium points of the system are:

$$(0, 0), \quad (2, 1).$$

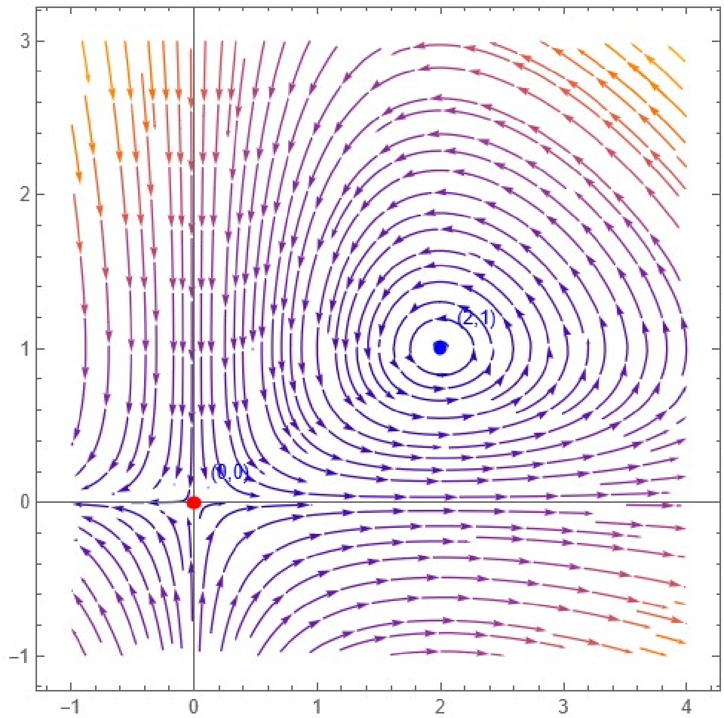


Figure 1.7: Phase portrait of system (1.8).

1.5.1 The Jacobian Matrix and Linearization

In order to analyze the behavior of trajectories near equilibrium points, it is common practice to consider the linearization of system (1.1) and then relate the trajectories of the nonlinear system to those of its linear counterpart.

Definition 1.10. [39] Let $J(x_0, y_0)$ be the Jacobian matrix of the vector field near an equilibrium point (x_0, y_0), which is defined as

$$J(x_0, y_0) = \begin{pmatrix} \frac{\partial P}{\partial x}(x_0, y_0) & \frac{\partial P}{\partial y}(x_0, y_0) \\ \frac{\partial Q}{\partial x}(x_0, y_0) & \frac{\partial Q}{\partial y}(x_0, y_0) \end{pmatrix}.$$

Then, the linearized form of system (1.1) near the equilibrium point (x_0, y_0) is given in matrix form by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.8}$$

Definition 1.11. [39] A singular point (x_0, y_0) is defined as hyperbolic if the Jacobian matrix $J(x_0, y_0)$ has eigenvalues with non-zero real parts. Conversely, if at least one eigenvalue has a zero real part, the point is classified as non-hyperbolic.

Example 1.9. Consider the following nonlinear system

$$\begin{cases} \dot{x} = 2x^2 + 2y, \\ \dot{y} = 3xy - 3. \end{cases} \tag{1.10}$$

To find the equilibrium points, we solve the system,

$$\begin{cases} 2x^2 + 2y = 0, \\ 3xy - 3 = 0. \end{cases}$$

Divide the first equation by 2 ,

$$x^2 + y = 0 \quad \Rightarrow \quad y = -x^2,$$

from the second equation,

$$3xy = 3 \quad \Rightarrow \quad xy = 1,$$

substitute $y = -x^2$ into $xy = 1$

$$x(-x^2) = -x^3 = 1 \quad \Rightarrow \quad x = -1, \quad y = -1,$$

hence, the system has a unique equilibrium point at,

$$(x_0, y_0) = (-1, -1).$$

Now, we compute the Jacobian matrix. Let

$$P(x, y) = 2x^2 + 2y, \quad Q(x, y) = 3xy - 3,$$

the Jacobian matrix is:

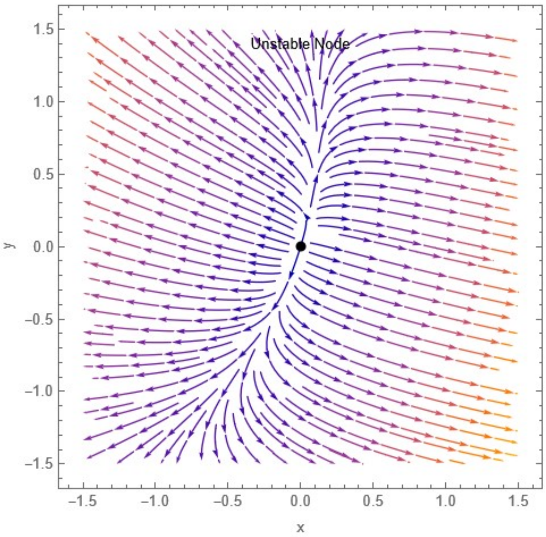
$$J(x, y) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} 4x & 2 \\ 3y & 3x \end{bmatrix},$$

evaluating the Jacobian at the equilibrium point $(x_0, y_0) = (-1, -1)$, we obtain:

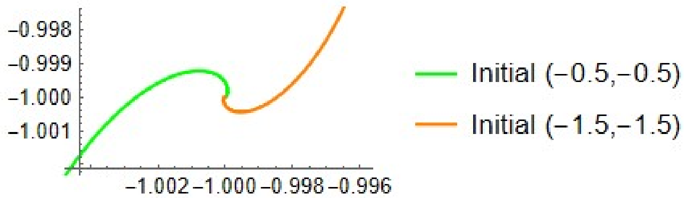
$$J(-1, -1) = \begin{bmatrix} -4 & 2 \\ -3 & -3 \end{bmatrix},$$

therefore, the linearization of system (1.4) at the point $(-1, -1)$ is:

$$\begin{cases} \dot{x} = -4x + 2y, \\ \dot{y} = -3x - 3y. \end{cases}$$



(a) For the Phase Portrait Plot of system(1.4)



(b) For the Solution Trajectories Plot of system (1.4)

Figure 1.8: Plot of system (1.4).

1.5.2 Classification of equilibrium points

Definition 1.12. [39] Consider the differential system (1.1) and let $J(x_0, y_0)$ be the Jacobian matrix associated with it at the equilibrium point (x_0, y_0). Let λ_1 and λ_2 be the eigenvalues of this matrix. The classification of equilibrium points is based on the following cases:

- Node: If λ_1 and λ_2 are real and have the same sign
  - If $\lambda_1 \leq \lambda_2 < 0$, the origin is a stable node.
  - If $\lambda_1 \geq \lambda_2 > 0$, the origin is an unstable node.
- Saddle:

  If λ_1 and λ_2 are real, nonzero, and of opposite signs, the origin is a saddle. A saddle is always unstable.
- Focus: If λ_1 and λ_2 are complex conjugates with $\mathrm{Re}(\lambda_{1,2}) \neq 0$
  - If $\mathrm{Re}(\lambda_{1,2}) < 0$, the origin is a stable focus.
  - If $\mathrm{Re}(\lambda_{1,2}) > 0$, the origin is an unstable focus.

- Center : If λ_1 and λ_2 are purely imaginary, the origin is a center. A center is stable but not asymptotically stable.

1.5.3 Hartman-Grobman Theorem

This theorem states that a dynamical system (1.1) near a hyperbolic singular point can be reduced to the study of a topologically equivalent linear system (1.8) near the origin. This theorem is a powerful tool in the analysis of dynamical systems, as it allows for the simplification of complex dynamics by examining a simpler linear model. It is particularly useful for understanding the local behavior of dynamical systems defined on an open subset of the plane.

Theorem 1.1. *[28] Suppose that the Jacobian matrix at the equilibrium point (x_0, y_0) has two eigenvalues such that $\mathrm{Re}(\lambda_1) \neq 0$ and $\mathrm{Re}(\lambda_2) \neq 0$. Then, the solutions of the nonlinear system (1.1) can be approximated by the solutions of the linearized system (1.8) in a neighborhood of the equilibrium point.*

In other words, the phase portrait of the linearized system (1.8) provides a good approximation of that of the nonlinear system (1.1) near this equilibrium point through a continuous transformation.

Remark 1.4. [39] In the case where $\mathrm{Re}(\lambda_1) = 0$ and $\mathrm{Re}(\lambda_2) = 0$, the linearization method does not provide sufficient information about the behavior of the nonlinear system. Specifically, if the equilibrium point (x_0, y_0) is a center for the linearized system (1.8), determining whether it remains a center or becomes a focus in the nonlinear system (1.1) requires further investigation. This is known as the center problem.

1.5.4 Topological Equivalence

Definition 1.13. [37] A function $h : \mathbb{R}^2 \to \mathbb{R}^2$ is called a *homeomorphism* if it is a continuous bijection with a continuous inverse.

Definition 1.14. [1] Two autonomous systems in the plane

$$\text{(S1)}: \quad \begin{cases} \dot{x} = P_1(x(t), y(t)), \\ \dot{y} = Q_1(x(t), y(t)). \end{cases} \qquad \text{(S2)}: \quad \begin{cases} \dot{x} = P_2(x(t), y(t)), \\ \dot{y} = Q_2(x(t), y(t)). \end{cases}$$

Defined on two open sets V and W respectively, are said to be *topologically equivalent* if there exists a homeomorphism $h : V \to W$ such that h maps the orbits of (S1) onto the orbits of (S2) and preserves the direction of motion.

Remark 1.5. [28] Topological equivalence via a homeomorphism allows for a classification primarily based on the stability or instability of the equilibrium. Two linear systems are topologically equivalent if they have the same number of eigenvalues, with real parts of the same signs.

Remark 1.6. Consider the differential system (1.1), and let $J(x_0, y_0)$ be the Jacobian matrix associated with this system at the equilibrium point (x_0, y_0). Let λ_1 and λ_2 be the eigenvalues of this matrix.

1. A singular point is said to be *elementary* if at least one eigenvalue of $J(x_0, y_0)$ is nonzero. If both eigenvalues vanish ($\lambda_1 = \lambda_2 = 0$), the point is called *non-elementary.* In this case:

- The singular point is referred to as *degenerate* if the linear part is identically zero ($J(x_0, y_0) = 0$).
- If the linear part is nonzero, the singular point is called *nilpotent* (see [15], Theorem 3.5).

2. A singular point is said to be *semi-hyperbolic* if exactly one of its eigenvalues is zero while the other is nonzero. The phase portraits of such points are well known (see [15], Theorem 2.19).

3. A singular point (x_0, y_0) is called a *center* if there exists a neighborhood V around it such that for every point $p \in V$ (with $P^2(p)+Q^2(p) \neq 0$), the orbits passing through p are closed and surround (x_0, y_0), indicating closed orbits and periodic dynamics.

1.5.5 Stability of the equilibrium

There may be more than one equilibrium point in a nonlinear system, and these points may be unstable or stable. Ensuring the stability of an equilibrium point is crucial in various situations. The following is a definition of stability

Let (x_0, y_0) be an equilibrium point of system (1.1).
We denote $X(t) = (P(x, y), Q(x, y))$ end $X_0 = (P(x_0, y_0), Q(x_0, y_0))$.

Definition 1.15. [28] We say that

1. (x_0, y_0) is *stable* if and only if

$$\forall \varepsilon > 0, \exists \eta > 0 \text{ such that } \|(x, y) - (x_0, y_0)\| < \eta \Rightarrow \forall t > 0, \|X(t) - X_0\| < \varepsilon.$$

2. (x_0, y_0) is *asymptotically stable* if and only if it is stable and

$$\lim_{t \to \infty} \|X(t) - X_0\| = 0.$$

Example 1.10. Consider the following system

$$\begin{cases} \dot{x} = 8x - 2y + x^2 + y^2, \\ \dot{y} = -2x + 2y - xy. \end{cases} \tag{1.9}$$

The Jacobian matrix of system (1.10) is defined as,

$$J(x, y) = \begin{pmatrix} 8 + 2x & -2 + 2y \\ -2 - y & 2 - x \end{pmatrix},$$

at the equilibrium point $(x_0, y_0) = (0, 0)$, the Jacobian becomes,

$$J(0, 0) = \begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix},$$

by solving $\det(J(0, 0) - \lambda I) = 0$, we obtain the characteristic equation,

$$\lambda^2 - 10\lambda + 12 = 0,$$

thus, the eigenvalues are:

$$\lambda_1 = 5 + \sqrt{13}, \quad \lambda_2 = 5 - \sqrt{13},$$

since both eigenvalues are real and positive, the origin is classified as an *unstable node* (source).

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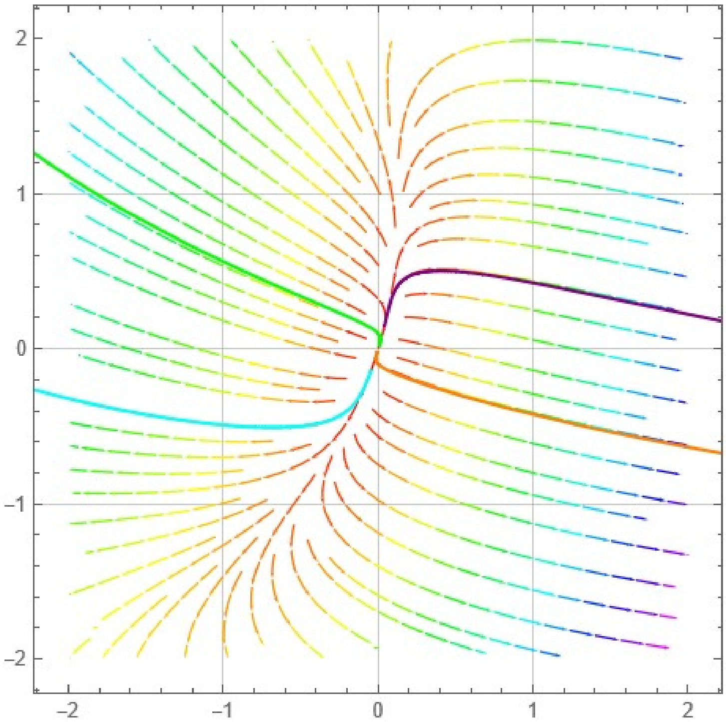


Figure 1.9: Phase portrait of the system (1.10).

Example 1.11. Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = \dfrac{1}{2}x + 4y + x^2 + y^2, \\ \dot{y} = 6x + 2y - 3xy. \end{cases} \tag{1.10}$$

The Jacobian matrix of system (1.10) is defined as,

$$J(x, y) = \begin{pmatrix} \frac{1}{2} + 2x & 4 + 2y \\ 6 - 3y & 2 - 3x \end{pmatrix},$$

at the equilibrium point $(x_0, y_0) = (0, 0)$, the Jacobian becomes,

$$J(0, 0) = \begin{pmatrix} \frac{1}{2} & 4 \\ 6 & 2 \end{pmatrix},$$

we compute the eigenvalues by solving the characteristic equation,

$$\det(J - \lambda I) = \begin{vmatrix} \frac{1}{2} - \lambda & 4 \\ 6 & 2 - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)(2 - \lambda) - 24,$$

expanding the determinant,

$$\left(\frac{1}{2} - \lambda\right)(2 - \lambda) = \lambda^2 - \frac{5}{2}\lambda + 1,$$

thus, the characteristic equation is:

$$\lambda^2 - \frac{5}{2}\lambda - 23 = 0,$$

multiplying through by 2 to eliminate the fraction,

$$2\lambda^2 - 5\lambda - 46 = 0,$$

using the quadratic formula,

$$\lambda = \frac{5 \pm \sqrt{393}}{4},$$

so the eigenvalues are :

$$\lambda_1 = \frac{5 + \sqrt{393}}{4}, \quad \lambda_2 = \frac{5 - \sqrt{393}}{4}.$$

Since one eigenvalue is positive and the other is negative, the origin is a *saddle point*, and therefore unstable.

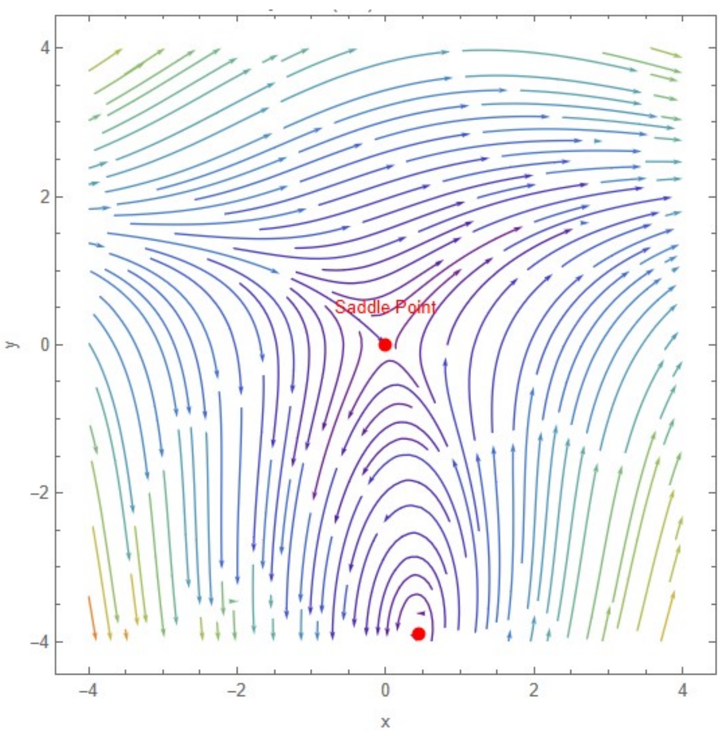


Figure 1.10: Phase portraits of system (1.11).

Example 1.12. Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = -3x + y - x^2, \\ \dot{y} = -2x - 2y. \end{cases} \tag{1.11}$$

we define the Jacobian matrix of the system (1.11) as,

$$J(x, y) = \begin{pmatrix} -3 - 2x & 1 \\ -2 & -2 \end{pmatrix}.$$

At the equilibrium point $(x_0, y_0) = (0, 0)$, the Jacobian becomes,

$$J(0, 0) = \begin{pmatrix} -3 & 1 \\ -2 & -2 \end{pmatrix},$$

by solving $\det(J(0,0) - \lambda I) = 0$, we obtain the characteristic equation,

$$\lambda^2 + 5\lambda + 8 = 0,$$

thus, the eigenvalues are:

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{-7}}{2} = \frac{-5}{2} \pm \frac{\sqrt{7}}{2} i,$$

since the eigenvalues have negative real parts and non-zero imaginary parts, the origin is classified as a *stable spiral point* (asymptotically stable focus).

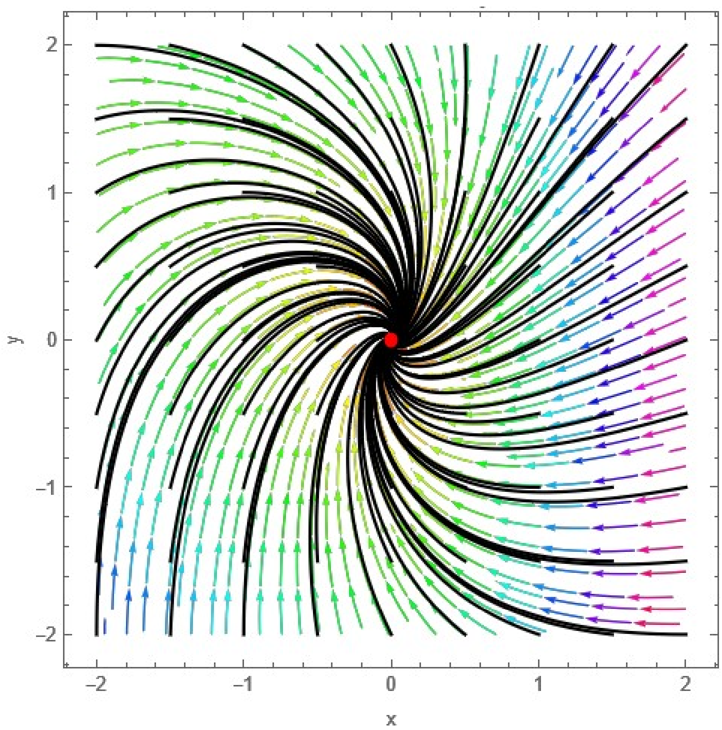


Figure 1.11: Phase portrait of the system (1.12).

1.6 Invariant curves

Invariant algebraic curves are a fundamental tool in studying the integrability of planar polynomial differential systems, as they are used to identify the existence of periodic solutions and limit cycles.

Definition 1.16. [15] We call an *invariant curve* of the system (1.1) any curve defined by the equation $U(x, y) = 0$ in the phase plane for which there exists a function $K = K(x, y)$, called the cofactor of the invariant curve $U = 0$, such that:

$$P(x, y)\frac{\partial U(x, y)}{\partial x} + Q(x, y)\frac{\partial U(x, y)}{\partial y} = K(x, y)U(x, y). \tag{1.12}$$

Equality (1.12) shows that on the invariant curve, the gradient $\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)$ of U is orthogonal to the vector field $\mathcal{X} = (P, Q)$. This means that at every point on the invariant curve, the vector field is tangent to the curve, and consequently, the curve is formed by the solutions (or trajectories) of the vector field $\mathcal{X}$.

Example 1.13. Consider the nonlinear system

$$\begin{cases} \dot{x} = x(2-y), \\ \dot{y} = y(x-2). \end{cases} \tag{1.13}$$

Assume that $U(x, y) = xy$ is an invariant curve for the system(1.14)

$$\begin{aligned} \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} &= y \cdot x(2-y) + x \cdot y(x-2) \\ &= xy\big[(2-y) + (x-2)\big] = xy(x-y). \end{aligned}$$

Thus, the curve $xy = 0$ is an invariant curve for the system, with the associated function

$$K(x, y) = x - y.$$

Definition 1.17. [21] An invariant curve $U(x, y) = 0$ is called algebraic of degree m if $U(x, y)$ is a polynomial of degree m. Otherwise, it is called non-algebraic.

Definition 1.18. [21] An algebraic curve $U(x, y) = 0$ is said to be irreducible if $U(x, y)$ is a polynomial that cannot be factored into polynomials of lower degrees in the ring $\mathbb{R}[x, y]$.

Remark 1.7. [15] When the cofactor $k(x, y)$ is a polynomial, the invariant curve defined by $U(x, y) = 0$ is said to have a polynomial cofactor. This allows us to apply algebraic techniques specific to polynomials in its analysis.

Theorem 1.2. *[20] Consider the system* (1.1) *and let* $\Gamma(t)$ *be a periodic orbit with period* $T > 0$*. Suppose that* $U : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve such that:*

$$\Gamma(t) = \{(x, y) \in \Delta \mid U(x, y) = 0\},$$

and let $K(x, y) \in C^1$ *be the cofactor associated with the invariant curve* $U(x, y) = 0$ *as given in equation* (1.12) *. If there exists a point* $p \in \Delta$ *such that* $U(p) = 0$ *and* $\nabla U(p) \neq 0$*, then the following holds:*

$$\int_0^T div(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt.$$

Remark 1.8. The condition $\nabla U(p) \neq 0$ ensures that the invariant curve $U(x, y) = 0$ does not contain singular points, meaning that the periodic orbit does not pass through critical points of the system.

1.7 Integrability of polynomial differential systems

In the qualitative analysis of polynomial differential systems, the concept of *integrability* plays a fundamental role. A polynomial differential system is said to be *integrable* if it admits a *first integral*, as defined below. However, determining a first integral for a given differential system is often a challenging task. The significance of the existence of a first integral lies in the fact that it completely characterizes the phase portrait of the system, providing a comprehensive understanding of its global dynamics.

1.7.1 First integral

Definition 1.19. [15] Let $H : \Delta \to \mathbb{R}$ be a C^1 function that is not locally constant. We say that H is a *first integral* of the differential system (1.1) in Δ if it remains constant along every trajectory of the system that is contained in Δ. In other words, H is a first integral if

$$\frac{dH(x,y)}{dt} = P(x,y)\frac{\partial H(x,y)}{\partial x} + Q(x,y)\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

The general solution of this equation is given by $H(x,y) = k$, where k is an arbitrary constant. Therefore, the system (1.1) is said to be *integrable* in Δ if it possesses a first integral H in Δ.

Example 1.14. we start with the given system

$$\begin{cases} \dfrac{dx}{dt} = 2xy, \\ \dfrac{dy}{dt} = y^2 - x^2. \end{cases} \tag{1.14}$$

The condition for a first integral $H(x,y)$ is :

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(2xy) + \frac{\partial H}{\partial y}(y^2 - x^2) = 0,$$

assuming a polynomial form for H

$$H(x,y) = ax^2 + by^2 + cx^3 + dxy^2,$$

computing the partial derivatives,

$$\frac{\partial H}{\partial x} = 2ax + 3cx^2 + dy^2, \qquad \frac{\partial H}{\partial y} = 2by + 2dxy,$$

substituting into the condition,

$$(2ax + 3cx^2 + dy^2)(2xy) + (2by + 2dxy)(y^2 - x^2) = 0,$$

expanding and simplifying,

$$4ax^2y + 6cx^3y + 2dxy^3 + 2by^3 + 2dxy^3 - 2bx^2y - 2dx^3y = 0,$$

grouping like terms,

$$(4a - 2b)x^2y + (6c - 2d)x^3y + (4d + 2b)xy^3 = 0,$$

setting each coefficient to zero yields the system,

$$4a - 2b = 0, \qquad 6c - 2d = 0, \qquad 2b + 4d = 0,$$

solving this system,

$$b = 2a, \qquad d = -a, \qquad c = -3a,$$

taking $a = 1$, the first integral becomes,

$$H(x, y) = x^2 + 2y^2 - 3x^3 - xy^2.$$

Verification shows this satisfies $\frac{dH}{dt} = 0$
a simpler equivalent form is:

$$H(x, y) = x^2 + y^2 - \frac{3}{2}x^3,$$

which also satisfies the conservation condition,

$$(2x - 3x^2)(2xy) + (2y)(y^2 - x^2) = 0.$$

This first integral represents a conserved quantity that remains constant along the solution trajectories of the system. It provides crucial insight into the system's dynamics without requiring the full solution of the differential equations. The existence of such an integral facilitates analysis of the solution behavior and stability properties.

1.7.2 Darboux integrability

Definition 1.20. [36] A Darboux function is a function of the form

$$f(x, y) = f_1(x, y)^{\lambda_1} f_2(x, y)^{\lambda_2} \dots f_p(x, y)^{\lambda_p} \exp\left(\frac{g(x, y)}{h(x, y)}\right),$$

where $f_i(x, y)$ for $i = 1, \dots, p$, $g(x, y)$, and $h(x, y)$ are polynomials in $\mathbb{C}[x, y]$ and the λ_i for $i = 1, \dots, p$ are complex numbers.

Definition 1.21. System (1.1) is called Darboux integrable if it has a first integral which is a Darboux function.

Definition 1.22. [16] A function that can be represented by quadratures of elementary functions is known as a Liouvillian function.

Determining whether a given class of functions has an integrating factor or an inverse integrating factor is another aspect of studying the integrability problem.

1.7.3 Integrating factors

Definition 1.23. [39] On the open subset $\Delta \subseteq \mathbb{R}^2$, the function $R(x, y)$ is an integrating factor of differential system (1.1).

$$\operatorname{div}(RP, RQ) = 0 \quad \text{or} \quad P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R \operatorname{div}(P, Q),$$

if $R \in C^1(U)$, $R \neq 0$ on U and

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}.$$

As is customary,

$$\operatorname{div}(X) = \operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

defines the divergence of the vector field X.

It is evident that the function H that satisfies

$$\frac{\partial H}{\partial x} = RQ, \quad \frac{\partial H}{\partial y} = -RP,$$

is a first integral, then the first integral H associated to the integrating factor R is given by

$$H(x, y) = -\int R(x, y)P(x, y)\, dy + h(x),$$

$$H(x, y) = \int R(x, y)Q(x, y)\, dx + h(y).$$

Inverse integrating factor

Definition 1.24. [23] If a nonzero function $V : \Delta \to \mathbb{R}$ of class $C^1(\Delta)$ satisfies the following linear partial differential equation and is not locally null:

$$Q\frac{\partial V}{\partial y} + P\frac{\partial V}{\partial x} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right),$$

then V is called an *inverse integrating factor* of system (1.1).

It is simple to confirm that an integrating factor in $\Delta \setminus \{V = 0\}$ of the system is defined by the function:

$$R = \frac{1}{V}.$$

2 Limit Cycles in Differential Systems

2.1 Introduction

Limit cycles are isolated periodic solutions of planar nonlinear differential systems. They correspond to closed trajectories in the phase plane, which nearby trajectories may either approach or diverge from over time. The concept was first introduced by Henri Poincaré in his 1882 memoir "Sur les courbes définies par une équation différentielle", where he explored the qualitative behavior of differential equations [40]. Limit cycles are essential in describing self-sustained oscillations, such as those observed in chemical reactions, biological rhythms, and electronic circuits. These oscillations arise naturally from the internal dynamics of the system, without the need for any external periodic forcing. This chapter outlines the mathematical theory of limit cycles.

Definition 2.1. [31] A limit cycle is it the solution of the system is any periodic orbit that is isolated from the set of all periodic orbits of the system.

An *isolated periodic orbit* means that nearby trajectories are not closed, instead, they spiral around the limit cycle, either moving away from it or approaching it.

Definition 2.2. [31]

The periodic solution of system (1.1) is called an algebraic limit cycle if it is a limit cycle and contained within an irreducible algebraic invariant curve $U(x, y) = 0$ of system (1.1). Otherwise, it is referred to as a non-algebraic limit cycle.

Example 2.1. In [40] of his foundational work, Henri Poincaré presented the first known example of a limit cycle. the studied system is a planar polynomial differential system of degree three

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1), \\ \dot{y} = y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1). \end{cases} \tag{2.1}$$

This system has a unique singular point at the origin, which is a focus. There are no singular points on the circle $x^2 + y^2 = 1$, which acts as a characteristic trajectory and therefore constitutes an isolated limit cycle. Hence, the unit circle

$$x^2 + y^2 = 1,$$

is the only limit cycle in the system, as observed by Poincaré.

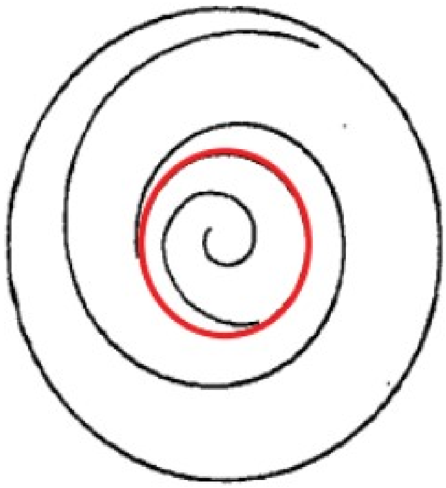


Figure 2.1: [31]First algebraic limit cycle from Poincaré's example,for the system (2.1).

Example 2.2. Consider the planar polynomial differential system

$$\begin{cases} \dot{x} = y + x(3 - x^2 - y^2), \\ \dot{y} = -x + y(1 - 2x^2 - 5y^2). \end{cases}$$

This system has a unique singular point at the origin $(0, 0)$.
To analyze the behavior near the origin and possible limit cycles, observe the following at the origin, the linearized system is:

$$\dot{x} = y + 3x, \quad \dot{y} = -x + y,$$

the nonlinear terms include cubic polynomials in x and y.
Consider the circle:

$$x^2 + y^2 = r^2,$$

for some $r > 0$ to check if a limit cycle exists near a certain radius, examine the radial component of the vector field. Multiply the system by the vector (x, y),

$$x\dot{x} + y\dot{y} = x\left[y + x(3 - x^2 - y^2)\right] + y\left[-x + y(1 - 2x^2 - 5y^2)\right],$$

simplify the expression,

$$\begin{aligned} x\dot{x} + y\dot{y} &= xy + 3x^2 - x^4 - x^2y^2 - xy + y^2 - 2x^2y^2 - 5y^4 \\ &= 3x^2 + y^2 - x^4 - 3x^2y^2 - 5y^4, \end{aligned}$$

rewrite in terms of $r^2 = x^2 + y^2$, this expression gives the rate of change of r^2.
If this quantity changes sign at some radius r, it indicates the possible presence of a limit cycle.

2.2 Existence and non existence of limit cycles in the plane

Theorem 2.1. *[12] If system (1.1) has no singular points, then it does not have any limit cycles.*

Theorem 2.2. *[23] Let (P, Q) be a C^1 vector field defined in the open subset $\Delta \subset \mathbb{R}^2$, $(u(t), v(t))$ a periodic solution of period T of the system (1.1), and $R : \Delta \to \mathbb{R}$ a C^1 map such that*

$$\int_0^T R(u(t), v(t))\, dt \neq 0,$$

and let $U = U(x, y)$ be a C^1 solution of the linear partial differential equation (1.12). Then the closed trajectory

$$\gamma = \{(u(t), v(t)) \in \Delta : t \in [0, T]\},$$

is contained in the set

$$\Sigma = \{(x, y) \in \Delta : U(x, y) = 0\},$$

and γ is not contained in a periodic annulus of the vector field (P, Q). Moreover, if the vector field (P, Q) and the functions R and U is analytic, then γ is a limit cycle.

Theorem 2.3. *[23] Let (P, Q) be a C^1 vector field defined on a non-empty open set $\Omega \subset \mathbb{R}^2$. Let $V = u(x, y)$ be a C^1 solution of the partial differential equation*

$$P(x, y)\frac{\partial V}{\partial x}(x, y) + Q(x, y)\frac{\partial V}{\partial y}(x, y) = \left(\frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)\right) V(x, y).$$

If Γ is a limit cycle of system (1.1), then Γ is contained in

$$\Sigma = \{(x, y) \in \Omega : V(x, y) = 0\}.$$

Theorem 2.4. *[31] Let Ω be a connected domain in $\mathbb{R}^2$. If*

$$\frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y),$$

does not vanish or keeps a constant sign on Ω, then the differential system admits no limit cycle entirely contained in Ω.

2.2.1 Stability of limit cycles

Let γ denote the trajectory corresponding to the limit cycle of system (1.1). Although the neighboring trajectories are not closed, they tend to follow a similar path to γ. The nature of γ as a limit cycle whether it is stable, semi-stable, or unstable is determined by the behavior of these nearby trajectories: they may spiral inward toward γ, spiral outward away from it, or do both. This distinction classifies the limit cycle based on whether surrounding trajectories converge to, diverge from, or partially approach and partially move away from γ.

Theorem 2.5. *[31] Consider a closed trajectory γ representing a limit cycle in a nonlinear dynamical system. The stability characteristics of γ can be described as follows:*

a. Stable (attractive): *If all trajectories in the vicinity of γ, both inside and outside, spiral towards γ as $t \to +\infty$.*

b. Unstable (repulsive): *If all neighboring trajectories spiral towards γ as $t \to -\infty$.*

c. Semi-Stable: *If trajectories inside γ approach it as $t \to +\infty$ while those outside approach it as $t \to -\infty$, or vice versa.*

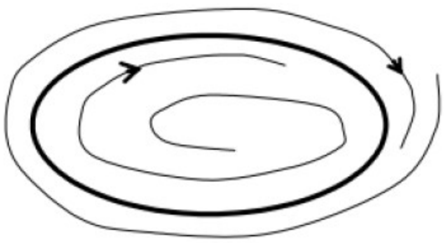


Figure 2.2: Stable limit cycle.

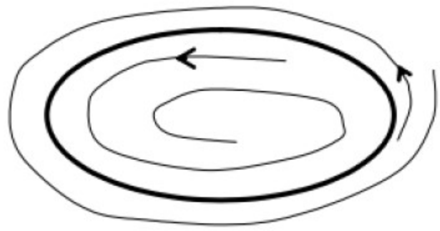


Figure 2.3: Unstable limit cycle.

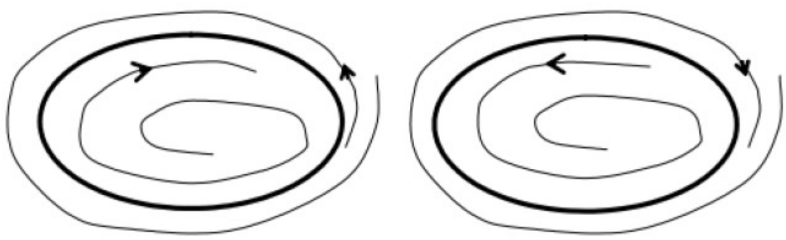


Figure 2.4: Semi-stable limit cycle.

Example 2.3. We consider the following planar system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(2 - 4x) - (4x - 2)y. \end{cases} \tag{2.2}$$

This system exhibits a closed and stable trajectory surrounding an equilibrium point, indicating the existence of a limit cycle in the dynamical behavior of the solutions. The accompanying figure shows the phase portrait of the system, where a closed and stable limit cycle appears surrounding the equilibrium point, confirming the periodic nature of the solutions in this model:

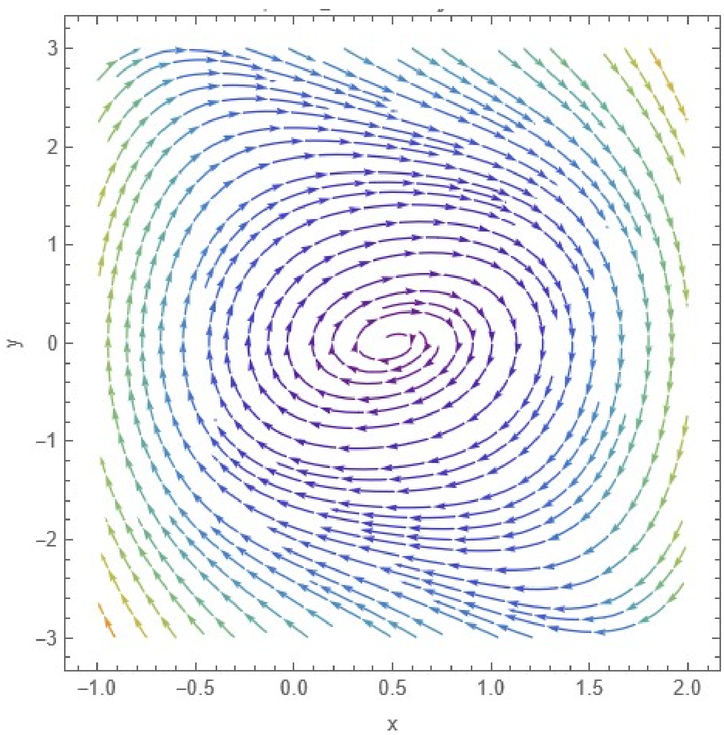


Figure 2.5: Phase portrait of system (2.2) .

Example 2.4. on consider its dynamic behavior in the phase plane (x, y), we define the auxiliary system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = y - y^3 - x. \end{cases} \tag{2.3}$$

This system introduces a second dimension by treating y as the velocity of x, and models the evolution of y with a cubic nonlinearity and a coupling to x. The resulting system allows for phase portrait analysis and the detection of potential limit cycles or equilibrium points.To visualize the dynamics and detect potential closed trajectories, the phase portrait of the planar system is given below:

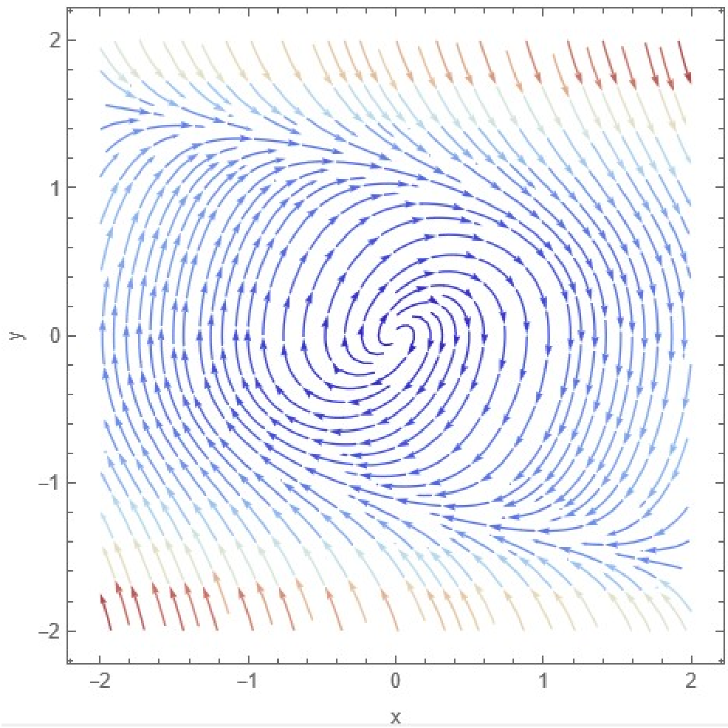


Figure 2.6: phase portrait of the system (2.3).

2.3 Poincaré Map

One of the most fundamental tools for analyzing the stability and bifurcations of periodic orbits is the *Poincaré map* (also known as the *first return map*), introduced by Henri Poincaré in 1881 [40].

The concept behind the Poincaré map is straightforward: if r is a periodic orbit of system

$$\dot{x} = f(x), \tag{2.4}$$

passing through a point x_0, and E is a hyperplane orthogonal to r at x_0, then for any point $x \in E$ sufficiently close to x_0, the solution $\varphi_t(x)$ of the system at $t = 0$ will eventually intersect E again at a point $P(x)$, which lies near x_0, see Figure 2.7. The mapping

$$x \mapsto \Pi(x),$$

is called the *Poincaré map*.

This map can also be defined when E is a smooth surface passing through $x_0 \in r$, provided that it is not tangent to the orbit r at x_0. In such a case, the surface E is said to intersect the orbit r *transversally* at x_0.

The following theorem guarantees the existence and continuity of the Poincaré map $\Pi(x)$, as well as the existence and continuity of its first derivative $D\Pi(x)$.

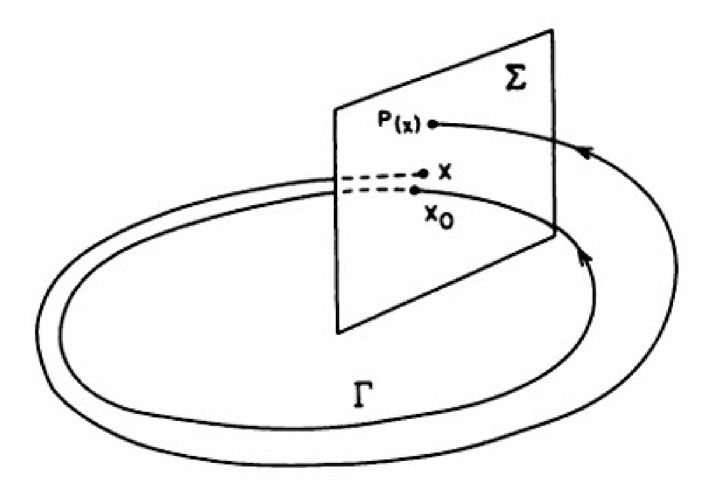


Figure 2.7: [40]The Poincare map.

Theorem 2.6. *[39] Let* Δ *be an open subset of* $\mathbb{R}^n$ *and Let the vector field of system (1.1). Suppose that* $\varphi_t(X_0)$*, is a periodic solution of (1.1) of period* T*, and that the cycle*

$$\gamma = \{X \in \mathbb{R}^n \mid X = \varphi_t(X_0), 0 \le t \le T\}$$

, is contained in Δ *.Let* Σ *be the hyperplane orthogonal to* γ *at* X_0*, i.e., let*

$$\Sigma = \{X \in \mathbb{R}^n \mid (X - X_0) \cdot (P(X_0), Q(X_0)) = 0\}.$$

Then there exists $\delta > 0$ *and a unique function* $\tau(X)$*, defined and continuously differentiable for* $X \in N_\delta(X_0)$*, such that*

$$\tau(X_0) = T \quad \text{and} \quad \varphi_t(X) \in \Sigma \quad \text{for all } X \in N_\delta(X_0).$$

Theorem 2.7. *[39] Let* $\gamma(t)$ *be a periodic solution of (1.1) of period* T*. Then the derivative of the Poincaré map* $\Pi(s)$ *along a straight line* Σ *normal to* $\gamma = \{X \in \mathbb{R}^2 \mid X = \gamma(t) - \gamma(0) \quad 0 \le t \le T\}$ *at* $X = (0,0)$ *is given by*

$$\prod = \exp \int_0^T div \cdot (P(\gamma(t), Q(\gamma(t))dt.$$

Definition 2.3. [39] Let $P(s)$ be the Poincaré map for a cycle r of a planar analytic system , and let

$$d(s) = \Pi(x) - s,$$

be the displacement function. Then if

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0 \quad \text{and} \quad d^{(k)}(0) \neq 0,$$

r is called a *multiple limit cycle* of multiplicity k. If $k = 1$, then r is called a *simple limit cycle.*

Theorem 2.8 (Poincaré). *[39] A planar analytic system cannot have an infinite number of limit cycles which accumulate on a cycle of system.*

For planar analytic systems, it is convenient at this point to discuss the Poincaré map in the neighborhood of a focus and to define what we mean by a multiple focus. *These* *results will prove useful in Chapter 4, where we discuss the bifurcation of limit cycles from a multiple focus.*

Suppose that the planar analytic system has a focus at the origin and that

$$\det Df(0) \neq 0.$$

Then system is linearly equivalent to the system

$$\begin{cases} \dot{x} = ax - by + p(x,y), \\ \dot{y} = bx + ay + q(x,y). \end{cases} \tag{2.4}$$

With $b \neq 0$, where the power series expansions of p and q begin with second or higher degree terms.

In polar coordinates, this system has the form

$$\dot{r} = ar + \mathcal{O}(r^2), \quad \dot{\theta} = b + \mathcal{O}(r).$$

Let $r(t, r_0, \theta_0), \theta(t, r_0, \theta_0)$ be the solution of this system satisfying

$$r(0, r_0, \theta_0) = r_0, \quad \theta(0, r_0, \theta_0) = \theta_0.$$

Then for $r_0 > 0$ sufficiently small and $b > 0$, $\theta(t, r_0, \theta_0)$ is a strictly increasing function of t. Let $t(\theta, r_0, \theta_0)$ be the inverse of this strictly increasing function, and for a fixed θ_0, define the function

$$\Pi(r_0) = r(t(\theta_0 + 2\pi, r_0, \theta_0), r_0, \theta_0).$$

Then for all sufficiently small $r_0 > 0$, $P(r_0)$ is an analytic function of r_0, which is called the Poincaré map *for the focus at the origin of system .*

Similarly, for $b < 0$, $\theta(t, r_0, \theta_0)$ is a strictly decreasing function of t, and the formula

$$\Pi(r_0) = r(t(\theta_0 - 2\pi, r_0, \theta_0), r_0, \theta_0),$$

is used to define the Poincaré map for the focus at the origin in this case.

Theorem 2.9. *[39] Let $\Pi(s)$ be the Poincaré map for a focus at the origin of the planar analytic system with $b \neq 0$, and suppose that $P(s)$ is defined for $0 < s < \delta_0$. Then there exists a $\delta > 0$ such that $P(s)$ can be extended to an analytic function defined for $|s| < \delta$. Furthermore,*

$$\Pi(0) = 0, \quad \Pi'(0) = \exp\left(\frac{2\pi a}{|b|}\right),$$

and if $d(s) = P(s) - s$, then

$$d(s)d(-s) < 0 \quad \text{for } 0 < |s| < \delta.$$

The fact that $d(s)d(-s) < 0$ for $0 < |s| < \delta$ can be used to show that if

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0,$$

then k is odd, i.e., $k = 2m + 1$. The integer $m = \frac{k-1}{2}$ is called the multiplicity of the focus. *If $m = 0$, the focus is called a* simple focus, *and it follows from the above theorem that system , with $b \neq 0$, has a simple focus at the origin if $a \neq 0$.*

The sign of $d'(0)$, i.e, the sign of a, determines the stability of the origin in this case. If $a < 0$, the origin is a stable focus*, and if $a > 0$, the origin is an* unstable focus.

If $d'(0) = 0$, i.e., if $a = 0$, then system has a multiple focus *or* center *at the origin. If $d'(0) = 0$, then the first nonzero derivative*

$$v := d^{(k)}(0) \neq,$$

is called the Liapunov number *for the focus. If $a < 0$, then the focus is stable, and if $a > 0$, it is unstable.*

If $d'(0) = d''(0) = 0$ and $d'''(0) \neq 0$, then the Liapunov number for the focus at the origin of system is given by the formula

$$a - d'''(0) = \frac{1}{26}\Big\{ [3(a_{30} + b_{03}) + (a_{12} + b_{21})] \\ - [2(a_{20}b_{20} - a_{02}b_{02}) - a_{11}(a_{02} + a_{20}) + b_{11}(b_{02} + b_{20})] \Big\}. \quad (2.5)$$

Here, the functions $p(x, y)$ and $q(x, y)$ from system are expressed as:

$$p(x, y) = \sum_{i+j\geq 2} a_{ij}x^i y^j, \quad q(x, y) = \sum_{i+j\geq 2} b_{ij}x^i y^j.$$

This information will be useful in this Section, where we shall see that m limit cycles can be made to bifurcate from a multiple focus of multiplicity m under a suitable small perturbation of system .

Example 2.5. In this example we consider here a particular case of the general system , by setting $a = -1$ and $b = 0$. The resulting system in Cartesian coordinates becomes

$$\begin{cases} \dfrac{dx}{dt} = x(x^2 + y^2 - 1)^2 + (-x + 2y + xy^2 + x^3)(-x^2 - y^2), \\ \dfrac{dy}{dt} = y(x^2 + y^2 - 1)^2 + (-2x - y + x^2y + y^3)(-x^2 - y^2). \end{cases} \quad (2.6)$$

We convert the system to polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$
computing the radial derivative $\frac{dr}{dt}$,

$$\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = r(r^2 - 1)^2 - r^3(r^2 - 1)\sin\theta\cos\theta\cos 2\theta,$$

computing the angular derivative $\frac{d\theta}{dt}$,

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{r^2} = -1 + r^2(2\cos 2\theta - \cos 4\theta),$$

to analyze the limit cycle at $r = 1$, we compute the partial derivative,

$$\left.\frac{\partial \dot{r}}{\partial r}\right|_{r=1} = -1 - \sin\theta\cos\theta\cos 2\theta.$$

Now compute the integral of the partial derivative over the interval $[0, 2\pi]$,

$$\int_0^{2\pi} (-1 - \sin\theta\cos\theta\cos 2\theta)\, d\theta = -2\pi,$$

since the value of the integral is negative ($-2\pi < 0$), the limit cycle at $r = 1$ is stable,

to analyze the stability of the limit cycle more precisely, we compute the second derivative

$$\left.\frac{\partial^2 \dot{r}}{\partial r^2}\right|_{r=1} = 8 - 6\sin\theta\cos\theta\cos 2\theta,$$

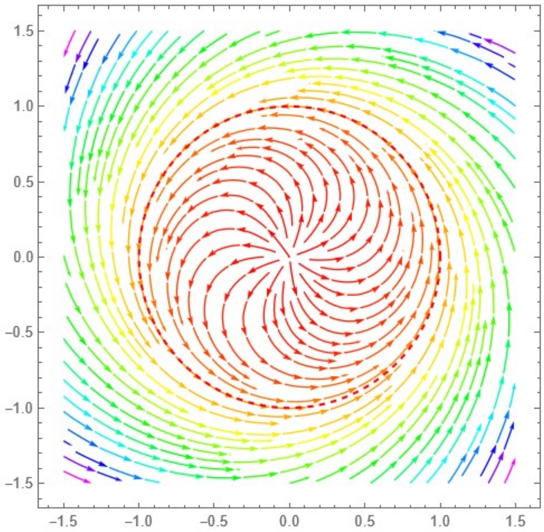
and compute its integral:

$$\int_0^{2\pi} (8 - 6\sin\theta\cos\theta\cos 2\theta)\, d\theta = 16\pi,$$

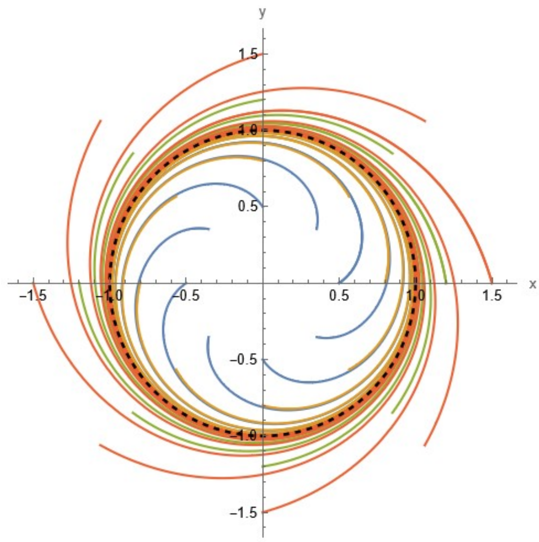
since the integral is positive ($16\pi > 0$), the limit cycle is stable and attracting.

Finally, the angular equation of motion for the system at $r = 1$ is

$$\frac{d\theta}{dt} = -1 + (2\cos 2\theta - \cos 4\theta).$$



(a) Solution trajectories with the limit cycle



(b) Phase portrait with the limit cycle

Figure 2.8: Vector field of systeme (2.4).

figure (2.8) illustrates the behavior of the dynamical system for the values $a = -1$ and $b = 0$, where the solution trajectories are shown to be attracted toward the stable limit cycle at radius $r = 1$. The plot clearly demonstrates how all trajectories converge toward the red circle representing the limit cycle, while the complex angular motion around the cycle exhibits a periodic behavior governed by

$$\frac{d\theta}{dt} = -1 + (2\cos 2\theta - \cos 4\theta).$$

This visual representation confirms the analytical results, which showed the stability of the limit cycle through the analysis of partial derivatives and their integrals.

3 The Study of Rational Limit Cycles for Bernoulli and Riccati Equations

3.1 Rational limit cycles on Bernoulli equations

3.1.1 Introduction

Limit cycles are key features in the qualitative theory of planar differential systems, as they represent isolated periodic behaviors that can be either stable or unstable [15], [39], [28]. Studying their existence, number, and stability is a central problem in differential equations, among the nonlinear first-order equations, certain forms including Bernoulli and Abel equations have attracted attention due to their analytical properties and their potential to exhibit rational limit cycles [13], [2], [14], [22]. Recent research has established upper bounds for the number of such cycles, linking them to the equation's coefficients and degree of nonlinearity [44], [21], understanding these cycles contributes to a deeper grasp of nonlinear dynamics in both theoretical frameworks and real-world applications [42], [27] .

Definition 3.1. A rational limit cycle is an isolated periodic solution of a differential equation such that the solution is a rational function in x that is a ration of two polynomials and not itself a polynomial function.

3.1.2 The Bernoulli Equation

The Bernoulli equation is a first-order nonlinear differential equation of the form

$$\frac{dy}{dx} = A(x)y^n + B(x)y,$$

where $A(x)$ and $B(x)$ are polynomial functions and $n \geq 2$ is a positive integer.This equation can be transformed into a linear differential equation by the substitution $z = y^{1-n}$, which greatly facilitates the study of its solutions, including limit cycles.

Theorem 3.1. *When $n \geq 3$ is odd, there are at most two rational limit cycles of system, and we provide an example of system with two rational limit cycles. Moreover, when $n \geq 4$ is even, there is at most one rational limit cycle of system, and we provide an example of system with one rational limit cycle.*

The proof of Theorem is given in this section.
First, we prove an auxiliary lemma.

Lemma 3.1. *The rational function* $y = \frac{q(x)}{p(x)}$, *with* $p(x)$ *non-constant, is a periodic solution of system if and only if*

$$q(x) = c \in \mathbb{R} \setminus \{0\}, \quad p(0) = p(1), \quad \textit{and } p(x) \textit{ has no zero in } [0,1],$$

and

$$B(x)p(x)^{n-1} + p(x)^{n-2}p'(x) + c^{n-1}A(x) = 0. \tag{3.1}$$

We first prove an auxiliary lemma

Proof of Lemma

Assume that $y(x) = \dfrac{c}{p(x)}$ is a periodic solution of the system, where $p(x)$ is a non-constant, continuous, and non zero function on $[0,1]$, and satisfies $p(0) = p(1)$.
Define the auxiliary function

$$g(x,y) = p(x)y - c. \tag{3.2}$$

Then $g(x, y(x)) = 0$ for all $x \in [0,1]$ Differentiating this identity with respect to x gives:

$$\frac{d}{dx}g(x,y) = p'(x)y + p(x)\frac{dy}{dx} = 0,$$

substituting the differential equation $\dfrac{dy}{dx} = A(x)y^n + B(x)y$ into the expression yields:

$$p'(x)y + p(x)A(x)y^n + p(x)B(x)y = 0, \tag{3.3}$$

since $g(x,y)$ is irreducible and linear in y, we can express the left-hand side of equation (3.3) as $k(x,y)g(x,y)$, where $k(x,y)$ is a polynomial of degree $n-1$ in y. Let

$$k(x,y) = k_0(x) + k_1(x)y + \cdots + k_{n-1}(x)y^{n-1},$$

then

$$k(x,y)g(x,y) = (k_0 + k_1 y + \cdots + k_{n-1}y^{n-1})(p(x)y - c).$$

Expanding and matching coefficients of powers of y gives the identities:

$$\begin{aligned}
\text{Coefficient of } y^0: &\quad -k_0(x)c = 0,\\
\text{Coefficient of } y^1: &\quad p'(x) + p(x)B(x) = k_0(x)p(x) - k_1(x)c,\\
\text{Coefficient of } y^2: &\quad 0 = k_1(x)p(x) - k_2(x)c,\\
&\ \vdots\\
\text{Coefficient of } y^n: &\quad p(x)A(x) = k_{n-1}(x)p(x).
\end{aligned}$$

From the first equation, we get $k_0(x) = 0$, so $c \neq 0$. from the second equation

$$p'(x) + p(x)B(x) = -k_1(x)c \quad \Rightarrow \quad k_1(x) = -\frac{p'(x) + p(x)B(x)}{c},$$

from the recursion

$$k_2(x) = \frac{p(x)}{c}k_1(x), \quad \ldots, \quad k_{n-1}(x) = \frac{p(x)^{n-2}}{c^{n-2}}k_1(x),$$

therefore,

$$A(x)=\frac{k_{n-1}(x)}{p(x)}=\frac{k_1(x)}{c^{n-2}p(x)^{n-1}}=-\frac{p'(x)+p(x)B(x)}{c^{n-1}p(x)^{n-1}},$$

multiplying both sides by $c^{n-1}p(x)^{n-1}$, we obtain,

$$B(x)p(x)^{n-1}+p'(x)p(x)^{n-2}+c^{n-1}A(x)=0,$$

conversely, assume the identity

$$B(x)p(x)^{n-1}+p'(x)p(x)^{n-2}+c^{n-1}A(x)=0,$$

holds for a function $p(x)$ that satisfies $p(0)=p(1)$ and is nonzero on $[0,1]$.

define $y(x)=\dfrac{c}{p(x)}$ then

$$\frac{dy}{dx}=-\frac{cp'(x)}{p(x)^2},$$

and the right-hand side of the differential equation becomes

$$A(x)y^n+B(x)y=\frac{c^nA(x)}{p(x)^n}+\frac{cB(x)}{p(x)},$$

multiply both sides by $p(x)^2$

$$-cp'(x)=c^nA(x)p(x)^{2-n}+cB(x)p(x),$$

divide by c:

$$-p'(x)=c^{n-1}A(x)p(x)^{2-n}+B(x)p(x),$$

multiplying by $p(x)^{n-2}$ gives

$$-p'(x)p(x)^{n-2}=c^{n-1}A(x)+B(x)p(x)^{n-1},$$

which is the given identity in reverse.

Hence $y(x)=\dfrac{c}{p(x)}$ is a solution, and since $p(0)=p(1)$, we also have $y(0)=y(1)$, So the solution is periodic.

3.1.3 Proof of Theorem

Reduction to the case $c=1$:

by Lemma 3, it is not restrictive to assume $c=1$, and we consider rational limit cycles of the form:

$$y=\frac{1}{p(x)}\quad\text{or}\quad y=p(x),$$

where $p(x)$ satisfies:

$p(0)=p(1)$, $p(x)\neq 0$ on $[0,1]$, and $p(x)$ satisfies equation (3.1)

can be rewritten as

$$p^{n-2}(x)p'(x)=-A(x)-B(x)p(x)^{n-1}.$$

Let $u(x)=p(x)^{n-1}$ then, differentiating,

$$u'(x)=(n-1)p(x)^{n-2}p'(x).$$

Substituting into the rewritten equation gives

$$\frac{1}{n-1}u'(x) = -A(x) - B(x)u(x),$$

or equivalently, multiplying through

$$u'(x) = -(n-1)A(x) - (n-1)B(x)u(x),$$

which is a linear ODE

suppose $u_1(x) = p_1(x)^{n-1}$ and $u_2(x) = p_2(x)^{n-1}$ are two solutions of the ODE. Then

$$u_1'(x) = -(n-1)A(x) - (n-1)B(x)u_1(x),$$

$$u_2'(x) = -(n-1)A(x) - (n-1)B(x)u_2(x).$$

Subtracting these equation

$$\frac{d}{dx}(u_1(x) - u_2(x)) = -(n-1)B(x)(u_1(x) - u_2(x)),$$

this last general solution

$$u_1(x) - u_2(x) = Ce^{-(n-1)\int B(x)dx},$$

where C is a constant.

Case 1: $B(x) \not\equiv 0$:

the difference $u_1(x) - u_2(x)$ is non-constant unless $C = 0$. However, the boundary condition $p(0) = p(1)$ implies $u(0) = u(1)$, so

$$C\left(e^{-(n-1)\int_0^1 B(x)dx} - 1\right) = 0,$$

which implies either $C = 0$ or $\int_0^1 B(x)dx = 0$. In both cases, $C = 0$, hence $u_1(x) = u_2(x)$.

Case 2: $B(x) \equiv 0$:

then the equation reduces to

$$u_1'(x) = u_2'(x) \Rightarrow u_1(x) = u_2(x) + C,$$

again, applying $u(0) = u(1)$, we get $C = 0$, so $u_1(x) = u_2(x)$.

Implications for $p(x)$

since $u = p^{n-1}$, uniqueness of u implies

$$p_1(x)^{n-1} = p_2(x)^{n-1} \Rightarrow p_1(x) = \alpha p_2(x),$$

for some constant α with $\alpha^{n-1} = 1$.

3.1.4 Example of Application

we consider the following differential equation

$$y' = -(x^2 - x + 1)^{n-2}(2x - 1),$$

we propose that the rational function,

$$y(x) = \frac{1}{x^2 - x + 1},$$

is a solution. Let us denote,

$$p(x) = x^2 - x + 1 \quad \text{so that} \quad y(x) = \frac{1}{p(x)},$$

we compute the derivative,

$$y'(x) = -\frac{p'(x)}{p(x)^2} = -\frac{2x - 1}{(x^2 - x + 1)^2}.$$

The right-hand side of the differential equation is:

$$-(x^2 - x + 1)^{n-2}(2x - 1),$$

now multiply $y'(x)$ by $p(x)^n$ to match degrees,

$$y'(x) \cdot p(x)^n = -\frac{2x - 1}{p(x)^2} \cdot p(x)^n = -(2x - 1) \cdot p(x)^{n-2},$$

his matches the right-hand side exactly. Thus, $y(x) = \frac{1}{x^2 - x + 1}$ is indeed a solution for any $n \geq 2$.

Now consider whether additional rational solutions of the form:

$$y(x) = \frac{\alpha}{x^2 - x + 1},$$

are also solutions. Taking the derivative,

$$y'(x) = -\frac{\alpha(2x - 1)}{(x^2 - x + 1)^2},$$

to match the right-hand side, we must have,

$$-\alpha \cdot \frac{2x - 1}{(x^2 - x + 1)^2} = -(x^2 - x + 1)^{n-2}(2x - 1) \Rightarrow \alpha = (x^2 - x + 1)^{n-2},$$

but since α is a constant, this only works if the function is identically constant in x. Therefore, to find which constants α are allowed, we impose:

$$\alpha^{n-1} = 1.$$

This equation determines how many rational limit cycles exist.

If $n \geq 4$ is even, the equation $\alpha^{n-1} = 1$ has only one real root $\alpha = 1$ so there is exactly one rational limit cycle.

$$y(x) = \frac{1}{x^2 - x + 1}.$$

If $n \geq 3$ is odd, the equation has two real roots $\alpha = \pm 1$ hence, there are two rational limit cycles

$$y(x) = \pm\frac{1}{x^2 - x + 1}.$$

If $n = 4$ (even), the differential equation becomes,

$$y' = -(x^2 - x + 1)^2(2x - 1),$$

the only rational solution is:

$$y(x) = \frac{1}{x^2 - x + 1}.$$

If $n = 3$ (odd), the equation becomes,

$$y' = -(x^2 - x + 1)(2x - 1),$$

then the rational solutions are:

$$y(x) = \pm\frac{1}{x^2 - x + 1}.$$

3.2 Rational limit cycles on Riccati equations

3.2.1 Introduction

A limit cycle is a closed, isolated periodic orbit whose stability determines nearby trajectory behavior. It plays a key role in understanding long-term dynamics of differential systems across various fields [15], [39], [28], this concept is central to Hilbert's 16th problem, which concerns bounding the number of limit cycles in polynomial systems. Riccati equations, though simple, can exhibit rational limit cycles under certain conditions. The number of these is influenced by the degree of the polynomial coefficients [44], and their reducibility to second-order linear equations allows detailed analysis [2], [13].

3.2.2 The Riccati Equation

The *Riccati equation* is a first-order nonlinear differential equation of the form

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2,$$

where where $A_0(x)$, $A_1(x)$, and $A_2(x)$ are polynomial functions of the independent variable x.

This equation is significant in the theory of differential equations because, despite its nonlinear quadratic term, it can often be transformed into a second-order linear differential equation. This property makes it a valuable tool in both theoretical and applied contexts. Riccati equations appear frequently in various fields such as control theory, quantum mechanics, and mathematical biology, where understanding their solutions including possible limit cycles is essential [2], [13].

Theorem 3.2. *A maximum of one rational limit cycle of system, and we provide an example of system with one rational limit cycle.*

3.2.3 Proof of Theorem

We analyze the uniqueness of rational limit cycles in the system

$$y'(x) = A_0(x) + A_1(x)y(x) + A_2(x)y(x)^2.$$

Assume that $y_0(x) = \dfrac{q_0(x)}{p_0(x)}$ is a rational solution of the system. Define the perturbation function

$$w(x) = y(x) - y_0(x),$$

which represents the deviation from the known rational solution $y_0(x)$, since $y_0(x)$ is a solution, it satisfies

$$y_0'(x) = A_0(x) + A_1(x)y_0(x) + A_2(x)y_0(x)^2,$$

then the derivative of $w(x)$ is given by

$$w'(x) = y'(x) - y_0'(x),$$

using the system, we write

$$w'(x) = \left(A_0(x) + A_1(x)y(x) + A_2(x)y(x)^2\right) - y_0'(x).$$

Since $y(x) = w(x) + y_0(x)$, we substitute:

$$w'(x) = A_0(x) + A_1(x)(w(x) + y_0(x)) + A_2(x)(w(x) + y_0(x))^2 - y_0'(x),$$

expanding the quadratic term,

$$(w + y_0)^2 = w^2 + 2wy_0 + y_0^2,$$

we obtain

$$w'(x) = A_0(x)+A_1(x)w(x)+A_1(x)y_0(x)+A_2(x)w(x)^2+2A_2(x)w(x)y_0(x)+A_2(x)y_0(x)^2-y_0'(x).$$

Using the fact that $y_0(x)$ satisfies the system, the terms $A_0(x)+A_1(x)y_0(x)+A_2(x)y_0(x)^2$ cancel with $y_0'(x)$, and we are left with,

$$w'(x) = A_2(x)w(x)^2 + (A_1(x) + 2A_2(x)y_0(x))\, w(x).$$

Define

$$B(x) = A_1(x) + 2A_2(x)y_0(x),$$

so the equation becomes

$$w'(x) = A_2(x)w(x)^2 + B(x)w(x),$$

which is a Riccati-type (or Bernoulli-type) equation. Such an equation has at most one rational solution. Therefore, the original system can have at most one rational limit cycle.

As an illustrative example, consider the system

$$y' = 1 + xy + y^2.$$

A rational solution is

$$y_0(x) = -\frac{1}{x},$$

define $w(x) = y(x) + \frac{1}{x}$, then

$$w'(x) = w(x)^2 + \left(x - \frac{2}{x}\right) w(x).$$

The solution $w(x) = 0$ corresponds to the limit cycle $y(x) = -\frac{1}{x}$, confirming that it is the unique rational limit cycle.

In conclusion, by transforming the problem via a change of variable, we reduce it to a Bernoulli equation, which proves that system can have at most one rational limit cycle.

Lemma 3.2. *[44] The rational function $w(x) = \frac{r(x)}{s(x)}$, with $r(x)$ non-constant, is a periodic solution of system (6) if and only if*

$$s(x) = c \in \mathbb{R} \setminus \{0\}, \quad r(0) = r(1), \quad \textit{and } r(x) \textit{ has no zero in } [0, 1],$$

and

$$r'(x) + r(x)B(x) + A_2(x)c = 0.$$

Now we prove an auxiliary lemma

Proof of lemma

We consider the differential equation

$$w' = A_2(x)w^2 + B(x)w, \quad \text{where } B(x) = A_1(x) + 2A_2(x)y_0(x).$$

Let $w(x) = \frac{r(x)}{s(x)}$ be a rational function with $r(x)$ non-constant. The function $w(x)$ is a periodic solution if and only if the following conditions hold,

$$s(x) = c \in \mathbb{R} \setminus \{0\}, \quad r(0) = r(1), \quad r(x) \text{ has no zeros in } [0, 1],$$

and

$$r'(x) + r(x)B(x) + A_2(x)c = 0,$$

suppose $w(x) = \frac{s(x)}{r(x)}$ is a periodic solution. Since $w(x)$ is well-defined and periodic on $[0, 1]$, it follows that $r(x) \neq 0$ for all $x \in [0, 1]$, define:

$$g(x, w) = r(x)w - s(x),$$

the total derivative of g along the solution must vanish, giving:

$$\frac{d}{dx} g(x, w) = r'(x)w + r(x)w' - s'(x) = 0,$$

substituting $w' = A_2(x)w^2 + B(x)w$, we obtain:

$$r'(x)w + r(x)(A_2(x)w^2 + B(x)w) - s'(x) = 0.$$

Since $g(x, w)$ is irreducible, there exists a polynomial $k(x, w) = k_0(x) + k_1(x)w$ such that

$$r'(x)w + r(x)(A_2(x)w^2 + B(x)w) - s'(x) = k(x, w)(r(x)w - s(x)),$$

expanding both sides, we compare coefficients of powers of w. This yields three equations

$$-s'(x) = -k_0(x)s(x),$$

$$r'(x) + r(x)B(x) = k_0(x)r(x) - k_1(x)s(x),$$

$$r(x)A_2(x) = k_1(x)r(x),$$

from the first equation, $s'(x) = k_0(x)s(x)$, which implies $s(x)$ must be a non-zero constant c, because a non-constant polynomial cannot divide its own derivative. The third equation gives $k_1(x) = A_2(x)$. Substituting into the second equation yields

$$r'(x) + r(x)B(x) = -A_2(x)c.$$

The periodicity condition $w(0) = w(1)$ translates to $\frac{c}{r(0)} = \frac{c}{r(1)}$, hence $r(0) = r(1)$,

conversely, assume $s(x) = c \neq 0$, $r(0) = r(1)$, $r(x) \neq 0$ on $[0, 1]$, and

$$r'(x) + r(x)B(x) + A_2(x)c = 0,$$

we verify that $w(x) = \frac{c}{r(x)}$ satisfies the differential equation. Differentiating,

$$w' = -\frac{cr'(x)}{r^2(x)},$$

substituting $r'(x) = -r(x)B(x) - A_2(x)c$, we obtain

$$w' = \frac{c(r(x)B(x) + A_2(x)c)}{r^2(x)} = A_2(x)\left(\frac{c}{r(x)}\right)^2 + B(x)\left(\frac{c}{r(x)}\right),$$

which matches the original equation. The periodicity $w(0) = w(1)$ follows directly from $r(0) = r(1)$,

the condition $r(x) \neq 0$ on $[0, 1]$ ensures $w(x)$ remains finite and smooth. The differential relation,

$$r'(x) + r(x)B(x) + A_2(x)c = 0,$$

enforces consistency between $r(x)$ and the coefficients $A_2(x)$, $B(x)$. The requirement $s(x) = c$ ensures $w(x)$ is a non-degenerate rational function. The periodicity is encoded in $r(0) = r(1)$, guaranteeing $w(0) = w(1)$.

The lemma establishes a precise correspondence between periodic rational solutions $w(x) = \frac{c}{r(x)}$ and the Riccati equation through three fundamental constraints.

Simplicity: the denominator must be constant ($s(x) \equiv c \neq 0$), reducing the problem to solving the first-order linear ODE:

$$r'(x) + B(x)r(x) + A_2(x)c = 0,$$

to check when the function $w(x) = c\,r(x)$ is a valid periodic solution to the differential equation

$$w' = A_2(x)w^2 + B(x)w,$$

the following conditions must hold

1. $r'(x) + B(x)\,r(x) + A_2(x)\,c = 0,$
2. $r(x) \neq 0 \quad \text{for all } x \in [0, 1],$

3. $r(0) = r(1)$.

Verification:
Plugging $w = c\,r$ into the differential equation gives:

$$w' = c\,r' = A_2(x)\,c^2\,r^2 + B(x)\,c\,r,$$

rewriting this as,

$$c\,r' = -A_2(x)\,c^2\,r^2 - B(x)\,c\,r,$$

and dividing both sides by $c\,r^2$ (which is valid since $r(x) \neq 0$) yields:

$$-\frac{r'}{r^2} = A_2(x)\,c + \frac{B(x)}{r},$$

multiplying back by r^2 and simplifying,

$$-c\,r' = A_2(x)\,c^2\,r^2 + B(x)\,c\,r,$$

which confirms that the equation holds when the condition $r'(x)+B(x)\,r(x)+A_2(x)\,c = 0$ is satisfied,

going the other way, if r satisfies all three conditions, then

$$w' = c\,r' = -c(B\,r + A_2\,c) = A_2\,w^2 + B\,w,$$

and periodicity of w follows since $r(0) = r(1)$ and $w = c\,r$.

$$\begin{aligned}&\text{Periodic solutions exist when:}\\ &r' + B\,r + A_2\,c = 0,\\ &r(x) \neq 0 \quad \text{for all } x \in [0,1],\\ &r(0) = r(1).\end{aligned}$$

The next part the Proof of Theorem 1 assume two distinct rational limit cycle solutions exist

$$w_1(x) = \frac{1}{r_1(x)}, \tag{3.4}$$

and

$$w_2(x) = \frac{1}{r_2(x)}. \tag{3.5}$$

. Substituting into equation, we obtain the pair of differential equations

$$r_1'(x) + B(x)r_1(x) + A_2(x) = 0, \qquad r_2'(x) + B(x)r_2(x) + A_2(x) = 0,$$

subtracting these equations yields,

$$(r_1'(x) - r_2'(x)) + B(x)(r_1(x) - r_2(x)) = 0,$$

this is a first-order linear differential equation for the difference $\Delta r(x) = r_1(x) - r_2(x)$, whose general solution is:

$$\Delta r(x) = K \exp\left(-\int B(x)\,dx\right),$$

for some constant $K \in \mathbb{R}$.

The periodic boundary conditions $r_1(0) = r_1(1)$ and $r_2(0) = r_2(1)$ imply

$$\Delta r(0) = \Delta r(1) \quad \Rightarrow \quad K = K \exp\left(-\int_0^1 B(x)\,dx\right),$$

this equality holds only if either

$B(x) \equiv 0$, or $K = 0$.

If $B(x) \equiv 0$, then $\Delta r(x) = K$ is constant However, periodicity again gives:

$$r_1(0) - r_2(0) = r_1(1) - r_2(1) \Rightarrow K = K,$$

which is always true. Yet, since $r_1(x) \not\equiv r_2(x)$ by assumption, the only consistent possibility is $K = 0$. Therefore, $r_1(x) \equiv r_2(x)$, contradicting the assumption of distinct solutions.

In the case where $B(x) \equiv 0$, the original equations reduce to

$$r_1'(x) = -A_2(x), \qquad r_2'(x) = -A_2(x),$$

implying $r_1(x) - r_2(x) =$ constant. Again, periodicity forces this constant to vanish, so $r_1(x) \equiv r_2(x)$. Recall that $B(x) = A_1(x) + 2A_2(x)y_0(x)$. If $B(x) \equiv 0$, then

$$A_1(x) = -2A_2(x)y_0(x),$$

using the relation $r_1'(x) = -A_2(x)$, this gives

$$A_1(x) = 2r_1'(x)y_0(x).$$

If $y_0(x) = \frac{q_0(x)}{p_0(x)}$, a rational function, we obtain the explicit expressions

$$A_2(x) = -r_1'(x), \qquad A_1(x) = \frac{2r_1'(x)q_0(x)}{p_0(x)}.$$

The condition $r_1(x) \neq 0$ on $[0,1]$ ensures that $w_1(x) = \frac{1}{r_1(x)}$ is regular on this interval, while the boundary condition $r_1(0) = r_1(1)$ ensures that $w_1(x)$ is periodic and hence a genuine limit cycle.

This complete derivation shows that any two rational limit cycle solutions must be identical, since their difference vanishes identically under the imposed conditions. Furthermore, the coefficient functions $A_1(x)$ and $A_2(x)$ are uniquely determined by the rational solution $r_1(x)$ and the particular solution $y_0(x)$, leaving no flexibility for the existence of distinct rational limit cycles. This uniqueness result is critical in the classification of rational periodic solutions to Riccati-type differential systems.

Assume two distinct periodic solutions $w_1(x) = \frac{1}{r_1(x)}$ and $w_2(x) = \frac{1}{r_2(x)}$ exist for the system,

$$w' = A_2(x)w^2 + B(x)w,$$

substituting into the differential equation yields,

$$r_1'(x) + B(x)r_1(x) + A_2(x) = 0,$$

$$r_2'(x) + B(x)r_2(x) + A_2(x) = 0,$$

subtracting these equations gives the linear ODE:

$$(r_1'(x) - r_2'(x)) + B(x)(r_1(x) - r_2(x)) = 0,$$

with general solution,

$$r_1(x) - r_2(x) = Ke^{-\int B(x)\,dx},$$

periodic boundary conditions $r_1(0) = r_1(1)$ and $r_2(0) = r_2(1)$ require

$$Ke^{-\int_0^1 B(x)\,dx} = K.$$

This holds only when either $B(x) \equiv 0$ or $K = 0$. In the case $B(x) \equiv 0$, the equation reduces to $r_1(x) = r_2(x) + C$, but periodicity enforces $C = 0$, so $r_1 \equiv r_2$. Thus, solution uniqueness is established,

the coefficients must satisfy

$$A_2(x) = -r_1'(x),$$
$$B(x) = A_1(x) + 2A_2(x)y_0(x) = 0,$$

which gives

$$A_1(x) = -2A_2(x)y_0(x) = 2r_1'(x)\frac{q_0(x)}{p_0(x)}.$$

For polynomial consistency, we require that $r_1'(x)$ factors as:

$$r_1'(x) = T(x)p_0(x),$$

making the coefficients

$$A_2(x) = -T(x)p_0(x), \qquad A_1(x) = 2T(x)q_0(x),$$

any additional solution of the form $w_2 = \frac{1}{r_1+\alpha}$ would require

$$\frac{r_1'(x)}{r_1(x) + \alpha} = T(x),$$

but since $r_1'(x) = T(x)p_0(x)$, this expression cannot be rational unless $\alpha = 0$. Thus, no additional distinct rational limit cycles can exist.

The non-vanishing condition $r_1(x) \neq 0$ on the interval $[0, 1]$ and $r_1'(x) \neq 0$ ensures the solution $w_1(x) = 1/r_1(x)$ is regular. The periodicity condition $r_1(0) = r_1(1)$ guarantees the limit cycle property.

This complete derivation proves the system admits at most one non-trivial rational limit cycle solution under the given constraints.

Given the Riccati equation

$$w' = A_2(x)w^2 + B(x)w,$$

we consider rational solutions of the form

$$w(x) = \frac{1}{r(x)},$$

substitution yields the linear equation

$$r'(x) + B(x)r(x) + A_2(x) = 0.$$

For a particular solution $y_0(x) = \frac{q_0(x)}{p_0(x)}$, we establish the expression for $B(x)$ using the identity

$$B(x) = A_1(x) + 2A_2(x)y_0(x),$$

assuming $r'(x) = T(x)p_0(x)$, the coefficient functions become

$$A_2(x) = -T(x)p_0(x), \qquad A_1(x) = 2T(x)q_0(x),$$

the consistency condition for $A_0(x)$ requires

$$p_0(x) \mid \left[q_0'(x)p_0(x) - q_0(x)p_0'(x) - T(x)q_0(x)^2\right],$$

since $\gcd(q_0, p_0) = 1$, the divisibility condition implies

$$p_0(x) \mid p_0'(x),$$

which holds only when $p_0(x)$ is constant.

Consider a hypothetical second solution of the form

$$\frac{1}{r(x) + \alpha}.$$

This would imply

$$\frac{r'(x)}{r(x) + \alpha} = T(x) \quad \Rightarrow \quad r(x) + \alpha = \frac{r'(x)}{T(x)} = p_0(x),$$

so we require:

$$p_0(x) \mid r(x) + \alpha,$$

which is impossible for arbitrary α, unless $p_0(x)$ is constant and $\alpha = 0$. Thus, the solution is unique.

3.2.4 Example of Application

Consider the Riccati equation

$$\frac{dy}{dx} = (2 - 4x)y^2 + (4x - 2)y.$$

Let $y_0(x) = 1$. Substituting into the differential equation

$$(2 - 4x)(1)^2 + (4x - 2)(1) = 2 - 4x + 4x - 2 = 0,$$

hence, $y_0(x) = 1$ is indeed a particular solution.

Set $w(x) = \frac{1}{r(x)}$, so that the Riccati equation becomes

$$w' = (2 - 4x)w^2 + (4x - 2)w,$$

substituting $w = \frac{1}{r}$ and computing w' gives

$$-\frac{r'}{r^2} = (2 - 4x)\left(\frac{1}{r^2}\right) + (4x - 2)\left(\frac{1}{r}\right),$$

multiply both sides by r^2

$$-r' = (2-4x)(1) + (4x-2)r,$$

thus, we obtain the linear ODE:

$$r'(x) + (4x-2)r(x) = -(2-4x),$$

let the integrating factor be

$$\mu(x) = \exp\left(\int (4x-2)\,dx\right) = e^{2x^2-2x},$$

then,

$$r(x) = e^{-2x^2+2x}\left(C - \int (2-4x)e^{2x^2-2x}dx\right),$$

let us choose a solution directly

$$r(x) = e^{-2x^2+2x} + 1,$$

evaluate at the endpoints

$$r(0) = e^0 + 1 = 2, \quad r(1) = e^0 + 1 = 2.$$

So, $r(0) = r(1)$, satisfying periodicity
since $e^{-2x^2+2x} > 0$ for all $x \in [0,1]$, we have

$$r(x) > 0 \quad \forall x \in [0,1],$$

ensuring no poles in $w(x)$ or $y(x)$.
The rational solution is:

$$y(x) = \frac{1}{r(x)} = \frac{1}{e^{-2x^2+2x}+1},$$

let us verify

$$y' = \frac{(2-4x)e^{-2x^2+2x}}{(e^{-2x^2+2x}+1)^2},$$

and compare with the right-hand side of the original Riccati equation

$$(2-4x)y^2 + (4x-2)y = (2-4x)\left(\frac{1}{e^{-2x^2+2x}+1}\right)^2 + (4x-2)\left(\frac{1}{e^{-2x^2+2x}+1}\right),$$

which simplifies to the same expression as y', confirming that the solution is correct.
The Riccati equation

$$\frac{dy}{dx} = (2-4x)y^2 + (4x-2)y,$$

admits a rational, periodic, and non-vanishing solution:

$$y(x) = \frac{1}{e^{-2x^2+2x}+1},$$

with a particular solution $y_0(x) = 1$, and satisfies all required conditions for a rational limit cycle.

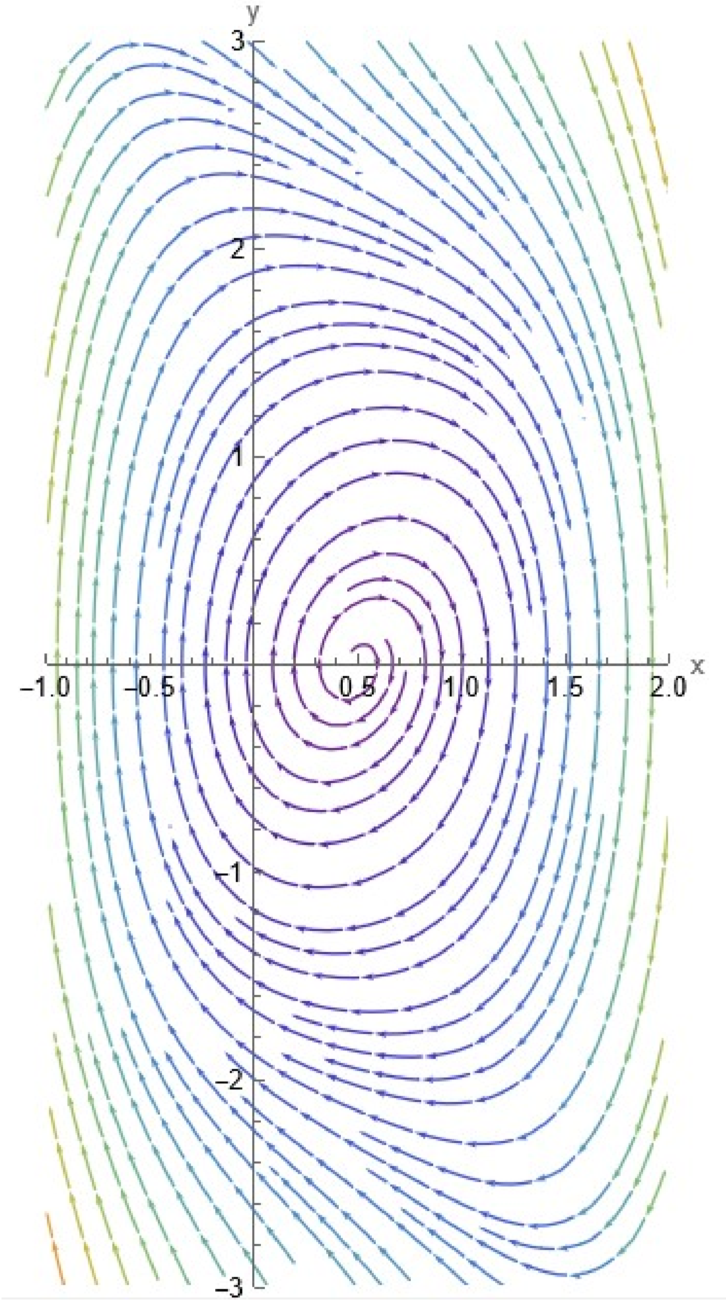


Figure 3.1: Phase Portrait with Limit Cycle

4 Conclusion

In this memoire , we have studied the rational limit cycles associated with two important classes of differential equations: Bernoulli equations and Riccati equations, focusing on cases where the coefficients are real polynomials. After presenting the necessary theoretical background, we examined key recent results in this area, particularly those found in the work of [44], which established sharp upper bounds for the number of possible rational limit cycles in these equations, work showed that Bernoulli equations can have at most two rational limit cycles when the exponent n is odd, and at most one when n is even. In the case of Riccati equations, it was proven that they can admit no more than two rational limit cycles. These results were further illustrated by concrete examples that demonstrate the optimality of the bounds. The significance of these findings lies not only in their theoretical contribution but also in their potential applications for understanding the periodic behavior of dynamical systems, particularly those modeled by differential equations that admit rational solutions. we hope that this work contributes to a deeper understanding of the structure of rational periodic solutions for Bernoulli and Riccati equations and serves as a foundation for future research aiming to extend this analysis to other types of equations or to explore the stability and dynamics associated with such solutions.

Technical Note

All graphs and illustrative plots presented in this thesis were generated using **Wolfram Mathematica**. The software was chosen for its precision and effectiveness in visualizing numerical solutions and analyzing the qualitative behavior of differential systems.

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Université de Ghardaïa
Faculté des Sciences et de la Technologie

جامعة غرداية
كلية العلوم والتكنولوجيا

قسم : رياضيات و اعلام الي

شعبة : رياضيات

تخصص: تحليل دالي و تطبيقات .

غرداية في : 17/12/2025

شهادة ترخيص بالتصحيح والايداع:

انا الاستاذ(ة) [illegible]

بصفتي المشرف المسؤول عن تصحيح مذكرة تخرج (ليسانس/ماستر/دكتورا) المعنونة بـ: The limite cycle of Bernolli and Riccati equation

من انجاز الطالب (الطلبة):

دحو باية

التي نوقشت/قييمت بتاريخ: 22 جوان 2025

اشهد ان الطالب/الطلبة قد قام /قاموا بالتعديلات والتصحيحات المطلوبة من طرف لجنة المناقشة وقد تم التحقق من ذلك من طرفنا

وقد استوفت جميع الشروط المطلوبة .

مصادقة رئيس القسم

رئيس قسم الرياضيات و الإعلام الآلي
الحاج موسى [illegible]

جامعة غرداية
قسم الرياضيات و الاعلام الآلي
كلية العلوم والتكنولوجيا

امضاء المسؤول عن التصحيح